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Estimation Of The Jensen Gap Via Quasi-Convex Functions*

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Abstract

If T is a convex function, ζ_s are in its domain and ω_s are nonnegative weights for $s = 1, \cdots, n$ with $\sum_{s=1}^{n} \omega_s = W > 0$, then the functional $J(\zeta, W, T)$ defined by

$$J(\zeta, W, T) = \frac{1}{W} \sum_{s=1}^{n} \omega_s T(\zeta_s) - T\left(\frac{1}{W} \sum_{s=1}^{n} \omega_s \zeta_s\right),$$

is known as the Jensen gap. In this paper, we derive a bound for J, when T is a function whose double derivative in absolute value is quasi-convex. Also, we demonstrate the bound for the integral version of J. We discuss some numerical examples. These examples not only demonstrate the sharpness of the bound but also addresses the fact that J can be estimated through quasi-convex functions when even the functions are not convex. Finally, we utilize the main result to deduce a Hölder type inequality, and propose an upper bound for the difference between the first and second term of the Hermite-Hadamard inequality.

1 Introduction

Within the context of current research in contemporary applied analysis, mathematical inequalities and their manifold manifestations hold a standard position regarding their applications in various fields of science, technology, and art [9, 10, 15–17, 20–23, 32, 33, 37–43]. The concept of convex functions arises from the field of mathematical inequalities. Theory of mathematical inequalities gives a virtuous frame work to the class of convex functions. The most important inequality for convex functions in the literature of applied mathematical inequalities is the Jensen inequality [2, 3, 6, 7, 11, 24–26]. The Jensen inequality in its classical version can be seen in [19], while its integral version has been given in [18]. It is fair to say that the Jensen inequality is one of the powerful mathematical inequalities, because this inequality gives rise to many other classical inequalities through some suitable manipulations. This inequality, and its variants have many implications in various aspects such as: they help to provide a qualitative theoretical background to integral and differential equations [24], play a remarkable role in the designing of electrical engineering phenomena, and several other physical phenomena in this discipline [11], in probability space they assist to deduce Rao-Blackwell estimates for the under considered parameters in their estimation processes [25], provide estimates for various entropies, distances, and divergences, and furnish a method to verify the non-negativity of Kullback-Leibler divergence [4, 5, 12, 14, 18, 27, 35, 36]. Moreover, this inequality allows for the exploration of dynamically consistent nonlinear assessments within a probability space and the utilization of super linear expectations in economic applications [26, 44]. This inequality has been presented for various classes of convex functions as well for example, s-convex [1], (α, m)-convex and m-convex [8], quasi-convex [28], strongly convex [29] and Q-class convex functions [30] etc.

It is notable that the class of convex functions is a subclass of the class of quasi-convex functions. Our purpose is to show that the Jensen inequality can be presented in a more general and standard framework for the class of quasi-convex functions. Dragomir and Pearce [13] presented a number of inequalities of the Jensen type pertaining to quasi-convex functions defined on some convex sets in the real linear spaces.

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Mihai and Niculescu [28] used some suitable signed constraints and discussed that the Jensen inequality for quasi-convex functions can be used in a more general form. Also, the authors removed the restriction on its trend towards probability measures.

Definition 1 ([34]) *Let $t \in [0,1]$, and $\vartheta_1, \vartheta_2 \in \mathbb{C}$, where $\mathbb{C}$ is a convex set. If the following inequality holds for the function $T : \mathbb{C} \to \mathbb{R}$, then it is referred to as a quasi-convex:*

$$T(t\vartheta_1 + (1-t)\vartheta_2) \leq \max\{T(\vartheta_1), T(\vartheta_2)\}.$$

Further in this section, we give a Green function and its related integral identity, which will be used in the process of obtaining the main results. The Green function defined on $[b,c] \times [b,c]$ is given by [31]:

$$G(\xi, x) = \begin{cases} \frac{(\xi-c)(x-b)}{c-b}, & b \leq x \leq \xi, \\ \frac{(x-c)(\xi-b)}{c-b}, & \xi \leq x \leq c, \end{cases} \tag{1}$$

and its related integral identity for a function $T \in C^2[b,c]$, is given by

$$T(\xi) = \frac{c-\xi}{c-b}T(b) + \frac{\xi-b}{c-b}T(c) + \int_b^c G(\xi,x)T''(x)dx. \tag{2}$$

It is worth noting that the Green function G is continuous and exhibits convexity with respect to both of the variables ξ and x.

