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Peregrination Of Minimal Topological Groups With A Relational Modeling *

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Abstract

In this paper, the notion of minimal topological group, a generalization of topological group was presented and obtain characterizations of minimal topological groups with their basic properties. Generalized Rough set approximation was used to model the inter-trading of industries.

1 Introduction

Topological Algebra is incorporation of an algebraic structure and a topology in which the operations are continuous. The circumstance of generalizing topological group by generalized open sets via decomposition of continuity and endowment of a structure [3, 4] is colossal and inevasible in the literature. The existence and characteristics of those generalized versions rely on the coherence between those generalized open sets. To dispense this, the theory of minimal open sets [8] and maximal open sets [2] can be used which are independent of those generalized open sets. One of the most endowed tool to model via a topology is the rough set approximation. Pawlak introduced the notion of Rough sets [9]. The equivalence relation is the establishment of its object identification where the upper and lower approximation operations are caused by equivalent relation. Pawlak Rough sets was extended to generalized rough sets by Lin [5] using neighbourhood systems and it was utilized on decision making [7]. In this paper, minimal topological group is defined and its characteristics are discussed.

2 Terminology

Through out this work, M connotes a group and Υa topology on space M with no separation axioms assumed. We follow mn instead of $m \cdot n$ for $m, n \in \mathsf{M}$ and m^{-1} indicates the inverse of m for $m \in \mathsf{M}$. Let

$$\mathsf{ST} = \{st : s \in \mathsf{S},\ t \in \mathsf{T}\} \quad \text{and} \quad \mathsf{S}^{-1} = \{s^{-1} : s \in \mathsf{S}\}$$

where $\mathsf{S}, \mathsf{T} \subseteq \mathsf{M}$. We denote Left translation, right translation and inversion by $\lambda_m(a) = ma$, $\rho_m(a) = am$, $i(m) = m^{-1}$ and S symmetric if $\mathsf{S} = \mathsf{S}^{-1}$. The power set of R is notated by $\mathsf{P}(\mathsf{R})$ for a set R. The notions $int(\mathsf{S})$, $cl(\mathsf{S})$ connotes the interior, closure of S in M and $\mathsf{O}(\mathsf{M})$ denotes the collection of open sets in M, respectively. $\mathsf{K} \subseteq \mathsf{M}$ is minimal-open [8] if either $int(\mathsf{K}) = \emptyset$ or $int(\mathsf{K}) = \mathsf{K}$. The minimal-open sets are closed under union and intersection of minimal-open with an open set is minimal open. The agglomeration of closed (respectively, minimal-open, minimal-closed) sets is signified by $\mathsf{C}(\mathsf{M})$ (respectively, $\mathsf{O}_{\mathsf{min}}(\mathsf{M}), \mathsf{C}_{\mathsf{min}}(\mathsf{M})$). For minimal interior and minimal closure we use $\mathsf{Int}_{\mathsf{min}}$ and $\mathsf{Cl}_{\mathsf{min}}$. A group M binded with a topology $\Upsilon$is denoted by the tuple (M, Υ). In addition, we set forth some preliminaries for the sequel.

Definition 1 *The function* $f : \mathsf{M} \mapsto \mathsf{N}$ *is*

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(i) ***minimal-continuous*** *if inverse image of an open set is minimal-open*

(ii) ***minimal open*** *if image of an open set is minimal-open*

(iii) ***minimal-irresolute*** *if inverse image of a minimal open set is minimal-open*

Definition 2 *The topological space* M *is*

(i) ***minimal-irresolute*** *if every minimal-open set of* M *is open.*

(ii) ***minimal-regular*** *if there is disjoint* $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *such that* $\mathsf{C} \subseteq \mathsf{E}$ *and* $k \in \mathsf{F}$ *for a closed set* C *of* M *and* $k \in \mathsf{M} \setminus \mathsf{C}$.

Lemma 1 *The following are equivalent in a topological space* M*:*

(i) M *is minimal-regular.*

(ii) There exists $\mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *of* x *such that* $\mathsf{Cl}_{min}(\mathsf{F}) \subseteq \mathsf{E}$ *for* $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ *and* $x \in \mathsf{E}$.

(iii) $\cap\{\mathsf{Cl}_{min}(\mathsf{F}) : \mathsf{C} \subseteq \mathsf{F}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})\} = \mathsf{C}$ *for* $\mathsf{C} \in \mathsf{C}(\mathsf{M})$.

(iv) There exists $\mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *such that* $\mathsf{K} \cap \mathsf{F} \neq \emptyset$ *and* $\mathsf{Cl}_{min}(\mathsf{F}) \subseteq \mathsf{E}$*, for* $\mathsf{K} \subseteq \mathsf{M}$ *and* $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ *with* $\mathsf{K} \cap \mathsf{E} \neq \emptyset$.

