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Inequalities Concerning The Derivatives Of Rational Functions With Prescribed Poles*

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Abstract

Let $r(z)$ be a rational function with at most n poles $a_1, a_2, \ldots, a_n$, where $|a_v| > 1$, $1 \leq v \leq n$. For rational functions, we use simple but elegant techniques to strengthen generalizations of certain results which extend some widely known polynomial inequalities to rational functions r. In return these reinforced results, in the limiting case, lead to the corresponding refinements of the said polynomial inequalities. In this paper we establish some inequalities that estimate the modulus of the derivative of a rational function $r(z)$ in the complex plane. We produce some sharper results by investigating an upper bound of $r(z)$ when all zeros lie in $\{z \in \mathbb{C} : |z| \geq k\}$.

1 Introduction

Let $\mathbb{P}_n$ denote the set of all complex polynomials $p(z)$ of degree at most n and $p'(z)$ is the derivative of $p(z)$. Let $T_k = \{z \in \mathbb{C} : |z| = k\}$ and D_{k-} and D_{k+} denote the regions inside and outside T_k. For $a_v \in \mathbb{C}$ with $v = 1, 2, \ldots, n$, let

$$w(z) = \prod_{v=1}^{n}(z - a_v)$$

and let $B(z) = \prod_{v=1}^{n}\left(\frac{1-\overline{a_v}z}{z-a_v}\right)$, $\mathbb{R}_n := \mathbb{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{p(z)}{w(z)} : p \in \mathbb{P}_n\right\}$, where product $B(z)$ is the finite Blaschke product. Then $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at infinty. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $z \in T_1$. Throughout this paper we assume that all poles are in D_{k+}, $k \geq 1$.

Definitions and Notations:

1) For $p(z) = \sum_{v=0}^{n} a_v z^v$, the conjugate transpose (reciprocal) p^* of p is defined by

$$p^*(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Therefore, if $p(z) = \prod_{v=1}^{n}(z - z_v)$, then $p^*(z) = \prod_{v=1}^{n}(1 - \overline{z_v}z)$.

2) For $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined by

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\overline{z}}\right)}.$$

Note that if $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, then $r^*(z) = \frac{p^*(z)}{w(z)}$, hence $r^*(z) \in \mathbb{R}_n$.

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3) For $w(z)=\prod_{v=1}^{n}(z-a_v)$, we denote by b the product of roots of $w(z)$, i.e., $b=a_1\times a_2\times\cdots\times a_n$.

4) If $p(z)=\sum_{v=0}^{n} b_v z^v$, then $\overline{p}(z)$ is defined as

$$\overline{p}(z)=\overline{b_0}+\overline{b_1}z+\overline{b_2}z^2+\cdots+\overline{b_n}z^n,$$

where, $\overline{p(\overline{z})}=\overline{p}(z)$.

In the past few years, several papers pertaining to derivatives of rational functions with prescribed poles have appeared in the study of rational approximation problems. Li, Mohapatra and Rodriguez [1] obtained Bernstein-type inequalities for rational functions with prescribed poles in the Chebyshev norm on the unit circle (for a function f defined on T_1 in complex plane, we write $\|f\|=\sup_{z\in T_1}|f(z)|$, the Chebyshev norm of f on T_1).

In 1995, Li, Mohapatra and Rodriguez [1] proved if $r(z)\neq 0$ in $T_1\cup D_{1-}$, then for $z\in T_1$ we have

$$|r'(z)|\leq \frac{|B'(z)|}{2}\|r\|, \tag{1}$$

whereas, if $r\in\mathbb{R}_n$ has exactly n zeros in $T_1\cup D_{1-}$, then for $z\in T_1$ we have

$$|r'(z)|\geq \frac{|B'(z)|}{2}\|r\|, \tag{2}$$

where $\|r\|=\max_{z\in T_1}|r(z)|$ and equality holds for $r(z)=aB(z)+b$ with $a,b\in T_1$.

