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On The Zeros Of Quaternionic Polynomials*

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Abstract

In this paper, we obtained results regarding the location of the zeros of quaternionic polynomials and the bound so obtained is without any restriction on the coefficients of the polynomial. An analogous result of Cauchy's classical result for the zeros of complex polynomial has been derived for quaternions.

1 Introduction

The study of the distribution of zeros of polynomials indeed holds a significant place in the history of mathematics, fostering theoretical advancements with broad applications. Limiting polynomials serve as a valuable tool for understanding zero behavior, particularly when traditional methods encounter difficulties. Pioneers like Gauss and Cauchy made seminal contributions to this field, laying the groundwork for subsequent developments. The introduction of complex numbers and their geometric interpretation marked a pivotal moment in this domain's evolution. Complex analysis, in particular, provided powerful tools for understanding the distribution of zeros and their behavior. Cauchy's classical result [2] on the distribution of zeros of complex polynomials is a cornerstone of this area of study. His theorem, elegantly phrased below, offers insights into the arrangement of zeros within the complex plane and remains foundational in contemporary mathematical analysis.

Theorem 1 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of p lie in*

$$|z| < 1 + \max_{1 \le v \le n-1} \left| \frac{a_v}{a_n} \right|.$$

The exploration of bounds for polynomial zeros represents a fundamental aspect of mathematical analysis, offering insights into the behavior of polynomial functions across the complex plane. While numerous results exist in the literature, Theorem 1 stands out for its remarkable ease of computation and its applicability without imposing restrictions on the polynomial coefficients. Indeed, the study of polynomial zeros, whether with or without constraints on coefficients, has garnered significant attention. The continuous relationship between the zeros and the coefficients of a polynomial underscores the complexity inherent in deriving precise bounds. Given this intricate relationship, imposing restrictions on polynomial coefficients often proves beneficial in attaining sharper bounds on zero locations. In this direction, the following elegant result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem (see [4], [5], [6]) which states that:

Theorem 2 *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $0 < a_0 \le a_1 \le ... \le a_n$, then all the zeros of p lie in $|z| \le 1$.*

In literature (see [5], [6]), there exists several generalizations and refinements of Theorem 1. By leveraging Theorem 1's computational simplicity and its lack of coefficient restrictions, mathematicians gain a powerful tool for analyzing polynomial zeros efficiently and effectively. This theorem's ability to provide accurate

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bounds without cumbersome computational burdens renders it invaluable in both theoretical investigations and practical applications and it will be interesting and useful if we shall be able to extend Theorem 1 to the class of quaternionic polynomials. The aim of this paper is to present an extension to the quaternion polynomials of some known results related to Cauchy's Classical theorem.

Background: With the interpretation of the complex numbers as a two-dimensional number system, Sir Rowan William Hamilton spent years trying to find a three dimensional number system. He failed at this, however he was fortunate in succeeding to find a four dimensional number system on 16th of October-1843. This number system is the quaternions which we denote as $\mathbb{H}$ in honour of Hamilton. We shall use the standard notation

$$\mathbb{H} = \{\alpha + \beta i + \gamma j + \delta k | \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$$

where i, j, k satisfy $i^2 = j^2 = k^2 = ijk = -1$. The quaternions are the standard example of non-commutative division ring and also forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis.

For $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$, the real part of q is α and β, γ, δ are the imaginary parts. The conjugate is $q^* = \alpha - \beta i - \gamma j - \delta k$ and modulus is

$$|q| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

The modulus is then a norm on $\mathbb{H}$. For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H} \mid |q| < r\}$.

We define the indeterminate for a quaternionic polynomial as q. Without commutativity we are left with the polynomial aq^n and the polynomial $a_0 q a_1 q \cdots q a_n$, where $a = a_0 a_1 \cdots a_n$, as different. To alleviate this problem, we adopt the standard that polynomials have indeterminate on the left and coefficients on the right so that we have the quaternionic polynomial $p_1(q) = \sum\limits_{l=0}^{m} q^l a_l$. For such a p_1 and $p_2(q) = \sum\limits_{l=0}^{n} q^l b_l$, the regular product of p_1 and p_2 is defined by

$$(p_1 * p_2)(q) = \sum_{i,j=0}^{n,m} q^{i+j} a_i b_j.$$

If p_1 has real coefficients then so called $*$ multiplication coincides with the usual point-wise multiplication. One should note that the product rule $*$ is associative and not commutative, in general. The absence of commutativity leads to a behaviour of polynomials rather unlike their behaviour in the real or complex case. For example, a real or complex polynomial of degree n can have at most n zeros, This follows from the Factor theorem which states that a being a zero of $p(z)$ is equivalent $z - a$ being a divisor of $p(z)$. However, the Factor Theorem only holds in a commutative ring (see Theorem III. 6.6 of [1]). In the Quaternion case, the second degree polynomial $q^2 + 1$ has an infinite number of zeros namely $q_0 = i, j, k$ and all those given by $w_0 = h^{-1} q_0 h \ \ \forall h \in \mathbb{H}$.

