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A Note On The Multiplicative KG-Sombor Index Of Trees*

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Abstract

The Sombor index is a newly introduced vertex-degree-based graph invariant with the ability to predict the enthalpy of vaporization and entropy of octane isomers. Recently, a variant of the Sombor index namely the multiplicative KG-Sombor index was put forward. The aim of this paper is to establish a sharp lower bound on the multiplicative KG-Sombor index for trees with a given order and maximum degree and to identify the extremal trees reaching the lower bound. The obtained results are extended to all connected graphs with a given order and maximum vertex degree.

1 Introduction

Consider a simple connected graph Θ with the vertex set $\mathcal{V}(\Theta)$ and the edge set $\mathcal{E}(\Theta)$. For $\alpha \in \mathcal{V}(\Theta)$, the set

$$\mathcal{N}_\Theta(\alpha) = \{\beta \in \mathcal{V}(\Theta) : \alpha\beta \in \mathcal{E}(\Theta)\},$$

is the open neighborhood of α in Θ and the degree $d_\Theta(\alpha)$ of α in Θ is the order of $\mathcal{N}_\Theta(\alpha)$. If $d_\Theta(\alpha) = 1$, we say α is a pendant vertex. By $\mathfrak{D} = \mathfrak{D}_\Theta$, we mean the maximum degree of Θ. For $\varrho = \alpha\beta \in \mathcal{E}(\Theta)$, the degree $d_\Theta(\varrho)$ is the number of edges incident to ϱ which equals $d_\Theta(\alpha) + d_\Theta(\beta) - 2$. The distance $d_\Theta(\alpha, \beta)$ is the number of edges in any shortest path connecting the vertices α and β in Θ.

Graph invariants are real numbers related to a graph which are invariant under all isomorphisms of the graph. One of the most important categories of graph invariants are *vertex-degree-based invariants* which are formulated based on the degree of vertices of graph. Various vertex-degree-based indices have been presented so far, from which we can mention the Zagreb indices [16, 17], Zagreb coindices [3, 4, 10], Randić connectivity index [18, 19, 28], harmonic index [11], Sum-connectivity index [2, 34], Albertson irregularity index [1, 5], forgotten topological index [13, 17], Lanzhou index [6, 8, 33], inverse sum indeg index [12, 32], etc. One of the newly introduced members of this category is the *Sombor index* which was introduced by Gutman [14] in 2021. It has found applications in various fields [30]. The Sombor index is formulated for a graph Θ by

$$SO(\Theta) = \sum_{\alpha\beta \in \mathcal{E}(\Theta)} \sqrt{d^2_\Theta(\alpha) + d^2_\Theta(\beta)}.$$

It has been shown in [29] that the Sombor index has the ability to predict the enthalpy of vaporization and entropy of octane isomers.

After the introduction of the Sombor index, certain variants of this index such as the reduced and increased Sombor indices [7, 14], reverse Sombor index [31], multiplicative Sombor index [22], modified Sombor index [25], and irregularity Sombor index [21, 9] have been proposed.

The *KG-Sombor index* is one of the variants of the Sombor index which was put forward by Kulli et al. [23] in 2022 as

$$KG(\Theta) = \sum_{\alpha\varrho} \sqrt{d^2_\Theta(\alpha) + d^2_\Theta(\varrho)},$$

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where the sum runs over all vertices $\alpha \in \mathcal{V}(\Theta)$ and all edges $\varrho \in \mathcal{E}(\Theta)$ that is incident to α. This index can also be written by

$$\begin{aligned} KG(\Theta) &= \sum_{\varrho=\alpha\beta\in\mathcal{E}(\Theta)} \left(\sqrt{d_\Theta^2(\alpha)+d_\Theta(\varrho)^2}+\sqrt{d_\Theta^2(\beta)+d_\Theta(\varrho)^2}\right) \\ &= \sum_{\alpha\beta\in\mathcal{E}(\Theta)} \left(\sqrt{d_\Theta^2(\alpha)+(d_\Theta(\alpha)+d_\Theta(\beta)-2)^2}+\sqrt{d_\Theta^2(\beta)+(d_\Theta(\alpha)+d_\Theta(\beta)-2)^2}\right). \end{aligned}$$

For further results on this invariant, see [15, 20, 26, 27].