2 Main Results

The forthcoming theorem puts forth an upper bound for the Jensen gap, which is the main result.

Theorem 1 *Assume a function $T \in C^2[b,c]$ such that $|T''|$ is quasi-convex and $\zeta_s \in [b,c]$, $\omega_s \geq 0$ for $s = 1, \cdots, n$ with the condition $\sum_{s=1}^n \omega_s = W > 0$, then*

$$\left| \frac{1}{W}\sum_{s=1}^n \omega_s T(\zeta_s) - T\left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s\right) \right| \leq \frac{\max\{|T''(b)|, |T''(c)|\}}{2} \left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s^2 - \left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s\right)^2 \right). \tag{3}$$

Proof. Using (2) in $\frac{1}{W}\sum_{s=1}^n \omega_s T(\zeta_s)$ and $T\left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s\right)$, we get

$$\frac{1}{W}\sum_{s=1}^n \omega_s T(\zeta_s) = \frac{1}{W}\sum_{s=1}^n \omega_s \left(\frac{c-\zeta_s}{c-b}T(b) + \frac{\zeta_s - b}{c-b}T(c) + \int_b^c G(\zeta_s, x)T''(x)dx \right) \tag{4}$$

and

$$\begin{aligned} T\left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s\right) &= \frac{c - \frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s}{c-b}T(b) + \frac{\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s - b}{c-b}T(c) \\ &\quad + \int_b^c G\left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s, x\right) T''(x)dx. \end{aligned} \tag{5}$$

Subtracting (5) from (4), we obtain

$$\begin{aligned} &\frac{1}{W}\sum_{s=1}^n \omega_s T(\zeta_s) - T\left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s\right) \\ &= \int_b^c \left(\frac{1}{W}\sum_{s=1}^n \omega_s G(\zeta_s, x) - G\left(\frac{1}{W}\sum_{s=1}^n \omega_s\zeta_s, x\right) \right) T''(x)dx. \end{aligned} \tag{6}$$

Taking absolute value of (6), we get

$$\begin{aligned}
&\left|\frac{1}{W}\sum_{s=1}^{n}\omega_s T(\zeta_s)-T\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s\zeta_s\right)\right| \\
=&\left|\int_b^c\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s G(\zeta_s,x)-G\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s\zeta_s,x\right)\right)T''(x)dx\right| \\
\leq&\int_b^c\left|\frac{1}{W}\sum_{s=1}^{n}\omega_s G(\zeta_s,x)-G\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s\zeta_s,x\right)\right||T''(x)|dx. \qquad (7)
\end{aligned}$$

Changing the variable $x=tb+(1-t)c,\ t\in[0,1]$ in above. Further due to the convexity of $G(\xi,x)$, we can deduce the following from (7)

$$\begin{aligned}
&\left|\frac{1}{W}\sum_{s=1}^{n}\omega_s T(\zeta_s)-T\left(\bar{\zeta}\right)\right| \\
\leq&(c-b)\int_0^1\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s G(\zeta_s,tb+(1-t)c)-G(\bar{\zeta},tb+(1-t)c)\right)|T''(tb+(1-t)c)|dt, \qquad (8)
\end{aligned}$$

where $\bar{\zeta}=\frac{1}{W}\sum_{s=1}^{n}\omega_s\zeta_s$.

The following inequality, based on the quasi-convexity of the function $|T''|$, is established from equation (8).

$$\begin{aligned}
&\left|\frac{1}{W}\sum_{s=1}^{n}\omega_s T(\zeta_s)-T\left(\bar{\zeta}\right)\right| \\
\leq&(c-b)\int_0^1\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s G(\zeta_s,tb+(1-t)c)-G(\bar{\zeta},tb+(1-t)c)\right) \\
&\times\max\left\{|T''(b)|,|T''(c)|\right\}dt \\
=&(c-b)\max\left\{|T''(b)|,|T''(c)|\right\}\left(\frac{1}{W}\sum_{s=1}^{n}\omega_s\int_0^1 G(\zeta_s,tb+(1-t)c)dt\right. \\
&\left.-\int_0^1 G(\bar{\zeta},tb+(1-t)c)dt\right). \qquad (9)
\end{aligned}$$