In 1997, inequalities (1) was improved by Aziz and Shah [3] under the same hypothesis. They proved if $r\in\mathbb{R}_n$ and $r(z)\neq 0$ in D_{1-}, then for $z\in T_1$,

$$|r'(z)|\leq \frac{|B'(z)|}{2}(\|r\|-m), \tag{3}$$

where, $m=\min_{z\in T_1}|r(z)|$. Equality holds for $r(z)=B(z)+he^{\iota\beta}$ where $h\leq 1$ and β is real.

Further, as a generalization of (1), Aziz and zargar [2] obtained that if $r\in\mathbb{R}_n$ and $r(z)\neq 0$ in D_{k-}, $k\geq 1$; then for $z\in T_1$ we have

$$|r'(z)|\leq \frac{1}{2}\left\{|B'(z)|-n\cdot\frac{k-1}{k+1}\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|. \tag{4}$$

Equality holds for $r(z)=\left(\frac{z+k}{z-a}\right)^n$ and $B(z)=\left(\frac{1-az}{z-a}\right)^n$ evaluated at $z=1$, where $a>1$, $\quad k\geq 1$.

Recently, Mir [5] refined inequality (1) by proving that if $r\in\mathbb{R}_n$ and $r(z)\neq 0$ in D_{1-}, then for $z\in T_1$

$$|r'(z)|\leq \frac{1}{2}\left\{|B'(z)|-\frac{|b_0|-|b_t|}{|b_0|+|b_t|}\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|. \tag{5}$$

Equality holds for $r(z)=\left(\frac{z+1}{z-a}\right)^n$ and $B(z)=\left(\frac{1-az}{z-a}\right)^n$ evaluated at $z=1$, where $a>1$.

2 Main Results

In this paper, we first estimate an upper bound of the modulus of the derivative of $r(z)$ on the unit circle when $r(z)$ has all the zeros in $T_k\cup D_{k+}$.

Theorem 1 *If $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$, $r(z) \neq 0$ in D_{k-}, $k \geq 1$ and $p(z) = \sum_{v=0}^{t} b_v z^v$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{k+1} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}\frac{|r(z)|^2}{\|r\|^2}\right]\|r\|, \tag{6}$$

where t is the number of zeros of r with counting multiplicity. Equality holds for $r(z) = \frac{(z+k)^t}{(z-a)^n}$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1$, $k \geq 1$.

Taking $k = 1$ in, (6) we get the following refinement of inequality (1).

Corollary 1 *If $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$, $r(z) \neq 0$ in D_{1-}, and $p(z) = \sum_{v=0}^{t} b_v z^v$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \left\{(n-t) + \frac{|b_0| - |b_n|}{|b_0| + |b_n|}\right\}\frac{|r(z)|^2}{\|r\|^2}\right]\|r\|, \tag{7}$$

where t is the number of zeros of r with counting multiplicity and equality holds for $r(z) = \frac{(z+1)^t}{(z-a)^n}$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1$.

In case $p(z)$ is a polynomial of degree n below result deducted from Theorem 1 directly sharpens (4).

Corollary 2 *If $r(z) = \dfrac{p(z)}{w(z)} \in \mathbb{R}_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$ having no zeros in D_{k-}, $k \geq 1$ and $p(z) = \sum_{v=0}^{n} b_v z^v$, then for $z \in T_1$*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \left\{\frac{n(k-1)}{k+1} + \frac{2}{k+1} \cdot \frac{|b_0| - k^n|b_n|}{|b_0| + k^n|b_n|}\right\}\frac{|r(z)|^2}{\|r\|^2}\right]\|r\|. \tag{8}$$

Equality holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1$, $k \geq 1$.

Remark 1 *Taking $k = 1$, Corollary 2 reduces to inequality 5.*

Next, we prove the following extension and sharpening of (3), which also provides a generalization of Theorem 1.