The multiplicative version of the KG-Sombor index (also called *multiplicative KG-Sombor index*) was considered by Kulli [24] in 2022 and formulated by

$$\begin{aligned} MKG(\Theta) &= \prod_{\varrho=\alpha\beta\in\mathcal{E}(\Theta)} \left(\sqrt{d_\Theta^2(\alpha)+d_\Theta(\varrho)^2}+\sqrt{d_\Theta^2(\beta)+d_\Theta(\varrho)^2}\right) \\ &= \prod_{\alpha\beta\in\mathcal{E}(\Theta)} \left(\sqrt{d_\Theta^2(\alpha)+(d_\Theta(\alpha)+d_\Theta(\beta)-2)^2}+\sqrt{d_\Theta^2(\beta)+(d_\Theta(\alpha)+d_\Theta(\beta)-2)^2}\right). \end{aligned}$$

This paper is concerned with studying some basic mathematical properties of the multiplicative KG-Sombor index. More precisely, we present a sharp lower bound for the multiplicative KG-Sombor index of trees with a given order and maximum degree and identify the trees that can achieve this lower bound.

We finish this section with the following observation which is an immediate consequence of the definition of the Multiplicative KG-Sombor index.

Observation 1 *For every $\varrho \notin \mathcal{E}(\Theta)$, $MKG(\Theta+\varrho) > MKG(\Theta)$.*

2 Main Results

A *tree* is a simple connected acyclic graph. A *rooted tree* is a tree in which a distinguished vertex is selected as the *root* of the tree. If a tree Υ contains at most one vertex σ of degree at least 3, then Υ is said a *starlike tree* (also called a *spider*) with the *center* σ. A path from σ to any pendant vertex of Υ is called a *leg*. Thus, S_η (a star with η vertices) can be seen as a starlike tree with $\eta-1$ legs (each of length 1) and P_η (a path with η vertices) is a starlike tree with 1 or 2 legs. In what follows, $\Upsilon_{\eta,\mathfrak{D}}$ stands for the collection of all trees with order η and maximum degree $\mathfrak{D}$.

Lemma 1 *Suppose that $\Upsilon \in \Upsilon_{\eta,\mathfrak{D}}$ is a rooted tree whose root is on a vertex ρ with $d_\Upsilon(\rho)=\mathfrak{D}$. If Υ has a vertex of degree at least 3 except ρ, then there exists a tree $\Upsilon' \in \Upsilon_{\eta,\mathfrak{D}}$ such that $MKG(\Upsilon') < MKG(\Upsilon)$.*

Proof. Among the non-root vertices of Υ with degree at least 3, let υ have the maximum distance from ρ and let $d_\Upsilon(\upsilon)=\kappa$. Assume that $\mathcal{N}_\Upsilon(\upsilon)=\{\upsilon_1,\upsilon_2,\ldots,\upsilon_\kappa\}$, where υ_κ is considered to be on the unique path connecting υ and ρ. Obviously, $d_\Upsilon(\upsilon_i)\le 2$ for each $1\le i\le \kappa-1$. Then three cases can be considered.

Case 1. At least two members of $\mathcal{N}_\Upsilon(\upsilon)$ are pendant vertices.
We may assume that, υ_1 and υ_2 are pendant vertices. Consider Υ' as the tree derived from Υ by deleting the edge $\upsilon\upsilon_1$ and adding the edge $\upsilon_1\upsilon_2$. Then