(v) There exists $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *such that* $\mathsf{K} \cap \mathsf{E} \neq \emptyset$, $\mathsf{C} \subseteq \mathsf{F}$ *and* $\mathsf{E} \cap \mathsf{F} = \emptyset$*, for* $\mathsf{K} \neq \emptyset$, $\mathsf{K} \subseteq \mathsf{M}$ *and* $\mathsf{C} \in \mathsf{C}(\mathsf{M})$ *with* $\mathsf{K} \cap \mathsf{C} = \emptyset$.

Proof. Let $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ with $x \in \mathsf{E}$. Then $\mathsf{D} = \mathsf{M} \setminus \mathsf{E} \in \mathsf{C}(\mathsf{M})$ and $x \notin \mathsf{D}$. By (i), there exist $\mathsf{F}, \mathsf{H} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ such that $x \in \mathsf{F} \cap \mathsf{D} \subseteq \mathsf{H}$ and $\mathsf{F} \cap \mathsf{H} = \emptyset$. Thus, $\mathsf{Cl}_{min}(\mathsf{F}) \cap \mathsf{H} = \emptyset$. So $x \in \mathsf{F} \subseteq \mathsf{Cl}_{min}(\mathsf{F}) \subseteq \mathsf{E}$. Hence $(i) \Rightarrow (ii)$.

Let $\mathsf{C} \in \mathsf{C}(\mathsf{M})$. Then

$$\cap\{\mathsf{Cl}_{min}(\mathsf{F}) : \mathsf{C} \subseteq \mathsf{F}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})\} \subseteq \mathsf{C}$$

since, a closed set is minimal closed. Conversely, let $x \notin \mathsf{C}$. Then $\mathsf{K} = \mathsf{M} \setminus \mathsf{C} \in \mathsf{O}(\mathsf{M})$ and $x \in \mathsf{K}$. Then there exists $\mathsf{R} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ such that $x \in \mathsf{R} \subseteq \mathsf{Cl}_{min}(\mathsf{R}) \subseteq \mathsf{K}$, by (ii). Let $\mathsf{F} = \mathsf{M} \setminus \mathsf{Cl}_{min}(\mathsf{R})$. Then $\mathsf{C} \subseteq \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ and $x \notin \mathsf{Cl}_{min}(\mathsf{F})$. Thus,

$$\cap\{\mathsf{Cl}_{min}(\mathsf{F}) : \mathsf{C} \subseteq \mathsf{F}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})\} = \mathsf{C}$$

and hence $(ii) \Rightarrow (iii)$.

Let $\mathsf{K} \subseteq \mathsf{M}$, $\mathsf{E} \in \mathsf{O}(\mathsf{M})$ such that $\mathsf{K} \cap \mathsf{E} \neq \emptyset$ and $x \in \mathsf{K} \cap \mathsf{E}$. Then $\mathsf{C} = \mathsf{M} \setminus \mathsf{E} \in \mathsf{C}(\mathsf{M})$ and $x \notin \mathsf{C}$. There exists $\mathsf{H} \in \mathsf{O}_{min}(\mathsf{M})$ such that $\mathsf{C} \subseteq \mathsf{H}$ and $x \in \mathsf{Cl}_{min}(\mathsf{H})$, by (iii). Let $\mathsf{F} = \mathsf{M} \setminus \mathsf{Cl}_{min}(\mathsf{H})$. Then $\mathsf{F} \in \mathsf{O}_{min}(\mathsf{M})$, $x \in \mathsf{F} \cap \mathsf{K}$ and

$$\mathsf{Cl}_{min}(\mathsf{F}) \subseteq \mathsf{Cl}_{min}(\mathsf{M} \setminus \mathsf{H}) = \mathsf{M} \setminus \mathsf{H} \subseteq \mathsf{E}.$$

Thus, $(iii) \Rightarrow (iv)$.

Let $\mathsf{K} \neq \emptyset$ and $\mathsf{C} \in \mathsf{C}(\mathsf{M})$ such that $\mathsf{K} \cap \mathsf{C} = \emptyset$. Since $\mathsf{M} \setminus \mathsf{C} \in \mathsf{O}(\mathsf{M})$ and $\mathsf{K} \neq \emptyset$, there exists $\mathsf{E} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ such that $\mathsf{K} \cap \mathsf{E} \neq \emptyset$ and $\mathsf{Cl}_{min}(\mathsf{E}) \subseteq \mathsf{M} \setminus \mathsf{C}$, by (iv). Let $\mathsf{F} = \mathsf{M} \setminus \mathsf{Cl}_{min}(\mathsf{E})$. Then $\mathsf{C} \subseteq \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ and $\mathsf{E} \cap \mathsf{F} = \emptyset$. Hence, $(iv) \Rightarrow (v)$.

