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Inequalities Of Hardy-Hilbert Type For Berezin Number Of Operators And Their Applications*

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Abstract

In this paper, we study Berezin number inequalities by using Hardy-Hilbert type inequality for self-adjoint operator in reproducing kernel Hilbert spaces. Also, we give an inequality for the Berezin symbol of self-adjoint operators in a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$. These results essentially improve and extend the most known results on this topic including some results.

1 Introduction and Notations

If $\{a_m\}$ and $\{b_n\}$ are two real sequences such that $0 < \sum_{m=0}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_m^2 < \infty$, then the Hardy-Hilbert inequality is given by

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right\}^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible (see [16]). Recently, a lot of results with applications of Hardy-Hilbert's inequality were obtained by mathematicians [14, 15, 20].

Let $T_p : \ell^p \to \ell^p$ be a linear operator such that for any non-negative sequence $a = \{a_m\}_{m=1}^{\infty} \in \ell^p$, then there exists $T_p a = c = \{c_n\}_{n=1}^{\infty} \in \ell^p$ satisfying

$$c_n = (T_p a)(n) = \sum_{m=0}^{\infty}\frac{a_m}{m+n+1}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{2}$$

Suppose that H is a separable Hilbert space and $T : H \to H$ is a bounded self-adjoint semi-positive define operator. In 2002, Zhang [32] proved the following inequality:

$$\langle a, Tb\rangle^2 \le \frac{\|T\|^2}{2}\left(\|a\|^2\|b\|^2 + \langle a, b\rangle^2\right) \tag{3}$$

where $\langle a, b\rangle$ is the inner product of a and b. Since the operator T_p defined by (2) (for $p = q = 2$) satisfies the condition of (3) (see [35]), we may improve (1) to the following form:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sqrt{2}}\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2 + \left(\sum_{n=0}^{\infty}a_n b_n\right)^2\right\}^{\frac{1}{2}}. \tag{4}$$

More information about Hardy and Hardy type inequalities the readers can be found, for example, in [7, 22, 24].

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Let R be a selfadjoint linear operator on a complex Hilbert space H. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathfrak{G}_p(R))$ of all continuous functions defined on the spectrum of R, denoted by $\mathfrak{G}_p(R)$, and the C^*-algebra $C^*(R)$ generated by R and the identity operator 1_H on H as follows [9]:

For any $f, g \in C(\mathfrak{G}_p(R))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| = \sup_{t \in \mathfrak{G}_p(R)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = R$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathfrak{G}_p(R)$.

With this concept we define $f(R) = \Phi(f)$ for all $f \in C(\mathfrak{G}_p(R))$ and we call it the continuous functional calculus for a selfadjoint operator R. If f is a real-valued continuous function on $\mathfrak{G}_p(R)$ and R is a selfadjoint operator, then $f(t) \geq 0$ for any $t \in \mathfrak{G}_p(R)$ means that $f(R) \geq 0$. Furthermore, if both f and g are real-valued functions on $\mathfrak{G}_p(R)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \mathfrak{G}_p(R) \text{ means that } f(R) \geq g(R)$$

in the operator order of $B(H)$.

Let Θ be an arbitrary set. Denote by $\mathcal{F}(\Theta)$ the set of all complex valued functions on Θ. A reproducing kernel Hilbert space (RKHS for short) on the set Θ is a Hilbert space $\mathcal{H} = \mathcal{H}(\Theta) \subset \mathcal{F}(\Theta)$ with a function $k_\mu : \Theta \times \Theta \to \mathcal{H}$, which is called the reproducing kernel enjoying the reproducing property $k_\mu := k(., \mu) \in \mathcal{H}$ for all $\mu \in \Theta$ and

$$f(\mu) = \langle f, k_\mu \rangle_{\mathcal{H}}$$

holds for all $\mu \in \Theta$ and all $f \in \mathcal{H}$ (see [25]). As it is known (see [1, 25]),

$$k_\mu(z) = \sum_{n=0}^{\infty} \overline{e_n(\mu)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n \geq 0}$ of the space $\mathcal{H}(\Theta)$.

Let $\widehat{k}_\mu = \frac{k_\mu}{\|k_\mu\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For any bounded linear operator R on $\mathcal{H}$, the Berezin symbol of R is the function $\widetilde{R}$ defined by (see [4, 23])

$$\widetilde{R}(\mu) := \left\langle R\widehat{k}_\mu, \widehat{k}_\mu \right\rangle_{\mathcal{H}} \ (\mu \in \Theta).$$

The Berezin symbol is a very useful tool in studying operators on the RKHS, including Hardy, Bergman and Fock spaces. For example, boundedness, invertibility, compactness and positivity of some operators are characterized or related with their Berezin symbols (see [5, 17, 19, 33, 34]).