2 Main Results

Let

$$\mathbb{P}_n := \left\{ p \ : \ p(q) = \sum_{l=0}^{n} q^l a_l, \quad q \in \mathbb{H} \right\}$$

denote the set of quaternionic polynomials with quaternion coefficients $a_l \in \mathbb{H}, \ \ 0 \le l \le n$. In general, an nth degree quaternionic polynomial has infinite number of zeros and to locate all those zeros will be interesting. In this direction, Carney et al. [3] proved the following extension of Theorem 2 for the quaternionic polynomial $p \in \mathbb{P}_n$. More precisely they proved the following result:

Theorem 3 *If $p \in \mathbb{P}_n$ is a quaternionic polynomial of degree n with real coefficients satisfying $0 < a_0 \le a_1 \le ... \le a_n$, then all the zeros of p lie in $|q| \le 1$.*

Recently, number of results have been obtained by various authors regarding the location of zeros of quaternionic polynomials (see [7], [8], [9]) and have extended various results of zeros of complex polynomial to quaternions. In this paper, we will prove the following result which extends Theorem 1 to the quaternionic polynomials and gives a bound for the zeros of quaternionic polynomial without any restriction on the coefficients:

Theorem 4 *All the zeros of the quaternion polynomial $p \in \mathbb{P}_n$ of degree n lie in $|q| < 1 + M$ where $M = \max_{1 \le l \le n} \left|\frac{a_{n-l}}{a_n}\right|$.*

Instead of proving Theorem 4, we prove following result for quaternionic polynomials with quaternionic coefficients which refines the Theorem 4.

Theorem 5 *All the zeros of the quaternion polynomial $p \in \mathbb{P}_n$ of degree n lie in $|q| < [(1+M)^n - 1]^{\frac{1}{n}}$, where M is given in Theorem 4.*

Now $[(1+M)^n - 1]^{\frac{1}{n}} < 1 + M$, if $(1+M)^n - 1 < (1+M)^n$, which is true, therefore Theorem 5 provides better bound than Theorem 4. Note that Theorem 3 holds for class of quaternionic polynomials with positive coefficients satisfying monotonicity , however Theorem 5 is true for all quaternionic polynomials without any restriction on the coefficients.

3 Lemma

Lemma 1 *For $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{H}$ such that $\sum\limits_{l=1}^{n} |\lambda_l| \le 1$, all the zeros of nth degree polynomial $p \in \mathbb{P}_n$ lie in the region $S = \{q \in \mathbb{H} | r_1 \le |q| \le r_2\}$, where*

$$r_1 = \min_{1 \le l \le n} \left|\lambda_l \frac{a_0}{a_l}\right|^{\frac{1}{l}} \quad \textit{and} \quad r_2 = \max_{1 \le l \le n} \left|\frac{1}{\lambda_l}\frac{a_{n-l}}{a_n}\right|^{\frac{1}{l}}.$$

Proof. From the definition of r_2, we have for all $1 \le l \le n$

$$\left|\frac{a_{n-l}}{a_n}\right| \le |\lambda_l| r_2^l$$

and hence

$$\sum_{l=1}^{n} \left|\frac{a_{n-l}}{a_n}\frac{1}{r_2^l}\right| \le \sum_{l=1}^{n} |\lambda_l|. \tag{1}$$

As quaternions do not satisfy commutative property, we have

$$\begin{aligned} p(q) &= q^n a_n + q^{n-1} a_{n-1} + q^{n-2} a_{n-2} \cdots + q a_1 + a_0 \\ &= q^n \Big(1 + \frac{1}{q}\frac{a_{n-1}}{a_n} + \frac{1}{q^2}\frac{a_{n-2}}{a_n} + \cdots + \frac{1}{q^{n-1}}\frac{a_1}{a_n} + \frac{1}{q^n}\frac{a_0}{a_n}\Big) a_n. \end{aligned} \tag{2}$$

Since modulus of an quaternion is real and real numbers do commute, we obtain from (2)

$$\begin{aligned} |p(q)| &= \left| q^n \Big(1 + \frac{1}{q}\frac{a_{n-1}}{a_n} + \frac{1}{q^2}\frac{a_{n-2}}{a_n} + \cdots + \frac{1}{q^{n-1}}\frac{a_1}{a_n} + \frac{1}{q^n}\frac{a_0}{a_n}\Big) a_n \right| \\ &\ge |q^n| \Big(1 - \Big|\frac{1}{q}\frac{a_{n-1}}{a_n} + \frac{1}{q^2}\frac{a_{n-2}}{a_n} + \cdots + \frac{1}{q^{n-1}}\frac{a_1}{a_n} + \frac{1}{q^n}\frac{a_0}{a_n}\Big|\Big) |a_n| \\ &\ge |q^n| \Big(1 - \Big\{\Big|\frac{1}{q}\Big|\Big|\frac{a_{n-1}}{a_n}\Big| + \Big|\frac{1}{q^2}\Big|\Big|\frac{a_{n-2}}{a_n}\Big| + \cdots + \Big|\frac{1}{q^{n-1}}\Big|\Big|\frac{a_1}{a_n}\Big| + \Big|\frac{1}{q^n}\Big|\Big|\frac{a_0}{a_n}\Big|\Big\} |a_n| \end{aligned}$$

$$= \big|q^n\big|\big|a_n\big|\Big\{1-\sum_{l=1}^{n}\Big|\frac{a_{n-l}}{a_n}\Big|\frac{1}{|q|^l}\Big\}.$$