By the Definition 2 of minimal-regular. Hence, $(v) \Rightarrow (i)$. ■

Lemma 2 *Let* (M, Υ) *be a minimal topological space.*

(i) For $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ *of* xy*, there exist* $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *of* x, y *with* $\mathsf{EF} \subseteq \mathsf{K}$ *if and only if* $f : \mathsf{M} \times \mathsf{M} \mapsto \mathsf{M}$ *by* $f(x, y) = xy$ *is minimal-continuous.*

(ii) For $\vartheta \in \mathsf{O}(\mathsf{M})$ *of* x^{-1}*, there exists* $\mathsf{Q} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *of* x *with* $\mathsf{Q}^{-1} \subseteq \vartheta$ *if and only if* $i : \mathsf{M} \mapsto \mathsf{M}$ *by* $i(x) = x^{-1}$ *is minimal-continuous.*

Proof. (I) Suppose, for $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ of xy, there exist $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of x, y with $\mathsf{EF} \subseteq \mathsf{K}$. Let $\mathsf{D} = \mathsf{E} \times \mathsf{F} \subseteq \mathsf{M} \times \mathsf{M}$. Then $int(\mathsf{D}) = int(\mathsf{E} \times \mathsf{F}) = int(\mathsf{E}) \times int(\mathsf{F})$ which is either $\emptyset$ or $\mathsf{E} \times \mathsf{F}$. In both cases, $int(\mathsf{D})$ is either $\emptyset$ or D. Thus, (x, y) is an element of $\mathsf{D} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M} \times \mathsf{M})$ and $f(\mathsf{D}) \subseteq \mathsf{K}$. Hence, for $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ of xy, the inverse image $\mathsf{D} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M} \times \mathsf{M})$ and thus f is minimal-continuous. The converse holds by reversing the argument.

(II) Suppose, for $\vartheta \in \mathsf{O}(\mathsf{M})$ of x^{-1}, there exists $\mathsf{Q} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of x with $\mathsf{Q}^{-1} \subseteq \vartheta$. Then $i(\mathsf{Q}) = \mathsf{Q}^{-1} \subseteq \vartheta$. Thus, i is minimal-continuous. The converse holds by reversing the argument. ■

3 Minimal Topological Group

Definition 3 *A pair* (M, Υ) *is* ***minimal topological group*** *if:*

- *for* $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ *of* xy*, there exist* $\mathsf{S}, \mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *of* x, y *with* $\mathsf{ST} \subseteq \mathsf{K}$.
- *for* $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ *of* x^{-1}*, there exists* $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *of* x *with* $\mathsf{T}^{-1} \subseteq \mathsf{S}$.

In a minimal topological group, multiplication and inversion are minimal-continuous (by Lemma 2).

Example 1 *Consider the addition modulo group* $(\mathbb{Z}_3, \oplus)$ *with* $\Upsilon = \{\emptyset, \{1, 2\}, \mathbb{Z}_3\}$*. Then,* $\mathsf{O}_{\mathsf{min}}(\mathbb{Z}_3) = \mathsf{P}(\mathbb{Z}_3) \setminus \{0\}$ *and thus* $(\mathbb{Z}_3, \Upsilon)$ *is a minimal topological group.*

Example 2 *Consider the symmetric group on three symbols* S_3 *with* $\Upsilon = \{\emptyset, \{e, (12), (13), (23)\}, S_3\}$*. Then,* (S_3, Υ) *is a minimal topological group.*

Proposition 1 *Let* (M, Υ) *be a minimal topological group. Then the following statements hold:*

(i) $\mathsf{K}^{-1} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *if and only if* $\mathsf{K} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$.

(ii) If $\mathsf{K} \in \mathsf{O}(\mathsf{M})$ *and* $\mathsf{N} \subseteq \mathsf{M}$*, then* KN *and* NK *are in* $\mathsf{O}_{\mathsf{min}}(\mathsf{M})$.

Proof. (I) Let $\mathsf{K} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$. Then either $int(\mathsf{K}) = \mathsf{K}$ or $int(\mathsf{K}) = \emptyset$. Suppose $int(\mathsf{K}) = \mathsf{K}$ then by Definition 3.1, $\mathsf{K}^{-1} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$. Suppose $int(\mathsf{K}) = \emptyset$ then $int(\mathsf{K}) = \emptyset = int(\mathsf{K}^{-1})$ and so $\mathsf{K}^{-1} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$.

(II) Let $n \in \mathsf{N}$, $a \in n\mathsf{K}$. Then $a = nk$ for some $k \in \mathsf{K}$. Now, $k = n^{-1}a$ and by Definition 3, there exist $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of n^{-1} and a such that $\mathsf{EF} \subseteq \mathsf{K}$. Thus, $a \in \mathsf{F} \subseteq n\mathsf{K}$ and hence $n\mathsf{K}$ is minimal-open. Now, $\mathsf{NK} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$, since minimal-open sets are closed under union. By the same token, $\mathsf{KN} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ can be proved. ■

Proposition 2 *Let* $\mathsf{C} \in \mathsf{C}(\mathsf{M})$ *be a minimal topological group* M*. Then* $a\mathsf{C}$ *and* $\mathsf{C}a$ *are minimal-closed, for* $a \in \mathsf{M}$.