The Berezin set and Berezin number for operator R are defined by (see Karaev [17, 18])

$$Ber(R) := Range\left(\widetilde{R}\right) = \left\{\widetilde{R}(\mu) : \mu \in \Theta\right\}$$

and

$$ber(R) := \sup\left\{\left|\widetilde{R}(\mu)\right| : \mu \in \Theta\right\},$$

respectively. Important results for the Berezin number of operators were obtained by authors in [2, 3, 10, 12, 26, 27].

The numerical range and numerical radius of R in $\mathcal{B}(\mathcal{H})$ are respectively defined by

$$W(R) := \{\langle Rf, f \rangle : f \in \mathcal{H}, \ \|f\| = 1\}$$

and

$$w(R) := \sup\{|\langle Rf, f\rangle| : f \in \mathcal{H},\ \|f\| = 1\}.$$

There is a relationship between Berezin set, Berezin number, numerical range and numerical radius as follows:

$$Ber(R) \subset W(R) \text{ and } ber(R) \leq w(R) \leq \|R\|.$$

For the numerical radius and its applications, we refer to [6, 8, 11, 21], and references therein. A fundamental inequality for the numerical radius is the power inequality

$$w(R^n) \leq w(R)^n,\ n \geq 1,$$

(see, e.g., [11, 13]). So, the following questions are natural:

Is it true that the above inequality is also provided for Berezin number of operators? For which operator classes, there exists a number $C > 0$ such that

$$ber(R)^n \leq C\,(ber(R^n)) \text{ for all } n? \tag{5}$$

Of course, an example with a nonzero nilpotent operator shows that there exists operators for which inequality (5) does not hold.

In this paper, we study inequality (5) by using Hardy-Hilbert type inequality for self-adjoint operator in reproducing kernel Hilbert spaces (Corollary 1 and Theorem 2). Also, we give an analog of inequality (2) for the Berezin symbol of self-adjoint operators in a RKHS $\mathcal{H} = \mathcal{H}(\Theta)$ (see Theorem 1). These results essentially improve and extend the most known results on this topic including some results of the papers [28, 29, 30, 31].

2 Hardy-Hilbert Type Inequality for Self-Adjoint Operators, Convex Functions and Berezin Number Inequalities

Now, we give an analog of inequality (4) for the Berezin symbol of self-adjoint operators on a RKHS $\mathcal{H} = \mathcal{H}(\Theta)$.

Theorem 1 *Let f, g be positive continuous functions on an interval $\Delta \subset (0,\infty)$. If $R, Q : \mathcal{H}(\Theta) \to \mathcal{H}(\Theta)$ are self-adjoint operators on a RKHS $\mathcal{H}(\Theta)$ with spectrum contained in Δ, then*

$$\begin{aligned}
&\widetilde{f(R)g(R)}(\mu) + \frac{1}{2}\widetilde{f(Q)}(\mu)\,\widetilde{g(R)}(\mu) + \frac{1}{2}\widetilde{f(R)}(\mu)\,\widetilde{g(Q)}(\mu) + \frac{1}{3}\widetilde{f(Q)g(Q)}(\mu) \\
<\ & \frac{\pi}{\sqrt{2}}\left\{\left(\widetilde{f^2(R)}(\mu) + \widetilde{f^2(Q)}(\mu)\right)\left(\widetilde{g^2(R)}(\mu) + \widetilde{g^2(Q)}(\mu)\right) + \left(\widetilde{f(R)g(R)}(\mu) + \widetilde{f(Q)g(Q)}(\mu)\right)^2\right\}^2
\end{aligned} \tag{6}$$

for $\mu, \nu \in \Theta$.