$$\begin{aligned} & MKG(\Upsilon)-MKG(\Upsilon') \\ = & \left(\sqrt{d_\Upsilon^2(\upsilon_1)+(d_\Upsilon(\upsilon)+d_\Upsilon(\upsilon_1)-2)^2}+\sqrt{d_\Upsilon^2(\upsilon)+(d_\Upsilon(\upsilon)+d_\Upsilon(\upsilon_1)-2)^2}\right) \\ & \times\left(\sqrt{d_\Upsilon^2(\upsilon_2)+(d_\Upsilon(\upsilon)+d_\Upsilon(\upsilon_2)-2)^2}+\sqrt{d_\Upsilon^2(\upsilon)+(d_\Upsilon(\upsilon)+d_\Upsilon(\upsilon_2)-2)^2}\right) \\ & \times\prod_{i=3}^{\kappa}\left(\sqrt{d_\Upsilon^2(\upsilon_i)+(d_\Upsilon(\upsilon)+d_\Upsilon(\upsilon_i)-2)^2}+\sqrt{d_\Upsilon^2(\upsilon)+(d_\Upsilon(\upsilon)+d_\Upsilon(\upsilon_i)-2)^2}\right) \end{aligned}$$

$$\begin{aligned}
& -\Big(\sqrt{d^2_{\Upsilon'}(\upsilon_1)+(d_{\Upsilon'}(\upsilon_1)+d_{\Upsilon'}(\upsilon_2)-2)^2}+\sqrt{d^2_{\Upsilon'}(\upsilon_2)+(d_{\Upsilon'}(\upsilon_1)+d_{\Upsilon'}(\upsilon_2)-2)^2}\Big)\\
& \times\Big(\sqrt{d^2_{\Upsilon'}(\upsilon_2)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_2)-2)^2}+\sqrt{d^2_{\Upsilon'}(\upsilon)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_2)-2)^2}\Big)\\
& \times\prod_{i=3}^{\kappa}\Big(\sqrt{d^2_{\Upsilon'}(\upsilon_i)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_i)-2)^2}+\sqrt{d^2_{\Upsilon'}(\upsilon)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_i)-2)^2}\Big)\\
\geq\ & \big(\sqrt{\kappa^2+(\kappa-1)^2}+\sqrt{1+(\kappa-1)^2}\ \big)^2-(\sqrt{2}+\sqrt{5})\big(\sqrt{2(\kappa-1)^2}+\sqrt{(\kappa-1)^2+4}\big)\\
=\ & \kappa^2+2(\kappa-1)^2+1+2\sqrt{\kappa^2+(\kappa-1)^2}\\
& \times\sqrt{1+(\kappa-1)^2}-2(\kappa-1)-\sqrt{10}(\kappa-1)-(\sqrt{2}+\sqrt{5})\sqrt{(\kappa-1)^2+4}\\
>\ & 7(\kappa-1)+1+2\sqrt{5}\sqrt{\kappa^2+(\kappa-1)^2}-(2+\sqrt{10})(\kappa-1)-(\sqrt{2}+\sqrt{5})\sqrt{(\kappa-1)^2+4}\\
>\ & 0.
\end{aligned}$$

Case 2. Exactly one of the members of $\mathcal{N}_\Upsilon(\upsilon)$ is a pendant vertex.

We may assume that, υ_1 is a pendant vertex and $\upsilon x_1x_2\dots x_l$, $l\geq 2$, be a path in Υ with $x_1=\upsilon_2$. Let Υ' denote the tree derived from Υ by deleting the edge $\upsilon\upsilon_1$ and adding the edge $x_l\upsilon_1$. Then