Proof. Let $x \in \mathsf{Cl}_{min}(a\mathsf{C})$, $b = a^{-1}x$ and $\mathsf{D} \in \mathsf{O}(\mathsf{M})$ of b. Then there exist $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a^{-1} and x in M such that $\mathsf{EF} \subseteq \mathsf{D}$, by Definition 3. Now, $\mathsf{F} \cap \mathsf{aC} \neq \emptyset$, since $x \in \mathsf{Cl}_{min}(a\mathsf{C})$. Let $c \in \mathsf{F} \cap a\mathsf{C}$. Then $a^{-1}c \in \mathsf{C} \cap \mathsf{EF} \subseteq \mathsf{C} \cap \mathsf{D}$ and so $\mathsf{C} \cap \mathsf{D} \neq \emptyset$. Thus b is a limit point of C. Since $\mathsf{C} \in \mathsf{C}(\mathsf{M})$ and $\mathsf{b} \in \mathsf{C}$, we have $x = ab$ and so $x \in a\mathsf{C}$. Thus, $\mathsf{Cl}_{min}(a\mathsf{C}) \subseteq a\mathsf{C}$ and $a\mathsf{C} = \mathsf{Cl}_{min}(a\mathsf{C})$, since $a\mathsf{C} \subseteq \mathsf{Cl}_{min}(a\mathsf{C})$ is trivial. Hence $a\mathsf{C} \in \mathsf{C}_{\mathsf{min}}(\mathsf{M})$. Proof of $\mathsf{C}a$ is similar. ■

Theorem 1 *Let* K *and* N *be subsets of minimal topological group* M*. Then* $\mathsf{Cl}_{min}(\mathsf{K}).\mathsf{Cl}_{min}(\mathsf{N}) \subseteq \mathsf{Cl}(\mathsf{KN})$.

Proof. Let $a \in \mathsf{Cl}_{min}(\mathsf{K}).\mathsf{Cl}_{min}(\mathsf{N})$ and $\mathsf{D} \in \mathsf{O}(\mathsf{M})$ of a where $a = kn$ for some $k \in \mathsf{Cl}_{min}(\mathsf{K})$ and $n \in \mathsf{Cl}_{min}(\mathsf{N})$. Then there exists $\mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ containing k and n, respectively such that $\mathsf{EF} \subseteq \mathsf{D}$, by Definition 3. Since $k \in \mathsf{Cl}_{min}(\mathsf{K})$ and $n \in \mathsf{Cl}_{min}(\mathsf{N})$ there exist $c \in \mathsf{K} \cap \mathsf{E}$ and $d \in \mathsf{N} \cap \mathsf{F}$. Now $cd \in (\mathsf{MN}) \cap (\mathsf{EF}) \subseteq \mathsf{KN} \cap \mathsf{D}$ and so $\mathsf{KN} \cap \mathsf{D} \neq \emptyset$. Hence $a \in \mathsf{Cl}(\mathsf{KN})$. ■

Definition 4 *A map* $f : \mathsf{S} \mapsto \mathsf{T}$ *is* ***minimal-homeomorphism*** *if* f *is bijective, minimal-continuous and minimal-open.*

Theorem 2 *Let* (M, Υ) *be a minimal topological group. Then left (right) translations and inversion are minimal-homeomorphisms.*

Proof. Let $a, b \in \mathsf{M}$ and $\mathsf{D}_1 \in \mathsf{O}(\mathsf{M})$ with $ab \in \mathsf{D}_1$. Then, for $\mathsf{D}_1 \in \mathsf{O}(\mathsf{M})$ of ab there exist $\mathsf{E}_1, \mathsf{F}_1 \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a and b such that $\mathsf{E}_1\mathsf{F}_1 \subseteq \mathsf{D}_1$, by Definition 3. Thus, $\mathsf{a}\mathsf{F}_1 \subseteq \mathsf{D}_1$ and left translation is minimal-continuous. Let $g \in \mathsf{M}$ and $\mathsf{D}_2 \in \mathsf{O}(\mathsf{M})$ of g. Now, g can be written as $g = a^{-1}ag$. Then, there exist $\mathsf{E}_2, \mathsf{F}_2 \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a^{-1} and ag such that $\mathsf{E}_2\mathsf{F}_2 \subseteq \mathsf{D}_2$, since left translation is minimal-continuous. Hence, left translation is minimal-homeomorphism. The proof is similar for right translations.

Let $\mathsf{S}_1 \in \mathsf{O}(\mathsf{M})$ a^{-1}. Then there exists $\mathsf{T}_1 \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a such that $\mathsf{T}_1^{-1} \subseteq \mathsf{S}_1$, by Definition 3. Thus, the inversion mapping is minimal-continuous. Let $\mathsf{S}_2 \in \mathsf{O}(\mathsf{M})$ of a. Then there exists $\mathsf{T}_2 \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a^{-1} such that $\mathsf{T}_2^{-1} \subseteq \mathsf{S}_2$,since inversion is minimal-continuous. Hence the inversion is minimal-homeomorphism. ■

Theorem 3 *Let* (M, Υ) *be a minimal topological group and* $\mathfrak{B}_e$ *be the base at identity* e *of* M. *Then there exists:*

(i) $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})_e$ *such that* $\mathsf{T}^2 \subseteq \mathsf{S}$, *for* $\mathsf{S} \in \mathfrak{B}_e$.