Proof. Suppose that $a_n, b_n = 0$ for all $n \geq 2$ in inequality (4). Since $a_n, b_n \geq 0$, we have from inequality (4) that

$$a_0 b_0 + \frac{a_1 b_0}{2} + \frac{a_0 b_1}{2} + \frac{a_1 b_1}{3} < \frac{\pi}{\sqrt{2}}\left\{\left(a_0^2 + a_1^2\right)\left(b_0^2 + b_1^2\right) + \left(a_0 b_0 + a_1 b_1\right)^2\right\}^{\frac{1}{2}}. \tag{7}$$

Let $z, w \in J$. Since $f, g \geq 0$ from hypothesis of theorem, we can put $a_0 = f(z)$, $a_1 = f(w)$, $b_0 = g(z)$, $b_1 = g(w)$ in (7). Hence, we get

$$\begin{aligned}
&f(z)g(z) + \frac{1}{2}f(w)g(z) + \frac{1}{2}f(z)g(w) + \frac{1}{3}f(w)g(w) \\
&< \frac{\pi}{\sqrt{2}}\left\{\left(f^2(z) + f^2(w)\right)\left(g^2(z) + g^2(w)\right) + \left(f(z)g(z) + f(w)g(w)\right)^2\right\}^{\frac{1}{2}}
\end{aligned} \tag{8}$$

for all $z, w \in \Delta$. Applying the functional calculus for R to the inequality (8), we obtain

$$\begin{aligned} & f(R)\,g(R) + \frac{1}{2}f(w)g(R) + \frac{1}{2}f(R)\,g(w) + \frac{1}{3}f(w)g(w) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{\left(f^2(R) + f^2(w)\right)\left(g^2(R) + g^2(w)\right) + \left(f(R)\,g(R) + f(w)\,g(w)\right)^2\right\}^{\frac{1}{2}} \end{aligned}$$

and hence

$$\begin{aligned} & \widetilde{f(R)\,g(R)}(\mu) + \frac{1}{2}f(w)\widetilde{g(R)}(\mu) + \frac{1}{2}\widetilde{f(R)}(\mu)\,g(w) + \frac{1}{3}f(w)g(w) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{\left(\widetilde{f^2(R)}(\mu) + f^2(w)\right)\left(\widetilde{g^2(R)}(\mu) + g^2(w)\right) + \left(\widetilde{f(R)}(\mu)\,\widetilde{g(R)}(\mu) + f(w)\,g(w)\right)^2\right\}^{\frac{1}{2}} \end{aligned}$$

for all $\mu \in \Theta$ and any $w \in \Delta$.

By applying the functional calculus for Q to the inequality (because Q is self-adjoint operator), we get

$$\begin{aligned} & \widetilde{f(R)\,g(R)}(\mu) + \frac{1}{2}f(Q)\widetilde{g(R)}(\mu) + \frac{1}{2}\widetilde{f(R)}(\mu)\,g(Q) + \frac{1}{3}f(Q)g(Q) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{\left(\widetilde{f^2(R)}(\mu) + f^2(Q)\right)\left(\widetilde{g^2(R)}(\mu) + g^2(Q)\right) + \left(\widetilde{f(R)\,g(R)}(\mu) + f(Q)\,g(Q)\right)^2\right\}^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have from above inequality that

$$\begin{aligned} & \widetilde{f(R)\,g(R)}(\mu) + \frac{1}{2}\widetilde{f(Q)}(\mu)\,\widetilde{g(R)}(\mu) + \frac{1}{2}\widetilde{f(R)}(\mu)\,\widetilde{g(Q)}(\mu) + \frac{1}{3}\widetilde{f(Q)g(Q)}(\mu) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{\left(\widetilde{f^2(R)}(\mu) + \widetilde{f^2(Q)}(\mu)\right)\left(\widetilde{g^2(R)}(\mu) + \widetilde{g^2(Q)}(\mu)\right) + \left(\widetilde{f(R)\,g(R)}(\mu) + \widetilde{f(Q)g(Q)}(\mu)\right)^2\right\}^{\frac{1}{2}} \end{aligned}$$

for all self-adjoint operators $R, Q \in \mathcal{B}(\mathcal{H})$ and all $\mu, \nu \in \Theta$. This gives the desired result. ■

Corollary 1

$$(ber\,(R))^2 < \left[2\pi - \frac{4}{3}\right] ber\left(f^2(R)\right)$$

for any self-adjoint operator $R \in \mathcal{B}(\mathcal{H})$ with spectrum contained in Δ.