$$\begin{aligned}
& MKG(\Upsilon)-MKG(\Upsilon')\\
=\ & \Big(\sqrt{d^2_{\Upsilon}(\upsilon_1)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_1)-2)^2}+\sqrt{d^2_{\Upsilon}(\upsilon)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_1)-2)^2}\Big)\\
& \times\Big(\sqrt{d^2_{\Upsilon}(x_l)+(d(x_l)+d_{\Upsilon}(x_{l-1})-2)^2}+\sqrt{d^2_{\Upsilon}(x_{l-1})+(d_{\Upsilon}(x_l)+d_{\Upsilon}(x_{l-1})-2)^2}\Big)\\
& \times\prod_{i=2}^{\kappa}\Big(\sqrt{d^2_{\Upsilon}(\upsilon_i)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_i)-2)^2}+\sqrt{d^2_{\Upsilon}(\upsilon)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_i)-2)^2}\Big)\\
& -\Big(\sqrt{d^2_{\Upsilon'}(\upsilon_1)+(d_{\Upsilon'}(\upsilon_1)+d_{\Upsilon'}(x_l)-2)^2}+\sqrt{d^2_{\Upsilon'}(x_l)+(d_{\Upsilon'}(\upsilon_1)+d_{\Upsilon'}(x_l)-2)^2}\Big)\\
& \times\Big(\sqrt{d^2_{\Upsilon'}(x_l)+(d_{\Upsilon'}(x_l)+d_{\Upsilon'}(x_{l-1})-2)^2}+\sqrt{d^2_{\Upsilon'}(x_{l-1})+(d_{\Upsilon'}(x_l)+d_{\Upsilon'}(x_{l-1})-2)^2}\Big)\\
& \times\prod_{i=2}^{\kappa}\Big(\sqrt{d^2_{\Upsilon'}(\upsilon_i)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_i)-2)^2}+\sqrt{d^2_{\Upsilon'}(\upsilon)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_i)-2)^2}\Big)\\
\geq\ & \Big(\sqrt{\kappa^2+(\kappa-1)^2}+\sqrt{1+(\kappa-1)^2}\Big)(\sqrt{2}+\sqrt{5})-2\sqrt{8}(\sqrt{2}+\sqrt{5})\\
\geq\ & (\sqrt{13}+\sqrt{5}-2\sqrt{8})\times(\sqrt{2}+\sqrt{5})\\
\approx\ & 0.1847\times(\sqrt{2}+\sqrt{5})\\
>\ & 0.
\end{aligned}$$

Case 3. None of the members of $\mathcal{N}_\Upsilon(\upsilon)$ are pendant vertices.
Let $\upsilon x_1x_2\dots x_t$ and $\upsilon y_1y_2\dots y_s$, $t,s\geq 2$, be two paths in Υ with $x_1=\upsilon_1$ and $y_1=\upsilon_2$. Let Υ' be the tree derived from Υ by deleting the edge $\upsilon\upsilon_1$ and adding the edge $y_s\upsilon_1$. Then

$$\begin{aligned}
& MKG(\Upsilon)-MKG(\Upsilon')\\
=\ & \Big(\sqrt{d^2_{\Upsilon}(\upsilon_1)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_1)-2)^2}+\sqrt{d^2_{\Upsilon}(\upsilon)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_1)-2)^2}\ \Big)\\
& \times\Big(\sqrt{d^2_{\Upsilon}(y_s)+(d_{\Upsilon}(y_s)+d_{\Upsilon}(y_{s-1})-2)^2}+\sqrt{d^2_{\Upsilon}(y_{s-1})+(d_{\Upsilon}(y_s)+d_{\Upsilon}(y_{s-1})-2)^2}\ \Big)\\
& \times\prod_{i=2}^{\kappa}\Big(\sqrt{d^2_{\Upsilon}(\upsilon_i)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_i)-2)^2}+\sqrt{d^2_{\Upsilon}(\upsilon)+(d_{\Upsilon}(\upsilon)+d_{\Upsilon}(\upsilon_i)-2)^2}\ \Big)\\
& -\Big(\sqrt{d^2_{\Upsilon'}(\upsilon_1)+(d_{\Upsilon'}(\upsilon_1)+d_{\Upsilon'}(y_s)-2)^2}+\sqrt{d^2_{\Upsilon'}(y_s)+(d_{\Upsilon'}(\upsilon_1)+d_{\Upsilon'}(y_s)-2)^2}\ \Big)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sqrt{d^2_{\Upsilon'}(y_s)+(d_{\Upsilon'}(y_s)+d_{\Upsilon'}(y_{s-1})-2)^2}+\sqrt{d^2_{\Upsilon'}(y_{s-1})+(d_{\Upsilon'}(y_s)+d_{\Upsilon'}(y_{s-1})-2)^2}\ \right)\\
&\times\prod_{i=2}^{\kappa}\left(\sqrt{d^2_{\Upsilon'}(\upsilon_i)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_i)-2)^2}+\sqrt{d^2_{\Upsilon'}(\upsilon)+(d_{\Upsilon'}(\upsilon)+d_{\Upsilon'}(\upsilon_i)-2)^2}\ \right)\\
\geq\ &\left(\sqrt{\kappa^2+\kappa^2}+\sqrt{4+\kappa^2}\right)\left(\sqrt{2}+\sqrt{5}\right)-2\sqrt{8}\left(\sqrt{2}+\sqrt{5}\right)\\
\geq\ &\left(\sqrt{18}+\sqrt{13}-2\sqrt{8}\right)\left(\sqrt{2}+\sqrt{5}\right)\\
>\ &0.
\end{aligned}
$$