Theorem 2 *If $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, where $p(z) = \sum_{v=0}^{t} b_v z^v$, $t \leq n$, r has exactly n poles at $a_1, a_2, \dots, a_n$ having no zeros in D_{k-}, $k \geq 1$ and $|b_0| \leq |c| \cdot |b_t|$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{k+1} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}\frac{\big(|r(z)| - m\big)^2}{\big(\|r\| - m\big)^2}\right](\|r\| - m),$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{z \in T_k} |r(z)|$ and equality holds for $r(z) = \dfrac{(z+k)^t}{(z-a)^n}$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$, where $a > 1$, $k \geq 1$.

Corollary 3 *If* $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, *where* $p(z) = \sum_{v=0}^{n} b_v z^v$, $|b_0| \le |c| \cdot |b_n|$ r *has exactly* n *poles at* $a_1, a_2, \ldots, a_n$ *having no zeros in* D_{k-}, $k \ge 1$, *then for* $z \in T_1$,

$$|r'(z)| \le \frac{1}{2}\left[|B'(z)| - \left\{\frac{n(k-1)}{k+1} + \frac{2}{k+1} \cdot \frac{|b_0| - k^n|b_n|}{|b_0| + k^n|b_n|}\right\} \frac{(|r(z)| - m)^2}{(\|r\| - m)^2}\right](\|r\| - m),$$

where $m = \min_{z \in T_k} |r(z)|$.

Taking, $k = 1$ and $t = n$ in Theorem 2 below corollary is obtained .

Corollary 4 *If* $r(z) = \frac{p(z)}{w(z)} \in \mathbb{R}_n$, *where* $p(z) = \sum_{v=0}^{n} b_v z^v$, r *has exactly* n *poles at* $a_1, a_2, \ldots, a_n$ *having no zeros in* D_{1-} *and* $|b_0| \le |c| \cdot |b_n|$ *then for* $z \in T_1$,

$$|r'(z)| \le \frac{1}{2}\left\{|B'(z)| - \frac{|b_0| - |b_n|}{|b_0| + |b_n|} \cdot \frac{(|r(z)| - m')^2}{(\|r\| - m')^2}\right\}(\|r\| - m'),$$

Equality holds for $r(z) = (\frac{z+1}{z-a})^n$ *and* $B(z) = \left(\frac{1-az}{z-a}\right)^n$ *evaluated at* $z = 1$, *where* $a > 1$.

Remark 2 *Let* $a_v = a > 1$ $\forall v = 1, 2, \ldots, n$. *Then* $w(z) = (z-a)^n$ *and* $r(z) = \frac{p(z)}{(z-a)^n}$, *so that*

$$r'(z) = \frac{-D_a p(z)}{(z-a)^{n+1}},$$

where, $D_a p(z) := np(z) + (a-z)p'(z)$ *denotes the polar derivative of a polynomial with respect to* a *and generalizes the ordinary derivative in the sense*

$$\lim_{|a| \to \infty} \frac{D_a p(z)}{a - z} = p'(z).$$

Also,

$$B(z) = \left(\frac{1-az}{z-a}\right)^n,$$

gives

$$B'(z) = \frac{n(1-\overline{a}z)^{n-1}(|a|^2 - 1)}{(z-a)^{n+1}}.$$

Further, let

$$\|r\| = \max_{z \in T_1}\left|\frac{p(z)}{(z-a)^n}\right|$$

be obtained at $z = e^{\iota\zeta}$, $0 \le \zeta < 2\pi$, *and*

$$m' = \min_{z \in T_1}|r(z)| = \min_{z \in T_1}\left|\frac{p(z)}{(z-a)^n}\right|$$

be obtained at $z = e^{\iota\beta}$, $0 \le \beta < 2\pi$. *Then clearly,*

$$\|r\| = \max_{z \in T_1}\left|\frac{p(z)}{(z-a)^n}\right| \le \left|\frac{p(e^{\iota\zeta})}{(e^{\iota\zeta} - a)^n}\right| \le \frac{\max_{z \in T_1}|p(z)|}{|(e^{\iota\zeta} - a)^n|} = \frac{\|p\|}{|(e^{\iota\zeta} - a)^n|}$$

and

$$m' = \min_{z \in T_1}|r(z)| = \min_{z \in T_1}\left|\frac{p(z)}{(z-a)^n}\right| = \left|\frac{p(e^{\iota\beta})}{(e^{\iota\beta} - a)^n}\right| \ge \frac{\min_{z \in T_1}|p(z)|}{(e^{\iota\beta} - a)^n} \ge \frac{m_p}{(e^{\iota\beta} - a)^n}.$$

where $m_p = \min_{z \in T_1}|p(z)|$.