Proof. We particularly obtain from inequality (6) for $Q = R$, $g = f$ and $\nu = \mu$ that

$$\begin{aligned} & \widetilde{f^2(R)}(\mu) + \frac{1}{2}\left[\widetilde{f(R)}(\mu)\right]^2 + \frac{1}{2}\left[\widetilde{f(R)}(\mu)\right]^2 + \frac{1}{3}\widetilde{f^2(R)}(\mu) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{4\left(\widetilde{f^2(R)}(\mu)\right)^2 + 4\left(\widetilde{f^2(R)}(\mu)\right)^2\right\}^{\frac{1}{2}} \end{aligned}$$

and thus

$$\left[\widetilde{f(R)}(\mu)\right]^2 < \left[2\pi - \frac{4}{3}\right]\widetilde{f^2(R)}(\mu)$$

for all $\mu \in \Theta$. Since $\widetilde{f(R)}(\mu) \geq 0$ $\left(\text{because } \widetilde{R}(\mu) \text{ is a real number for all } \mu \in \Theta\right)$ and $\widetilde{f^2(R)}(\mu) \geq 0$, we have from above inequality,

$$\sup_{\mu\in\Theta}\left[\widetilde{f(R)}(\mu)\right]^2 < \left[2\pi - \frac{4}{3}\right]\sup_{\mu\in\Theta}\widetilde{f^2(R)}(\mu)$$

for all $\mu \in \Theta$. This inequality clearly gives that

$$(ber\,(R))^2 < \left[2\pi - \frac{4}{3}\right] ber\left(f^2(R)\right)$$

for any self-adjoint operator $R \in \mathcal{B}(\mathcal{H})$ with spectrum contained in Δ, which proves the theorem. ■

Now, we give an Berezin number inequality for self-adjoint operators and convex functions on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Theorem 2 *Let $f : \Delta \to (0, \infty)$ be a convex and continuous function and $R : \mathcal{H}(\Theta) \to \mathcal{H}(\Theta)$ be self-adjoint operator on a RKHS $\mathcal{H}(\Theta)$ with spectrum contained in Δ. Then we have*

$$[f(ber\,(R))]^2 < \left(\pi - \frac{1}{6}\right) ber\left(f^2\,(R)\right).$$

Proof. We know from inequality (8) that

$$\begin{aligned} & f(z)g(z) + \frac{1}{2}f(w)g(z) + \frac{1}{2}f(z)g(w) + \frac{1}{3}f(w)g(w) \\ < \; & \frac{\pi}{\sqrt{2}} \left\{ \left(f^2\,(z) + f^2\,(w)\right)\left(g^2\,(z) + g^2\,(w)\right) + \left(f\,(z)\,g\,(z) + f\,(w)\,g\,(w)\right)^2 \right\}^{\frac{1}{2}} \end{aligned} \tag{9}$$

for all $z, w \in \Delta$. By putting $z = \widetilde{R}\,(\mu)$ in (9), we have that

$$\begin{aligned} & f(\widetilde{R}\,(\mu))g(\widetilde{R}\,(\mu)) + \frac{1}{2}f(w)g\left(\widetilde{R}\,(\mu)\right) + \frac{1}{2}f\left(\widetilde{R}\,(\mu)\right)g(w) + \frac{1}{3}f(w)g(w) \\ < \; & \frac{\pi}{\sqrt{2}} \left\{ \left(f^2\left(\widetilde{R}\,(\mu)\right) + f^2\,(w)\right)\left(g^2\left(\widetilde{R}\,(\mu)\right) + g^2\,(w)\right) \right. \\ & \left. + \left(f\left(\widetilde{R}\,(\mu)\right)g\left(\widetilde{R}\,(\mu)\right) + f\,(w)\,g\,(w)\right)^2 \right\}^{\frac{1}{2}} \end{aligned}$$

for all $\mu \in \Theta$ and any $w \in \Delta$. By applying the functional calculus for Q to the inequality (because Q is self-adjoint operator), we get

$$\begin{aligned} & f\left(\widetilde{R}\,(\mu)\right)g\left(\widetilde{R}\,(\mu)\right) + \frac{1}{2}f\,(Q)\,g\left(\widetilde{R}\,(\mu)\right) + \frac{1}{2}f\left(\widetilde{R}\,(\mu)\right)g(Q) + \frac{1}{3}f(Q)g(Q) \\ < \; & \frac{\pi}{\sqrt{2}} \left\{ \left(f^2\left(\widetilde{R}\,(\mu)\right) + f^2\,(Q)\right)\left(g^2\left(\widetilde{R}\,(\mu)\right) + g^2\,(Q)\right) \right. \\ & \left. + \left(f\left(\widetilde{R}\,(\mu)\right)g\left(\widetilde{R}\,(\mu)\right) + f\,(Q)\,g\,(Q)\right)^2 \right\}^{\frac{1}{2}} \end{aligned}$$