With the help of $x=tb+(1-t)c$, $t\in[0,1]$, we establish that

$$\int_0^1 G(\zeta_s,tb+(1-t)c)dt=\frac{(\zeta_s^2-c\zeta_s-b\zeta_s+bc)}{2(c-b)}. \qquad (10)$$

Replacing ζ_s by $\bar{\zeta}$ in (10), we get

$$\int_0^1 G(\bar{\zeta},tb+(1-t)c)dt=\frac{((\bar{\zeta})^2-c\bar{\zeta}-b\bar{\zeta}+bc)}{2(c-b)}. \qquad (11)$$

Substituting the values from (10) and (11) in (9) and simplifying, we get (3). ∎

Remark 1 *Instead of G in Theorem 1, employing the Green functions G_1-G_4 as outlined in [31], we acquire the same result (3).*

Here, we present a theorem that outlines the integral interpretation of Theorem 1.

Theorem 2 *Assume that the function $|T''|$ is quasi-convex, for $T \in C^2[b,c]$. Also, assume that h_1, h_2 are real valued functions whose domain is $[a_1, a_2]$. Further assume that $h_1(y) \in [b,c]$ for all $y \in [a_1, a_2]$ and the functions $(T \circ h_1)h_2, h_2, h_1 h_2$ are integrable on $[a_1, a_2]$. Then for $h_2(y) \geq 0$ with $\int_{a_1}^{a_2} h_2(y)dy := H > 0$, the following inequality holds*

$$\left| \frac{1}{H} \int_{a_1}^{a_2} (T \circ h_1)(y) h_2(y) dy - T\left(\frac{1}{H} \int_{a_1}^{a_2} h_1(y) h_2(y) dy \right) \right|$$
$$\leq \frac{\max\{|T''(b)|, |T''(c)|\}}{2} \left(\frac{1}{H} \int_{a_1}^{a_2} h_1^2(y) h_2(y) dy - \left(\frac{1}{H} \int_{a_1}^{a_2} h_1(y) h_2(y) dy \right)^2 \right). \tag{12}$$

Proof. The result (12) can be obtained by adopting the procedure of Theorem 1. ∎

Example 1 *Let $T(y) = \frac{y^2}{4}(2\ln y - 3)$, $h_1(y) = y$, $h_2(y) = 1 \ \forall \ y \in [1,2]$. Then $T''(y) = \ln y \geq 0$, $\forall$ $y \in [1,2]$. From this, we conclude that for T as a convex function, $|T''|$ is quasi-convex but not classical convex. Further we ensure that $h_1(y) \in [1,2] \ \forall \ y \in [1,2]$, therefore taking $[b,c] = [a_1,a_2] = [1,2]$, we obtain $H = \int_{a_1}^{a_2} h_2(y)dy = 1$. Hence, we derive*

$$\int_1^2 T(h_1(y))dy - T(1.5) = -1.2147 + 1.2314 = 0.0167,$$

when we evaluate left hand side of inequality (12) for these values. This reveals the level of precision inherent in the Jensen inequality. Upon computing the right-hand side of inequality (12) to be 0.0289*, From our investigation, we arrive at the following deduced result:*

$$0.0167 < 0.0289.$$

The difference of 0.0122*, resulting from subtracting* 0.0167 *from* 0.0289*, when examining the inequality from above, underscores the striking proximity of bound for the Jensen gap in inequality (12) to its true value.*

Example 2 *Let $T(y) = \frac{4}{15} y^{\frac{5}{2}}$, $h_1(y) = y^2$, $h_2(y) = 1 \ \forall \ y \in [0,1]$. Then $T''(y) = \sqrt{y} \geq 0$, $\forall \ y \in [0,1]$. From this, we conclude that for T as a convex function, $|T''|$ is quasi-convex but not classical convex. Further we ensure that $h_1(y) \in [0,1] \ \forall \ y \in [0,1]$, therefore taking $[b,c] = [a_1,a_2] = [0,1]$, we find that $H = \int_{a_1}^{a_2} h_2(y)dy = 1$. Hence, one can verify that*

$$\int_0^1 T(h_1(y))dy - T(0.3333) = 0.0273,$$

when the left side of inequality (12) is evaluated for these facts. This reveals the level of precision inherent in the Jensen inequality. Upon computing the right hand side of inequality (12) to be 0.0445*, we can now deduce the next result:*

$$0.0273 < 0.0445.$$

The minute difference of 0.0172 *between the right and left sides of the inequality above underscores the striking accuracy of the Jensen's gap bound, as presented in inequality (12), in approximating the actual value.*

The proposition that follows introduces a re-imagined form of the Hölder inequality, directly stemming from the insights of Theorem 1.