(ii) $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})_e$ *such that* $\mathsf{T}^{-1} \subseteq \mathsf{S}$, *for* $\mathsf{S} \in \mathfrak{B}_e$.

(iii) $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})_e$ *such that* $g.\mathsf{T} \subseteq \mathsf{S}$ $(\mathsf{T}.g \subset \mathsf{S})$, *for* $\mathsf{S} \in \mathfrak{B}_e, g \in \mathsf{S}$.

Proof. (I) Let $\mathsf{S} \in \mathfrak{B}_e$. Then $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ of e. Then there exist $\mathsf{O}, \mathsf{P} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of e such that $\mathsf{OP} \subseteq \mathsf{S}$, by Definition 3. Let $\mathsf{T} \in \{\mathsf{O}, \mathsf{P}\}$ such that $\mathsf{T}^2 \subseteq \mathsf{S}$.

(II) Let $\mathsf{S} \in \mathfrak{B}_e$. Then $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ of e. Then, there exists $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of e such that $\mathsf{T}^{-1} \subseteq \mathsf{S}$, since inversion is minimal-continuous on M.

(III) Let $\mathsf{S} \in \mathfrak{B}_e$ and $g \in \mathsf{S}$. Now, $g = g.e$ $(g = e.g)$. Then there exist $\mathsf{P} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of g and $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of e such that $\mathsf{PT}(\mathsf{TP}) \subseteq \mathsf{S}$, by Definition 3. Thus, there is a $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})_e$ such that $g\mathsf{T} \subseteq \mathsf{S}$ $(\mathsf{T}g \subseteq \mathsf{S})$, for $g \in \mathsf{S}$. ■

Theorem 4 *Let* $(\mathsf{M}, \Upsilon, \mathcal{P})$ *be a minimal topological group and* K *a subgroup of* M.

(i) If $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ *and* $\mathsf{S} \subseteq \mathsf{K}, \mathsf{S} \neq \emptyset$ *then* $\mathsf{K} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$.

(ii) An open subgroup K *of* M *is minimal-closed and itself is a minimal topological group.*

Proof. (I) Suppose $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ and $\mathsf{S} \subseteq \mathsf{K}, \mathsf{S} \neq \emptyset$. Then, $\mathsf{S}m \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ for $m \in \mathsf{M}$, by Proposition 1. Then $\mathsf{K} = \cup_{m \in \mathsf{K}} \mathsf{S}m \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$, since minimal-open sets are closed under union.

(II) Let K be an open subgroup of M. Then $\gamma = \{\mathsf{K}a_i : a_i \in \mathsf{M}\}$ is the family of right cosets of K which is disjoint minimal-open covering of M. Thus, $\mathsf{M} = \cup_{a_i \in \mathsf{M}} \mathsf{K}a_i$ and so $\mathsf{K}a_i = \left(\cup_{a_j \neq a_i \in \mathsf{M}} \mathsf{K}a_j\right)^c$. Therefore an element of γ is both minimal-open and minimal-closed. In particular, $\mathsf{K} = \mathsf{K}e$ is minimal-closed in M. Now, We have to show that for $a, b \in \mathsf{K}$ and $\mathsf{D} \in \mathsf{O}(\mathsf{K})$ of ab^{-1} in K, there exist $\mathsf{S} \in \mathsf{O}_{\mathsf{min}}(\mathsf{K})$ of a and $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{K})$ of b such that $\mathsf{ST}^{-1} \subseteq \mathsf{D}$. There exist $\mathsf{E} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a and $\mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of b such that $\mathsf{EF}^{-1} \subseteq \mathsf{D}$, by Definition 3. The sets $\mathsf{S} = \mathsf{K} \cap \mathsf{E}, \mathsf{T} = \mathsf{K} \cap \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$, since $\mathsf{K} \in \mathsf{O}(\mathsf{M})$. Thus, $\mathsf{ST}^{-1} \subseteq \mathsf{EF}^{-1} \subseteq \mathsf{D}$. ■

Theorem 5 *Let* M *and* S *be minimal topological groups,* S *minimal irresolute and* f *be a homomorphism which is minimal-irresolute at identity* e_{M}. *Then* f *is minimal-irresolute.*

Proof. Let $a \in \mathsf{M}$ and $\mathsf{E} \in \mathsf{O}_{\mathsf{min}}(\mathsf{S})$ with $f(a) = b \in \mathsf{E}$. Since S is minimal irresolute, $\mathsf{E} \in \mathsf{O}(\mathsf{S})$. Thus $b^{-1}\mathsf{E} \in \mathsf{O}_{\mathsf{min}}(\mathsf{S})$ with $e_{\mathsf{S}} \in b^{-1}\mathsf{E}$. Then there exists $\mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of e_{M} such that $f(\mathsf{F}) \subseteq b^{-1}\mathsf{E}$, since f is minimal-irresolute at e_{M}. It follows that $f(a\mathsf{F}) = f(a)f(\mathsf{F}) \subseteq \mathsf{E}$, by f is homomorphism and hence f is minimal-irresolute. ■