Hence the result follows. ■

Lemma 2 *Let Σ be a starlike tree of order η and $\mathfrak{D}\geq 3$ legs. If Σ has a leg of length 1 and another leg of length at least 3, then there exists a starlike tree Σ' of order η and $\mathfrak{D}$ legs for which $MKG(\Sigma)>MKG(\Sigma')$.*

Proof. Assume that σ is the center of Σ and $N_\Sigma(\sigma)=\{\sigma_1,\sigma_2,\ldots,\sigma_{\mathfrak{D}}\}$. Root Σ at σ. We may assume that $d_\Sigma(\sigma_1)=1$ and $\sigma_2y_1y_2\ldots y_t$, $t\geq 2$ is a longest leg of Σ. Let Σ' show the tree obtained from Σ by deleting the edge y_ty_{t-1} and adding the edge σ_1y_t. Then

$$
\begin{aligned}
MKG(\Sigma)-MKG(\Sigma')=&2\sqrt{8}\left(\sqrt{\mathfrak{D}^2+(\mathfrak{D}-1)^2}+\sqrt{1+(\mathfrak{D}-1)^2}\right)\\
&-\left(\sqrt{2\ \mathfrak{D}^2}+\sqrt{4+\mathfrak{D}^2}\ \right)\left(\sqrt{2}+\sqrt{5}\right)>0,
\end{aligned}
$$

from which we get the desired result. ■

Here is the main theorem of the paper.

Theorem 1 *For any tree $\Upsilon\in\Upsilon_{\eta,\mathfrak{D}}$ of order $\eta\geq 3$,*

$$MKG(\Upsilon)\geq(\sqrt{\mathfrak{D}^2+4}+\sqrt{2}\ \mathfrak{D})^{\mathfrak{D}}(\sqrt{5}+\sqrt{2})^{\mathfrak{D}}(4\sqrt{2})^{\eta-2\mathfrak{D}-1},$$

when $\mathfrak{D}\leq\frac{\eta-1}{2}$ and

$$MKG(\Upsilon)\geq(\sqrt{\mathfrak{D}^2+(\mathfrak{D}-1)^2}+\sqrt{1+(\mathfrak{D}-1)^2})^{2\mathfrak{D}+1-\eta}(\sqrt{\mathfrak{D}^2+4}+\sqrt{2}\ \mathfrak{D})^{\eta-\mathfrak{D}-1}(\sqrt{5}+\sqrt{2})^{\eta-\mathfrak{D}-1},$$

when $\mathfrak{D}>\frac{\eta-1}{2}$. The equality occurs if and only if Υ is a starlike tree whose all legs are of a length less than 3 or all legs are of length more than 1.

Proof. Let $\Upsilon^*\in\Upsilon_{\eta,\mathfrak{D}}$ be a tree with $MKG(\Upsilon^*)\leq MKG(\Upsilon)$ for each $\Upsilon\in\Upsilon_{\eta,\mathfrak{D}}$. Select a vertex ρ of Υ^* of degree $\mathfrak{D}$ as the root of Υ^*. If $\mathfrak{D}=2$, then $\Upsilon\cong P_\eta$ and

$$MKG(\Upsilon)=MKG(P_\eta)=(4\sqrt{2})^{(\eta-3)}(\sqrt{2}+\sqrt{5})^2,$$

as desired. Let $\mathfrak{D}\geq 3$. By the selection of Υ^* and Lemma 1, we conclude that Υ^* is a starlike tree centered at ρ. By Lemma 2, we deduce that all legs of Υ^* are of length more than 1 or all are of a length less than 3. In the first case when all legs of Υ^* are of length more than 1, we have $\mathfrak{D}\leq\frac{\eta-1}{2}$ and