Using, these facts in Corollary 4, we get for $|a| > 1$ and $z \in T_1$,

$$\left|\frac{D_a p(z)}{(z-a)^{n+1}}\right| \leq \frac{1}{2}\Bigg\{\left|\frac{n(1-\overline{a}z)^{n-1}(|a|^2-1)}{(z-a)^{n+1}}\right| -$$
$$\frac{|b_0|-|b_n|}{|b_0|+|b_n|}\left(\frac{|p(z)|/|(z-a)^n| - m_p/|e^{\iota\beta}-a|^n}{\|p\|/|e^{\iota\zeta}-a|^n - m_p/|e^{\iota\beta}-a|^n}\right)\Bigg\}\left(\frac{\|p\|}{|e^{\iota\zeta}-a|^n} - \frac{-m_p}{|e^{\iota\beta}-a|^n}\right).$$

Now letting $|a| \to \infty$, we get the following.

Corollary 5 *If $p(z) = \sum_{v=0}^{n} b_v z^v \in \mathbb{P}_n$ and $p(z) \neq 0$ in D_{1-}, then for $z \in T_1$,*

$$|p'(z)| \leq \frac{1}{2}\left[n - \frac{|b_0|-|b_n|}{|b_0|+|b_n|}\cdot\frac{|p|-m_p}{\|p\|-m_p}\right](\|p\|-m_p),$$

where $m_p = \min_{z\in T_1}|r(z)|$.

3 Lemmas

To prove above theorems, we need following lemmas. The first lemma is due to Li, Mohapatra and Rodriguez [1].

Lemma 1 *If $r \in \mathbb{R}_n$ and $z \in T_1$, then*

$$|r^{*\prime}(z)| + |r'(z)| \leq |B'(z)| \max_{z\in T_1}|r(z)|.$$

Equality holds for $r(z) = uB(z)$ with $u \in T_1$.

Lemma 2 *If x_v, with $v = 1, 2, \ldots, n$ is a sequence of reals such that $x_v \geq 1$ for all $v \in \mathbb{N}$, then*

$$\sum_{v=1}^{n}\frac{x_v-1}{x_v+1} \geq \frac{\prod_{v=1}^{n} x_v - 1}{\prod_{v=1}^{n} x_v + 1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. *The claim follows easily by induction on n. Here we omit details.* ∎

Lemma 3 ([2]) *If $z \in T_1$, then*

$$\Re\left(z\frac{w'(z)}{w(z)}\right) = \frac{n-|B'(z)|}{2}.$$

Lemma 4 *Assume that $r(z) = \dfrac{p(z)}{w(z)} \in \mathbb{R}_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$, $r(z) \neq 0$ in $z \in D_{k-}$, $k \geq 1$ $p(z) = \sum_{v=0}^{n} a_v z^v$. Let t be the number of zeros of r with counting multiplicity. Then for each point on $z \in T_1$, with $r(z) \neq 0$*

$$\Re\left(\frac{zr'(z)}{r(z)}\right) \leq \left[\frac{1}{1+k}\left\{t - \frac{|b_0|-k^t|b_t|}{|b_0|+k^t|b_t|}\right\} - \frac{n-|B'(z)|}{2}\right].$$

Proof. *Since $r(z)$ has all its zeros in $T_k \cup D_{k+}$, $k \geq 1$, let $z_1, z_2, \ldots, z_t$ be the zeros (may not be distinct) of $p(z)$, then $t \leq n$. We have*

$$r(z) = \frac{p(z)}{w(z)}.$$

This implies

$$\frac{zr'(z)}{r(z)} = \frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)}.$$