and hence

$$\begin{aligned} & f\left(\widetilde{R}\,(\mu)\right)g\left(\widetilde{R}\,(\mu)\right) + \frac{1}{2}\widetilde{f\,(Q)}\,(\nu)\,g\left(\widetilde{R}\,(\mu)\right) + \frac{1}{2}f\left(\widetilde{R}\,(\mu)\right)\widetilde{g\,(Q)}\,(\nu) + \frac{1}{3}\widetilde{f\,(Q)\,g\,(Q)}\,(\nu) \\ < \; & \frac{\pi}{\sqrt{2}} \left\{ \left(f^2\left(\widetilde{R}\,(\mu)\right) + \widetilde{f^2\,(Q)}\,(\nu)\right)\left(g^2\left(\widetilde{R}\,(\mu)\right) + \widetilde{g^2\,(Q)}\,(\nu)\right) \right. \\ & \left. + \left(f\left(\widetilde{R}\,(\mu)\right)g\left(\widetilde{R}\,(\mu)\right) + \widetilde{f\,(Q)\,g\,(Q)}\,(\nu)\right)^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{10}$$

By considering power functions and convexity of f, g, we get

$$f\left(\widetilde{Q}\,(\nu)\right) \leq \widetilde{f\,(Q)}\,(\nu)\,,\;\; g\left(\widetilde{Q}\,(\nu)\right) \leq \widetilde{g\,(Q)}\,(\nu) \tag{11}$$

and

$$f^2\left(\widetilde{R}\,(\mu)\right) \leq \widetilde{f^2\,(R)}\,(\mu) \quad \text{and } g^2\left(\widetilde{R}\,(\mu)\right) \leq \widetilde{g^2\,(R)}\,(\mu)\,. \tag{12}$$

By putting (11) and (12) in (10), we get @@@

$$\begin{aligned} & f\left(\widetilde{R}(\mu)\right) g\left(\widetilde{R}(\mu)\right)+\frac{1}{2} f\left(\widetilde{Q}(\nu)\right) g\left(\widetilde{R}(\mu)\right)+\frac{1}{2} f\left(\widetilde{R}(\mu)\right) g\left(\widetilde{Q}(\nu)\right)+\frac{1}{3} f \widetilde{(Q) g(Q)}(\nu) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{\left(\widetilde{f^2(R)}(\mu)+\widetilde{f^2(Q)}(\nu)\right)\left(\widetilde{g^2(R)}(\mu)+\widetilde{g^2(Q)}(\nu)\right)\right. \\ & \left.+\left(f \widetilde{(R) g(R)}(\mu)+f \widetilde{(Q) g(Q)}(\nu)\right)^2\right\}^{\frac{1}{2}} \end{aligned} \tag{13}$$

for all $\mu, \nu \in \Theta$.

We particularly obtain from inequality (13) for $Q = R$, $g = f$ and $\nu = \mu$ that

$$\begin{aligned} & \left[f\left(\widetilde{R}(\mu)\right)\right]^2+\frac{1}{2}\left[f\left(\widetilde{R}(\mu)\right)\right]^2+\frac{1}{2}\left[f\left(\widetilde{R}(\mu)\right)\right]^2+\frac{1}{3} \widetilde{f^2(R)}(\mu) \\ < \; & \frac{\pi}{\sqrt{2}}\left\{4\left(\widetilde{f^2(R)}(\mu)\right)^2+4\left(\widetilde{f^2(R)}(\mu)\right)^2\right\}^{\frac{1}{2}} \end{aligned}$$

and hence

$$\left[f\left(\widetilde{R}(\mu)\right)\right]^2<\left(\pi-\frac{1}{6}\right) \widetilde{f^2(R)}(\mu)$$

for all $\mu \in \Theta$. Since $\left[f(\widetilde{R}(\mu))\right]^2 \geq 0$ and $\widetilde{f^2(R)}(\mu) \geq 0$, by taking supremum in the last inequality, we have,

$$[f(ber(R))]^2<\left(\pi-\frac{1}{6}\right) ber\left(f^2(R)\right)$$

for all self-adjoint operator R and $\mu \in \Theta$, as required. ■

References

[1] N. Aronzajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68(1950), 337–404.

[2] M. Bakherad, Some Berezin number inequalities for operator matrices, Czechoslovak Math. J., 68(2018), 997–1009.

[3] M. Bakherad, M. Hajmohamadi, R. Lashkaripour and S. Sahoo, Some extensions of Berezin number inequalities on operators, Rocky Mountain J. Math., 51(2021), 1941–1951.

[4] F. A. Berezin, Covariant and contravariant symbols for operators, Math. USSR-Izv., 6(1972), 1117–1151.