Theorem 6 *Let* M *be a minimal topological group with base* $\mathfrak{B}_e$ *at identity* e *such that for* $\mathsf{S} \in \mathfrak{B}_e$ *there is* $\vartheta \in \mathsf{O}(\mathsf{M})$ *of* e *which is symmetric with* $\vartheta^2 \subseteq \mathsf{S}$. *Then* M *satisfies minimal-regularity at* e.

Proof. Let $\mathsf{S} \in \mathsf{O}(\mathsf{M})$ with $e \in \mathsf{S}$. By hypothesis, there is $\vartheta \in \mathsf{O}(\mathsf{M})$ of e which is symmetric with $\vartheta^2 \subseteq \mathsf{S}$. Let $a \in \mathsf{Cl}_{min}(\vartheta)$. Then $a\vartheta \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a, and $a\vartheta \cap \vartheta \neq \emptyset$. Thus, there exist $b, c \in \vartheta$ with $c = ab$ and so $a = cb^{-1} \in \vartheta\vartheta^{-1} = \vartheta\vartheta \subseteq \mathsf{S}$. Thus $\mathsf{Cl}_{min}(\vartheta) \subseteq \mathsf{S}$. ■

Definition 5 *A minimal topological group* M *is minimal-connected if it is impossible to split* M *as union of two disjoint non-void minimal-open sets in* M.

Theorem 7 *Let* M *be a minimal topological group which is minimal irresolute and* K *be a subgroup of* M*. If* K*,* M/K *are minimal-connected, then* M *is minimal-connected.*

Proof. Suppose M is not minimal-connected. Assume $\mathsf{M} = \mathsf{E} \cup \mathsf{F}$ with $\mathsf{E}, \mathsf{F} \neq \emptyset, \mathsf{E} \cap \mathsf{F} = \emptyset, \mathsf{E}, \mathsf{F} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$. Now, coset of K is a subset of either E or F, since K is minimal-connected. Thus,

$$\mathsf{M}/\mathsf{K} = \{a\mathsf{K} : a\mathsf{K} \subseteq \mathsf{E}\} \cup \{a\mathsf{K} : a\mathsf{K} \subseteq \mathsf{F}\} = \{a\mathsf{K} : a \in \mathsf{E}\} \cup \{a\mathsf{K} : a \in \mathsf{F}\}$$

and so M/K is not minimal connected, a contradiction. Thus, M is minimal-connected. ■

Theorem 8 *Let* M *be a minimal topological group which is minimal-connected and minimal irresolute with identity* e*. If* $\mathsf{S} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ *of* e*, then* M *is induced by* S.

Proof. Let $\mathsf{S} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of e. Denote $\mathsf{S}^n = s_1.s_2....s_n$ where $s_i \in \mathsf{S}, n \in \mathbb{N}$. Let $\mathsf{T} = \cup_{n=1}^{\infty}\mathsf{S}^n$. Since $\mathsf{S}^n \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ for all $n \in \mathbb{N}$, $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$. Now, let $a \in \mathsf{Cl}_{min}(\mathsf{T})$. Since $a\mathsf{S}^{-1} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of a, $\mathsf{T} \cap a\mathsf{S}^{-1} \neq \emptyset$. Let $b \in \mathsf{T} \cap a\mathsf{S}^{-1}$. Then $b = a.s^{-1}$ for some $s \in \mathsf{S}$, since $b \in a\mathsf{S}^{-1}$. Also, $b \in \mathsf{S}^n$ for some $n \in \mathbb{N}$ since $b \in \mathsf{T}$ and so $b = s_1s_2...s_n$ with $s_i \in \mathsf{S}$. Now, $a = s_1s_2...s_n.s$. Thus, $a \in \mathsf{S}^{n+1} \subseteq \mathsf{T}$. and hence $\mathsf{T} \in \mathsf{C}_{\mathsf{min}}(\mathsf{M})$. Since M is minimal-connected with $\mathsf{T} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ and $\mathsf{T} \in \mathsf{C}_{\mathsf{min}}(\mathsf{M})$, we see that $\mathsf{T} = \mathsf{M}$. Thus, M is induced by S. ■

Theorem 9 *If* M *is a minimal topological group which is minimal-connected and minimal irresolute with* H*, a discrete invariant subgroup of* M*, then* $\mathsf{H} \subseteq \mathsf{Z}(\mathsf{M})$*, with* $\mathsf{Z}(\mathsf{M})$ *indicates the center of* M.