Hence,

$$\Re\left(\frac{zr'(z)}{r(z)}\right) = \Re\left(\frac{zp'(z)}{p(z)}\right) - \Re\left(\frac{zw'(z)}{w(z)}\right). \tag{9}$$

Now

$$p(z) = b_t \prod_{v=1}^{t} (z - z_v).$$

This implies

$$\frac{zp'(z)}{p(z)} = \sum_{v=1}^{t} \frac{z}{z - z_v}.$$

Now for the points $e^{i\theta}$, with $0 \le \theta \le 2\pi$ and $p(e^{i\theta}) \ne 0$, we get

$$\begin{aligned}\Re\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right) &= \sum_{v=1}^{t} \Re\left(\frac{e^{i\theta}}{e^{i\theta} - z_v}\right) \\ &\le \sum_{v=1}^{t} \frac{1}{1 + |z_v|} \\ &\le \frac{t}{1+k} - \frac{1}{1+k} \sum_{v=1}^{t} \frac{|z_v| - k}{|z_v| + k} \\ &\le \frac{t}{1+k} - \frac{1}{1+k} \sum_{v=1}^{t} \frac{\frac{|z_v|}{k} - 1}{\frac{|z_v|}{k} + 1}.\end{aligned}$$

Since $\frac{|z_v|}{k} \ge 1$, with $v = 1, 2, \ldots, n$, and by Lemma 2, we get

$$\begin{aligned}\Re\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right) &\le \frac{t}{1+k} - \frac{1}{1+k}\left(\frac{\prod_{v=1}^{t} \frac{|z_v|}{k} - 1}{\prod_{v=1}^{t} \frac{|z_v|}{k} + 1}\right) \\ &= \frac{t}{1+k} - \frac{1}{1+k}\left(\frac{\frac{|b_0|}{k^t|b_t|} - 1}{\frac{|b_0|}{k^t|b_t|} + 1}\right).\end{aligned}$$

From (9), substituting above value and using Lemma 3, we get.

$$\Re\left(\frac{zr'(z)}{r(z)}\right) \le \left[\frac{1}{1+k}\left\{t - \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\} - \frac{n - |B'(z)|}{2}\right].$$

This completes the proof. ■

4 Proofs of Main Results

Proof of Theorem 1. Since $r(z)$ has no zeros in D_{k-}, $k \ge 1$, let $z_1, z_2, \ldots, z_t$ be the zeros (may not be distinct) of $p(z)$, then $t \le n$. Hence by Lemma 4, we have for $z \in T_1$

$$2\Re\left(\frac{zr'(z)}{r(z)}\right) \le 2\left[\frac{t}{1+k} - \left\{\frac{1}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|} + \frac{n - |B'(z)|}{2}\right\}\right]. \tag{10}$$

Note that

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\overline{z}}\right)} = B(z)\overline{r}\left(\frac{1}{z}\right).$$

Differentiating, both sides gives

$$(r^*(z))' = B'(z)\overline{r}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\overline{r}'\left(\frac{1}{z}\right).$$

Since $z \in T_1$, we have $\overline{z} = 1/z$, and so

$$|(r^*(z))'| = |(zB'(z)/B(z))\overline{r(z)} - \overline{zr'(z)}|. \tag{11}$$

By ([1]) and Lemma 1, we have

$$\frac{zB'(z)}{B(z)} = \left|\frac{zB'(z)}{B(z)}\right| = |B'(z)|.$$

Thus, from equation (11), we have

$$|(r^*(z))'| = ||B'(z)|r(z) - zr'(z)|.$$

Therefore, for $z \in T_1$ and using inequality (10), we obtain.

$$\begin{aligned}
\left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\
&= \left|\frac{zr'(z)}{r(z)}\right|^2 + |B'(z)|^2 - 2|B'(z)|\Re\left(\frac{zr'(z)}{r(z)}\right) \\
&\geq \left|\frac{zr'(z)}{r(z)}\right|^2 + |B'(z)|^2 - 2|B'(z)|\left[\frac{t}{1+k} - \left\{\frac{1}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|} + \frac{n - |B'(z)|}{2}\right\}\right] \\
&\geq \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\left[\frac{t}{1+k} - \left\{\frac{1}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|} + \frac{n}{2}\right\}\right] \\
&\geq \left|\frac{zr'(z)}{r(z)}\right|^2 + \left[\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right]|B'(z)|.
\end{aligned}$$