Now for $|q| > r_2$ so that $\frac{1}{|q|} < \frac{1}{r_2}$, we obtain from above

$$|p(q)| > |q^n|\,|a_n|\left\{1-\sum_{l=1}^{n}\left|\frac{a_{n-l}}{a_n}\right|\frac{1}{r_2^l}\right\}.$$

Using inequality (1) and noting the hypothesis $\sum\limits_{l=1}^{n}|\lambda_l| \leq 1$, we obtain for $|q| > r_2$

$$|p(q)| > \big|q^n\big|\big|a_n\big|\Big\{1-\sum_{l=1}^{n}|\lambda_l|\Big\} \geq 0.$$

This implies that $|p(q)| > 0$ for $|q| > r_2$, consequently all the zeros of $p(q)$ lie in $|q| \leq r_2$. This proves the second part of Lemma 1.

We shall now prove first part of the Lemma 1. If $a_0 = 0$, then $r_1 = 0$. Therefore, there is nothing to prove and in this case Lemma 1 is proved completely. So we let $a_0 \neq 0$. Consider the reciprocal polynomial $R(q) = q^n * p(\frac{1}{q})$. Since q^n is an quaternionic polynomial of degree n with real coefficients, we have by the definition of regular product $*$

$$R(q) = q^n * p(\frac{1}{q}) = q^n a_0 + q^{n-1}a_1 + \cdots + qa_{n-1} + a_n.$$

By second part of the Lemma 1, all the zeros of $R(q)$ lie in

$$|q| \leq r_2 = \max_{1\leq l\leq n}\left|\frac{1}{\lambda_l}\frac{a_l}{a_0}\right|^{\frac{1}{l}} = \max_{1\leq l\leq n}\left|\frac{1}{\lambda_l}\frac{1}{\frac{a_0}{a_l}}\right|^{\frac{1}{l}} = \frac{1}{\min_{1\leq l\leq n}\left|\frac{\lambda_l a_0}{a_l}\right|^{\frac{1}{l}}}.$$

Replacing q by $\frac{1}{q}$ and noting that $p(q) = q^n * R(\frac{1}{q})$, it follows all the zeros of $p(q)$ lie in

$$|q| \geq \min_{1\leq l\leq n}\left|\frac{\lambda_l a_0}{a_l}\right|^{\frac{1}{l}}.$$

This proves first part of the lemma and hence the Lemma 1 is completely proved. ■

4 Proof of Theorem 5

For $1 \leq l \leq n$, we take

$$\mu_l = \left(\frac{(1+M)^n}{(1+M)^n-1}\right)\left(\frac{a_{n-l}}{a_n(1+M)^l}\right).$$

Since $M = \max_{1\leq l\leq n}\left|\frac{a_{n-l}}{a_n}\right|$, we obtain

$$\begin{aligned}\sum_{l=1}^{n}|\mu_l| &= \sum_{l=1}^{n}\left|\left(\frac{(1+M)^n}{(1+M)^n-1}\right)\left(\frac{a_{n-l}}{a_n(1+M)^l}\right|\right)\\ &\leq \left(\frac{(1+M)^n}{(1+M)^n-1}\right)\sum_{l=1}^{n}\frac{1}{(1+M)^l}\left|\frac{a_{n-l}}{a_n}\right|\end{aligned}$$

$$\leq \left(\frac{(1+M)^n}{(1+M)^n-1}\right) M \sum_{l=1}^{n} \frac{1}{(1+M)^l}$$
$$= 1.$$

Applying Lemma 1, it follows that all the zeros of $p(q)$ lie in

$$\begin{aligned}
|q| &\leq r_2 = \max_{1\leq l\leq n} \left|\frac{1}{\mu_l}\frac{a_{n-l}}{a_n}\right|^{\frac{1}{l}} \\
&= \max_{1\leq l\leq n} \left(\frac{(1+M)^n-1}{(1+M)^n}\right)^{\frac{1}{l}} (1+M) \\
&= (1+M) \max_{1\leq l\leq n} \left(1-\frac{1}{(1+M)^n}\right)^{\frac{1}{l}} \\
&= \left[\left(1+M\right)^n - 1\right]^{\frac{1}{n}}.
\end{aligned}$$

This completes the proof of Theorem 5.

5 Conclusions

The classic result of Cauchy has been extended to quaternionic polynomials and a refined bound for the zeros of quaternionic polynomial with quaternion coefficients has been found without any restriction on the coefficients of the polynomial.

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