Proposition 1 *Suppose that $[b,c]$ is a positive interval, given two positive n-tuples $(\alpha_1, \cdots, \alpha_n)$, $(\beta_1, \cdots, \beta_n)$ with both meeting the condition $\frac{\sum_{s=1}^n \alpha_s \beta_s}{\sum_{s=1}^n \beta_s^q}$, $\alpha_s \beta_s^{-\frac{q}{p}} \in [b,c]$ for $s = 1, \cdots, n$. Then for $q > 1$ and $p \in$*

$\mathbb{R}^+ - \{(2,3) \cup (0,1]\}$, *conforming to* $p^{-1} + q^{-1} = 1$, *the following result holds*

$$\left(\sum_{s=1}^{n} \alpha_s^p\right)^{\frac{1}{p}} \left(\sum_{s=1}^{n} \beta_s^q\right)^{\frac{1}{q}} - \sum_{s=1}^{n} \alpha_s \beta_s \leq \left(\frac{\max\{p(p-1)b^{p-2}, p(p-1)c^{p-2}\}}{2} \left(\frac{1}{\sum_{s=1}^{n} \beta_s^q} \sum_{s=1}^{n} \alpha_s^2 \beta_s^{1-\frac{q}{p}} - \left(\frac{1}{\sum_{s=1}^{n} \beta_s^q} \sum_{s=1}^{n} \alpha_s \beta_s\right)^2\right)\right)^{\frac{1}{p}} \sum_{s=1}^{n} \beta_s^q. \tag{13}$$

Proof. Let $T(x) = x^p,\ x \in [b, c]$, which is a convex function for given values of p. Also, we have $|T''|''^{p-4} > 0$. It is evident that the function $|T''|$ exhibits convex behavior. This guarantees the quasi-convex nature of the function $|T''|$, therefore using (3) for $T(x) = x^p,\ \omega_s = \beta_s^q$ and $\zeta_s = \alpha_s \beta_s^{-\frac{q}{p}}$, we derive

$$\left(\left(\sum_{s=1}^{n} \alpha_s^p\right) \left(\sum_{s=1}^{n} \beta_s^q\right)^{p-1} - \left(\sum_{s=1}^{n} \alpha_s \beta_s\right)^p\right)^{\frac{1}{p}} \leq \left(\frac{\max\{p(p-1)b^{p-2}, p(p-1)c^{p-2}\}}{2} \left(\frac{1}{\sum_{s=1}^{n} \beta_s^q} \sum_{s=1}^{n} \alpha_s^2 \beta_s^{1-\frac{q}{p}} - \left(\frac{1}{\sum_{s=1}^{n} \beta_s^q} \sum_{s=1}^{n} \alpha_s \beta_s\right)^2\right)\right)^{\frac{1}{p}} \sum_{s=1}^{n} \beta_s^q. \tag{14}$$

By utilizing the inequality $x^\kappa - y^\kappa \leq (x-y)^\kappa,\ 0 \leq y \leq x,\ \kappa \in [0,1]$ for $y = \left(\sum_{s=1}^{n} \alpha_s \beta_s\right)^p,\ x = \left(\sum_{s=1}^{n} \alpha_s^p\right) \left(\sum_{s=1}^{n} \beta_s^q\right)^{p-1}$ and $\kappa = \frac{1}{p}$, we obtain

$$\left(\sum_{s=1}^{n} \alpha_s^p\right)^{\frac{1}{p}} \left(\sum_{s=1}^{n} \beta_s^q\right)^{\frac{1}{q}} - \sum_{s=1}^{n} \alpha_s \beta_s \leq \left(\left(\sum_{s=1}^{n} \alpha_s^p\right) \left(\sum_{s=1}^{n} \beta_s^q\right)^{p-1} - \left(\sum_{s=1}^{n} \alpha_s \beta_s\right)^p\right)^{\frac{1}{p}}. \tag{15}$$

Now using (15) in (14), we get (13). ■

Remark 2 *Applying the methodology from Proposition 1, we can unveil a fresh interpretation of the Hölder inequality, this time in its integral form, as a direct outcome of Theorem 2.*

The subsequent outcome of Theorem 2 is an application in the form of a corollary, unveiling a novel upper bound for the Hermite-Hadamard gap.