This implies for $z \in T_1$,

$$\left[|r'(z)|^2 + \left\{\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}|B'(z)||r(z)|^2\right]^{\frac{1}{2}} \leq |(r^*(z))'|.$$

Above inequality becomes by Lemma 1

$$\left[|r'(z)|^2 + \left\{\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}\right]^{\frac{1}{2}} \leq \|r\||B'(z)| - |r'(z)|.$$

After manipulation, we get

$$|r'(z)|^2 + \left\{\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\} \leq \big(\|r\||B'(z)| - |r'(z)|\big)^2$$

Equivalently, for $z \in T_1$,

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \left\{\frac{n(k+1) - 2t}{k+1} - \frac{2}{k+1}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}\frac{|r(z)|^2}{\|r\|^2}\right]\|r\|.$$

This completes the proof. ∎

Remark 3 *We show our upper bound in Theorem 1 improves an upperbound in inequality (5) as follows. Since $t \le n$ and $k \ge 1$, we get that*

$$\left[\frac{n(k+1)-2t}{k+1}+\frac{2}{k+1}\cdot\frac{|b_0|-k^t|b_t|}{|b_0|+k^t|b_t|}\right]\frac{|r(z)|^2}{\|r\|^2} \ge \left[\frac{n(k+1)-2n}{k+1}+\frac{2}{k+1}\cdot\frac{|b_0|-k^n|b_n|}{|b_0|+k^n|b_n|}\right]\frac{|r(z)|^2}{\|r\|^2} \ge \frac{|b_0|-|b_n|}{|b_0|+|b_n|}\cdot\frac{|r(z)|^2}{\|r\|^2}.$$

Hence,

$$|B'(z)|-\left[\frac{n(k+1)-2t}{k+1}+\frac{2}{k+1}\cdot\frac{|b_0|-k^t|b_t|}{|b_0|+k^t|b_t|}\right]\frac{|r(z)|^2}{\|r\|^2} \le |B'(z)|-\frac{|b_0|-|b_n|}{|b_0|+|b_n|}\cdot\frac{|r(z)|^2}{\|r\|^2}.$$

In particular, if $t=n$ and $k=1$, then

$$|B'(z)|-\left[\frac{n(k+1)-2t}{k+1}+\frac{2}{k+1}\cdot\frac{|b_0|-k^t|b_t|}{|b_0|+k^t|b_t|}\right]\frac{|r(z)|^2}{\|r\|^2} = |B'(z)|-\frac{|b_0|-|b_n|}{|b_0|+|b_n|}\cdot\frac{|r(z)|^2}{\|r\|^2}.$$

Therefore, our upperbound in Theorem 1 improves an upperbound in inequality (5).

Proof of Theorem 2. Assume that all zeros of $r \in \mathbb{R}_n$ lie in $T_k \cup D_{k+}$, where $k \ge 1$. Let $m = \min_{z\in T_k}|r(z)|$ and t be the number of zeros of r with counting multiplicity.

If $r(z)$ has a zero on $z \in T_k$, then $m = 0$ and hence for every α with $|\alpha| < 1$, we get $r(z) - \alpha m = r(z)$. In this case Theorem 2 follows from Theorem 1 . Henceforth we assume, $r(z)$ has no zeros on $z \in T_k$, we have for every α with $|\alpha| < 1$ that

$$|-\alpha m| = |\alpha|.m < |r(z)| \quad \text{for } z \in T_k.$$

It follows from Rouche's theorem that $R(z) = r(z) - \alpha m$ and $r(z)$ have the same number of zeros in D_{k-}. That is, for every α with $|\alpha| < 1$, $R(z)$ has no zeros in D_{k-}. Since $R(z) \ne 0$, Lemma 4 yields for $z \in T_1$,

$$\Re\left(\frac{zR'(z)}{R(z)}\right) \le \frac{t}{1+k}-\left\{\frac{1}{1+k}\cdot\frac{|b_0+(-1)^{t+1}\alpha\cdot m\cdot c|-k^t|b_t-\alpha\cdot m|}{|b_0+(-1)^{t+1}\alpha\cdot m|+k^t|b_t-\alpha\cdot m|}+\frac{n-|B'(z)|}{2}\right\}. \tag{12}$$