$$MKG(\Upsilon^*)=(\sqrt{\mathfrak{D}^2+4}+\sqrt{2}\ \mathfrak{D})^{\mathfrak{D}}(\sqrt{5}+\sqrt{2})^{\mathfrak{D}}(4\sqrt{2})^{\eta-2\mathfrak{D}-1},$$

as desired. In the second case when all legs of Υ^* are of a length less than 3, by the previous case, it can be assumed that Υ^* contains a leg of length 1. If $\Upsilon^*\cong S_\eta$, then there is nothing to prove; otherwise the number of pendant vertices adjacent to ρ is $2\mathfrak{D}+1-\eta$ and we have

$$MKG(\Upsilon^*)=(\sqrt{\mathfrak{D}^2+(\mathfrak{D}-1)^2}+\sqrt{1+(\mathfrak{D}-1)^2})^{2\mathfrak{D}+1-\eta}(\sqrt{\mathfrak{D}^2+4}+\sqrt{2}\mathfrak{D})^{\eta-\mathfrak{D}-1}(\sqrt{5}+\sqrt{2})^{\eta-\mathfrak{D}-1}.$$

This completes the proof of the theorem. ■

Figure 1 illustrates all starlike trees with 8 vertices and 3 legs.

Using Theorem 1 and Observation 1 we get the following corollary.

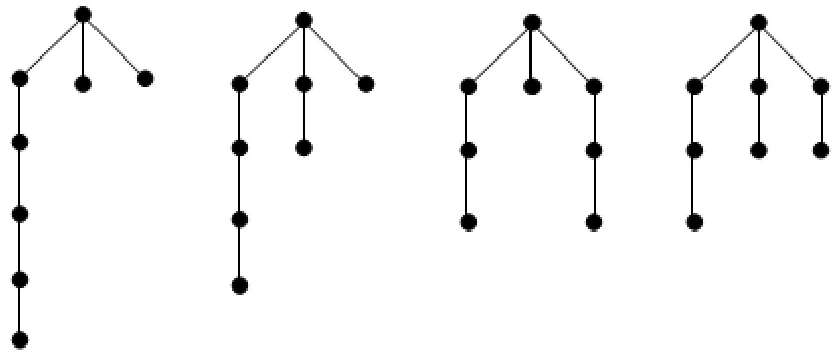

Figure 1: All starlike trees with 8 vertices and 3 legs.

Corollary 1 *For any connected graph Θ of order η and maximum degree $\mathfrak{D}$,*

$$MKG(\Theta) \geq (\sqrt{\mathfrak{D}^2+4}+\sqrt{2}\mathfrak{D})^{\mathfrak{D}}(\sqrt{5}+\sqrt{2})^{\mathfrak{D}}(4\sqrt{2})^{\eta-2\mathfrak{D}-1},$$

when $\mathfrak{D} \leq \frac{\eta-1}{2}$. And

$$MKG(\Theta) \geq (\sqrt{\mathfrak{D}^2+(\mathfrak{D}-1)^2}+\sqrt{1+(\mathfrak{D}-1)^2})^{2\mathfrak{D}+1-\eta}(\sqrt{\mathfrak{D}^2+4}+\sqrt{2}\ \mathfrak{D})^{\eta-\mathfrak{D}-1}(\sqrt{5}+\sqrt{2})^{\eta-\mathfrak{D}-1},$$

when $\mathfrak{D} > \frac{\eta-1}{2}$. The equality happens if and only if Θ is a starlike tree whose all legs are of a length less than 3 or all legs are of length more than 1.

3 Conclusion

In this paper, we calculated the minimum value of the multiplicative KG-Somber index in the class of trees with order n and maximum degree $\mathfrak{D}$. In addition, we proved that within the aforementioned class, this index attains its minimum value at the starlike trees whose all legs are of a length less than 3 or all legs are of length more than 1. The results were extended to all connected graphs with a given order and maximum vertex degree.

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