[5] N. Das and M. Sahoo, Positive Toeplitz operators on the Bergman space, Ann. Funct. Anal., 4(2013), 171–182.

[6] S. S. Dragomir, A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces, Banach J. Math. Anal., 1(2017), 154–175.

[7] D. E. Edmunds, V. Kokilashvili and A. Meskhi, Bounded and Compact Integral Operators, Kluwer Academic Publishers, Dordrecht, 2002.

[8] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators II., Studia Math., 182(2007), 133–140.

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, Mond-Pečarić Method in Operator Inequalities (Inequalities for bounded selfadjoint operators on a Hilbert space), Element, Zagreb, 2005.

[10] M. Guesba, Some Euclidean Berezin number inequalities of a pair of operators and their applications, Filomat, 37(2023), 8777–8790.

[11] K. E. Gustafson and D. K. M. Rao, Numerical Range, New York: Springer Verlag, 1997.

[12] M. Hajmohamadi, R. Lashkaripour and M. Bakherad, Improvements of Berezin number inequalities, Linear Multilinear Algebra, 68(2020), 1218–1229.

[13] P. R. Halmos, A Hilbert Space Problem Book, New York: Springer Verlag, 1982.

[14] F. Hansen, Non-commutative Hardy inequalities, Bull. Lond. Math. Soc., 41(2009), 1009–1016.

[15] F. Hansen, K. Krulić, J. Pečarić and L. E. Persson, Generalized noncommutative Hardy and Hardy-Hilbert type inequalities, Internat J Math., 21(2010), 1283–1295.

[16] G. Hardy, J. E. Littlewood and G. Polya, Inequalities. 2nd ed. Cambridge: Cambridge University Press, 1967.

[17] M. T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, J. Funct. Anal., 238(2006), 181–192.

[18] M. T. Karaev, Reproducing kernels and Berezin symbols techniques in various questions of operator theory, Complex Anal. Oper. Theory, 7(2013), 983–1018.

[19] M. T. Karaev, M. Gürdal and U. Yamancı, Special operator classes and their properties, Banach J. Math. Anal., 7(2013), 74–85.

[20] M. Kian, Hardy-Hilbert type inequalities for Hilbert space operators, Ann. Funct. Anal., 3(2012), 128–134.

[21] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia Math., 168(2005), 73–80.

[22] A. Kufner, L. Maligranda and L.E. Persson, The prehistory of the Hardy inequality, Amer. Math. Monthly., 113(2016), 715–732.

[23] E. Nordgren and P. Rosenthal, Boundary Values of Berezin symbols, Oper. Theory Adv. Appl., 73(1994), 362–368.

[24] B. Opic and A. Kufner, Hardy-Type Inequalities, Harlow: Pitman Reseach Notes in Mathematics, 1990.

[25] S. Saitoh and Y. Sawano, Theory of Reproducing Kernels and Applications, Springer Singapore, 2016.

[26] S. Sahoo, N. Das and N. C. Rout, On Berezin number inequalities for operator matrices, Acta. Math. Sin.-English Ser., 37(2021), 873–892.

[27] S. Sahoo and M. Bakherad, Some Extended Berezin Number Inequalities, Filomat, 35(2021), 2043–2053.

[28] U. Yamancı, M. Gürdal and M. T. Garayev, Berezin number inequality for convex function in reproducing kernel Hilbert space, Filomat, 31(2017), 5711–5717.

[29] U. Yamancı and M. Gürdal, On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space, New York J. Math., 23(2017), 1531–1537.

[30] U. Yamancı, M. Garayev and C. Çelik, Hardy-Hilbert type inequality in reproducing kernel Hilbert space: its applications and related results, Linear Multilinear Algebra, 67(2019), 830–842.

[31] U. Yamancı, Operator inequalities in reproducing kernel Hilbert spaces, Commun. Fac. Sci. Univ. Ank .Ser. A1 Math. Stat, 71(2022), 204–211.

[32] K. W. Zhang, A bilinear inequality, J. Math. Anal. Appl., 271(2002), 288–296.

[33] X. Zhao and D. Zheng, Invertibility of Toeplitz operators via Berezin transforms, J. Oper. Theory, 75(2016), 475–495.

[34] K. Zhu, Operator Theory in Function Spaces, 2nd edn. Mathematical Surveys and Monographs: American Mathematical Society, 2007.

[35] M. Wilhelm, On the spectrum of Hilbert's matrix, Amer. J. Math., 72(1950), 699–704.