Since $|b_0| \le |c|\cdot|b_n|$, we see that

$$|b_0|\cdot|\alpha|\cdot m \le |\alpha|\cdot m\cdot|c|\cdot|b_t|,$$

$$|b_n|\cdot|b_0|+|b_0|\cdot|\alpha|\cdot m \le |b_t|\cdot|b_0|+|\alpha|\cdot m\cdot|c|\cdot|b_t|,$$

and

$$\frac{|b_t|}{|b_0|} \ge \frac{|b_t|+|\alpha|\cdot m}{|b_0|+|\alpha|\cdot|c|\cdot m}.$$

Choosing argument of α in such a way that

$$|b_0+(-1)^{t+1}\cdot\alpha\cdot m\cdot c| = |b_0|+|\alpha|\cdot|c|\cdot m,$$

we get

$$\frac{|b_t|}{|b_0|} \ge \frac{|b_t-\alpha\cdot m|}{|b_0+(-1)^{t+1}\alpha\cdot c\cdot m|}.$$

Hence, it follows from inequality (12) that

$$\Re\left(\frac{zR'(z)}{R(z)}\right) \le \frac{t}{1+k}-\left\{\frac{1}{1+k}\cdot\frac{|b_0|-k^t|b_n|}{|b_0|+k^t|b_n|}+\frac{n-|B'(z)|}{2}\right\}. \tag{13}$$

Note that

$$R^*(z) = B(z)\overline{R\left(\frac{1}{\overline{z}}\right)} = B(z)\overline{R}\left(\frac{1}{z}\right).$$

Differentiating both sides gives

$$(R^*(z))' = B'(z)\overline{R}\left(\frac{1}{z}\right) - \frac{B(z)}{z^2}\overline{R}'\left(\frac{1}{z}\right).$$

Since $z \in T_1$, we have $\overline{z} = 1/z$, and so

$$|(R^*(z))'| = |(zB'(z)/B(z))\overline{R(z)} - \overline{zR'(z)}|. \tag{14}$$

By ([1]) and Lemma 1, we have

$$\frac{zB'(z)}{B(z)} = \left|\frac{zB'(z)}{B(z)}\right| = |B'(z)|.$$

Thus from (14), we have

$$|(R^*(z))'| = ||B'(z)|R(z) - zR'(z)|.$$

Then

$$\begin{aligned}
\left|\frac{z(R^*(z))'}{R(z)}\right|^2 &= \left||B'(z)| - \frac{zR'(z)}{R(z)}\right|^2 \\
&= \left|\frac{zR'(z)}{R(z)}\right|^2 + |B'(z)|^2 - 2|B'(z)|\Re\left(\frac{zR'(z)}{R(z)}\right) \\
&\geq \left|\frac{zR'(z)}{R(z)}\right|^2 + |B'(z)|^2 - 2|B'(z)|\left[\frac{t}{1+k} - \left\{\frac{1}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|} + \frac{n - |B'(z)|}{2}\right\}\right] \\
&\geq \left|\frac{zR'(z)}{R(z)}\right|^2 - 2|B'(z)|\left[\frac{t}{1+k} - \left\{\frac{1}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|} + \frac{n}{2}\right\}\right] \\
&\geq \left|\frac{zR'(z)}{R(z)}\right|^2 + \left[\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right]|B'(z)|.
\end{aligned}$$

where the inequality comes from (13).