Proof. Suppose $\mathsf{H} = \{e\}$, then it is obvious. Suppose $\mathsf{H} \neq \{e\}$. Let $h \neq e \in \mathsf{H}$. Then, by discreteness of H there is $\mathsf{D} \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of h with $\mathsf{D} \cap \mathsf{H} = \{h\}$. Now, $\mathsf{E}, \mathsf{E}.h \in \mathsf{O}_{\mathsf{min}}(\mathsf{M})$ of e and h with $(\mathsf{E}.h).\mathsf{E}^{-1} \subseteq \mathsf{D}$. Let $b \in \mathsf{E}$ be arbitrary. Since H is an invariant, $b.\mathsf{H} = \mathsf{H}.b$. Then $b.h \in \mathsf{H}.b$ and so $b.h.b^{-1} \in \mathsf{H}$. Also, $b.h.b^{-1} \in \mathsf{E}h\mathsf{E}^{-1} \subseteq \mathsf{D}$. Therefore, $b.h.b^{-1} \in \mathsf{D} \cap \mathsf{H} = \{h\}$ which implies $b.h.b^{-1} = h$. Thus, $b.h = h.b$ for $b \in \mathsf{E}$. Since E^n with $n \in \mathbb{N}$ covers M and M is minimal-connected then $a = b_1.b_2...b_n$ where $b_1, b_2, ..., b_n \in \mathsf{E}$ and $n \in \mathbb{N}$. Since h commutes with every element of E,

$$a.h = b_1.b_2....b_n.h = b_1.b_2....h.b_n = \cdots = b_1.h.b_2....b_n = h.b_1.b_2....b_n = h.a.$$

Hence $h \in \mathsf{H}$ and $h \in \mathsf{Z}(\mathsf{M})$. Since h is arbitrary, $\mathsf{H} \subseteq \mathsf{Z}(\mathsf{M})$. ■

4 Mathematical Modelling via Rough Sets

A country's economic growth rely on the decision changes of the sectors of an Industry and the decision making depends on the choices and availability. Let $\mathsf{M} = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_7\}$ be a collection of industries and K be the group of self trading relationships on M. Then K is a group under the composition of functions. Let Υbe the topology generated by minimal neighbourhood of $\Upsilon_i's$. Then, (K, Υ) is a minimal topological group.

Industries	Υ_1	Υ_2	Υ_3	Υ_4	Υ_5	Υ_6	Υ_7	Grown
Υ_1	×	✓	✓	×	✓	×	✓	Yes
Υ_2	✓	✓	×	×	✓	×	✓	Yes
Υ_3	✓	×	×	✓	×	✓	×	No
Υ_4	×	✓	✓	×	×	✓	×	Idle
Υ_5	✓	×	×	✓	✓	✓	×	Yes
Υ_6	×	✓	✓	✓	×	×	✓	No
Υ_7	✓	×	✓	✓	✓	×	×	Idle

From the above table, we obtain

$$V(\Upsilon_1) = \{\Upsilon_2, \Upsilon_3, \Upsilon_5, \Upsilon_7\},\ \ V(\Upsilon_2) = \{\Upsilon_1, \Upsilon_2, \Upsilon_5, \Upsilon_7\},$$
$$V(\Upsilon_3) = \{\Upsilon_1, \Upsilon_4, \Upsilon_6\},\ \ V(\Upsilon_4) = \{\Upsilon_2, \Upsilon_3, \Upsilon_6\},$$
$$V(\Upsilon_5) = \{\Upsilon_1, \Upsilon_4, \Upsilon_5, \Upsilon_6\},\ \ V(\Upsilon_6) = \{\Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_7\}$$

and

$$V(\Upsilon_7) = \{\Upsilon_1, \Upsilon_3, \Upsilon_4, \Upsilon_5\}.$$

Now, we define the relation R by $\Upsilon_i \sim \Upsilon_j$ if $|V(\Upsilon_i) \cap V(\Upsilon_j)| \geq 2$. Then,

$$\begin{aligned} R \ = \ & \{(\Upsilon_1,\Upsilon_1),(\Upsilon_1,\Upsilon_2),(\Upsilon_1,\Upsilon_4),(\Upsilon_1,\Upsilon_6),(\Upsilon_1,\Upsilon_7),(\Upsilon_2,\Upsilon_1),(\Upsilon_2,\Upsilon_2),(\Upsilon_2,\Upsilon_5),(\Upsilon_2,\Upsilon_6),\\ & (\Upsilon_2,\Upsilon_7),(\Upsilon_3,\Upsilon_3),(\Upsilon_3,\Upsilon_5),(\Upsilon_3,\Upsilon_7),(\Upsilon_4,\Upsilon_1),(\Upsilon_4,\Upsilon_4),(\Upsilon_4,\Upsilon_6),(\Upsilon_5,\Upsilon_2),(\Upsilon_5,\Upsilon_3),\\ & (\Upsilon_5,\Upsilon_5),(\Upsilon_5,\Upsilon_7),(\Upsilon_6,\Upsilon_1),(\Upsilon_6,\Upsilon_2),(\Upsilon_6,\Upsilon_4),(\Upsilon_6,\Upsilon_6),(\Upsilon_6,\Upsilon_7),(\Upsilon_7,\Upsilon_1),(\Upsilon_7,\Upsilon_2),\\ & (\Upsilon_7,\Upsilon_3),(\Upsilon_7,\Upsilon_5),(\Upsilon_7,\Upsilon_6),(\Upsilon_7,\Upsilon_7)\}. \end{aligned}$$