This implies that for $z \in T_1$,

$$\left[|R'(z)|^2 + \left\{\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}|B'(z)||R(z)|^2\right]^{\frac{1}{2}} \leq |(R^*(z))'|, \tag{15}$$

where

$$R^*(z) = B(z)\overline{R\left(\frac{1}{\overline{z}}\right)} = r^*(z) - \overline{\alpha}mB(z)$$

so that

$$\big(R^*(z)\big)' = \big(r^*(z)\big)' - \overline{\alpha}mB'(z) \quad \text{and} \quad R'(z) = (r(z) - \alpha m)' = r'(z).$$

Apply, these relations to (15), we obtain that,

$$\left[|r'(z)|^2 + \left\{\frac{n(k+1) - 2t}{k+1} + \frac{2}{1+k}\cdot\frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}|B'(z)||r(z) - \alpha m|^2\right]^{\frac{1}{2}} \leq |(r^*(z))' - \overline{\alpha}mB'(z)|. \tag{16}$$

for $z \in T_1$ and for α with $|\alpha| < 1$.

Choosing argument of α such that

$$|(r^*(z))' - \overline{\alpha}mB'(z)| = |(r^*(z))'| - m|\alpha||B'(z)|, \quad \text{for} \quad z \in T_1.$$

Triangle inequality yields that

$$|r(z) - m\alpha| \geq ||r(z)| - m|\alpha||.$$

Note that

$$||r(z)| - m|\alpha||^2 = \big(|r(z)| - m|\alpha|\big)^2 \quad \text{and} \quad |r(z) - m\alpha|^2 \geq (|r(z)| - m|\alpha|)^2.$$

Substituting above relations in (16), we get

$$\left[|r'(z)|^2 + \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}|B'(z)|(|r(z)| - m|\alpha|)^2\right]^{\frac{1}{2}} \leq |(r^*(z))'| - m|\alpha||B'(z)|.$$

Letting $|\alpha| \to 1$, we get

$$\left[|r'(z)|^2 + \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}(|r(z)| - m|)^2|B'(z)|\right]^{\frac{1}{2}} \leq |(r^*(z))'| - m|B'(z)|.$$

Lemma 1, implies that

$$\left[|r'(z)|^2 + \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}(|r(z)| - m|)^2|B'(z)|\right]^{\frac{1}{2}} \leq |B'(z)|.||r|| - |r'(z)| - m|B'(z)|.$$

Equivalently, for $z \in T_1$:

$$|r'(z)|^2 + \left[\frac{n(k+1)-2t}{k+1} + \frac{2}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right](|r(z)| - m|)^2|B'(z)| \leq \big[\big(\|r\| - m\big)|B'(z)| - r'(z)|\big]^2.$$

A simple manipulation gives, for $z \in T_1$

$$\begin{aligned} |r'(z)|^2 \leq{}& \big(\|r\| - m\big)^2|B'(z)|^2 + |r'(z)|^2 - 2(\|r\| - m)|B'(z)||r'(z)| \\ &- \left[\frac{n(k+1)-2t}{k+1} + \frac{2}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right](|r(z)| - m|)^2|B'(z)|. \end{aligned}$$

This implies

$$|r'(z)| \leq \frac{1}{2}\left[\frac{\big(\|r\| - m\big)^2|B'(z)|}{(\|r\| - m)} - \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{1+k} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}\frac{(|r(z)| - m)^2}{(\|r\| - m)^2}\right].$$

Hence

$$|r'(z)| \leq \quad \frac{1}{2}\left[|B'(z)| - \left\{\frac{n(k+1)-2t}{k+1} + \frac{2}{k+1} \cdot \frac{|b_0| - k^t|b_t|}{|b_0| + k^t|b_t|}\right\}\frac{(|r(z)| - m)^2}{(\|r\| - m)^2}\right](\|r\| - m).$$

where t is the number of zeros of r with counting multiplicity and $m = \min_{z \in T_k} |r(z)|$.

This proves the inequality for $R(z) \neq 0$. Therefore, inequality holds for all $z \in T_1$. This completes the proof. ■

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