Now, the minimal neighbourhoods of elements of M are

$$< \Upsilon_1 >= \{\Upsilon_1, \Upsilon_2, \Upsilon_4, \Upsilon_6, \Upsilon_7\},$$
$$< \Upsilon_2 >= \{\Upsilon_1, \Upsilon_2, \Upsilon_5, \Upsilon_6, \Upsilon_7\},$$
$$< \Upsilon_3 >= \{\Upsilon_3, \Upsilon_5, \Upsilon_7\},$$
$$< \Upsilon_4 >= \{\Upsilon_1, \Upsilon_4, \Upsilon_6\},$$
$$< \Upsilon_5 >= \{\Upsilon_2, \Upsilon_3, \Upsilon_5, \Upsilon_7\},$$
$$< \Upsilon_6 >= \{\Upsilon_1, \Upsilon_2, \Upsilon_4, \Upsilon_6, \Upsilon_7\}$$

and

$$< \Upsilon_7 >= \{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_5, \Upsilon_6, \Upsilon_7\}.$$

Case 1: Industries which are Grown $S = \{\Upsilon_1, \Upsilon_2, \Upsilon_5\}$. Then the upper, lower approximations and accuracy of S are given by $U_S = M$, $L_S = \{\emptyset\}$ and $\Omega(S) = \frac{0}{7} = 0$.

<table>
<tr><th>Removing attribute Υ_j</th><th>Upper Approximation</th><th>Lower Approximation</th><th>Accuracy</th><th>Dispensability</th></tr>
<tr><td>Υ_1</td><td>$\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_5, \Upsilon_6, \Upsilon_7$</td><td rowspan="4">$\emptyset$</td><td rowspan="4">0</td><td rowspan="4">No</td></tr>
<tr><td>Υ_2
Υ_3
Υ_4</td><td>M</td></tr>
<tr><td>Υ_5</td><td>$\Upsilon_1, \Upsilon_2, \Upsilon_4, \Upsilon_5, \Upsilon_6, \Upsilon_7$</td></tr>
<tr><td>Υ_6
Υ_7</td><td>M</td></tr>
</table>

Case 2: Industries which are not Grown $S = \{\Upsilon_3, \Upsilon_6\}$. Then the upper, lower approximations and accuracy of S are given by $U_S = M$, $L_S = \{\emptyset\}$ and $\Omega(S) = \frac{0}{7} = 0$.

<table>
<tr><th>Removing attribute Υ_j</th><th>Upper Approximation</th><th>Lower Approximation</th><th>Accuracy</th><th>Dispensability</th></tr>
<tr><td>Υ_1
Υ_2</td><td>M</td><td rowspan="5">$\emptyset$</td><td rowspan="5">0</td><td rowspan="5">No</td></tr>
<tr><td>Υ_3</td><td>$\Upsilon_1, \Upsilon_2, \Upsilon_4, \Upsilon_6, \Upsilon_7$</td></tr>
<tr><td>Υ_4
Υ_5</td><td>M</td></tr>
<tr><td>Υ_6</td><td>$\Upsilon_3, \Upsilon_5, \Upsilon_7$</td></tr>
<tr><td>Υ_7</td><td>M</td></tr>
</table>

Case 3: Industries which are Idle $S = \{\Upsilon_4, \Upsilon_7\}$. Then the upper, lower approximations and accuracy of S are given by $U_S = M$, $L_S = \{\emptyset\}$ and $\Omega(S) = \frac{0}{7} = 0$.

<table>
<tr><th>Removing attribute Υ_j</th><th>Upper Approximation</th><th>Lower Approximation</th><th>Accuracy</th><th>Dispensability</th></tr>
<tr><td>Υ_1
Υ_2
Υ_3</td><td>M</td><td rowspan="4">$\emptyset$</td><td rowspan="4">0</td><td rowspan="4">No</td></tr>
<tr><td>Υ_4</td><td>$\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_5, \Upsilon_6, \Upsilon_7$</td></tr>
<tr><td>Υ_5
Υ_6</td><td>M</td></tr>
<tr><td>Υ_7</td><td>$\Upsilon_1, \Upsilon_4, \Upsilon_6$</td></tr>
</table>

In all the above three cases, the accuracy on removal of attributes does not change and so there will be no change in the current following policies of the Industries.

5 Conclusion

In this article, we present the notion of generalized topological group by minimal open sets and also proved some characterizations. We observe that, endowment of such approach will bestow generalized topological groups with ambivalent properties. Similar generalized version of topological groups were discussed in [6, 7].

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