Corollary 1 *Under the condition that* $|\Gamma''|$ *is quasi-convex function for a function* $\Gamma \in C^2[e_1, e_2]$, *the following relationship holds*

$$\left|\frac{1}{e_2 - e_1} \int_{e_1}^{e_2} \Gamma(y) dy - \Gamma\left(\frac{e_1 + e_2}{2}\right)\right| \leq \frac{(e_2 - e_1)^2}{24} \max\{|\Gamma''(e_1)|, |\Gamma''(e_2)|\}. \tag{16}$$

Proof. Using (12) for $\Gamma = T,\ [b, c] = [e_1, e_2]$ and $h_2(y) = 1,\ h_1(y) = y$ for all $y \in [e_1, e_2]$, we get (16). ■

Corollary 2 *When defining the function* Γ *as outlined in Corollary 1, consider incorporating the following supplementary details:*

(i) If $|\Gamma''|$ *is increasing, then we obtain the following result directly from Corollary 1,*

$$\left|\frac{1}{e_2 - e_1} \int_{e_1}^{e_2} \Gamma(y) dy - \Gamma\left(\frac{e_1 + e_2}{2}\right)\right| \leq \frac{(e_2 - e_1)^2}{24} |\Gamma''(e_2)|. \tag{17}$$

(ii) If $|\Gamma''|$ is decreasing, then we obtain the following result directly from Corollary 1,

$$\left|\frac{1}{e_2-e_1}\int_{e_1}^{e_2}\Gamma(y)dy-\Gamma\left(\frac{e_1+e_2}{2}\right)\right|\leq\frac{(e_2-e_1)^2}{24}|\Gamma''(e_1)|. \tag{18}$$

Remark 3 *Sarikaya et al. have independently derived the results (16)–(18) as well [34].*

3 Conclusions

By applying Jensen's inequality, we expand upon the traditional understanding of convexity. This inequality establishes a valuable mathematical framework for understanding convex functions. Also, this inequality has been presented for generalized convex functions. Some mathematicians have worked out on Jensen type inequalities for quasi-convex functions. In this paper, we have associated the Jensen inequality to quasi-convex functions and derived a bound for its gap pertaining to functions whose double derivative in absolute values are quasi-convex. We have also demonstrated the accuracy of the bound through numerical examples. The examples show that the Jensen gap can be estimated for the quasi-convex functions even when they are not convex. Furthermore, we have uncovered a new variant of the Hölder inequality and a revised bound for the Hermite-Hadamard gap. The idea proposed in this article may inculcate further research in the field of modern applied analysis.

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References

[1] M. Adil Khan, S. Khan and Y.-M. Chu, New estimates for the Jensen gap using s-convexity with applications, Front. Phys., 8(2020), Article 313.

[2] M. Adil Khan, S. Khan and Y.-M. Chu, A new bound for the Jensen gap with applications in information theory, IEEE Access, 8(2020), 98001–98008.

[3] M. Adil Khan, S. Khan,D. Pečarić and J. Pečarić, New improvements of Jensen's type inequalities via 4-convex functions with applications, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 115(2021), Article No 43.

[4] M. Adil Khan, D. Pečarić and J. Pečarić, On Zipf-Mandelbrot entropy, J. Comput. Appl. Math., 346(2019), 192–204.

[5] K. Ahmad, M. Adil Khan, S. Khan, A. Ali and Y.-M. Chu, New estimates for generalized Shannon and Zipf-Mandelbrot entropies via convexity results, Results Phys., 18(2020).

[6] K. Ahmad, M. Adil Khan, S. Khan, A. Ali and Y.-M. Chu, New estimation of Zipf-Mandelbrot and Shannon entropies via refinements of Jensen's inequality, AIP Adv., 11(2021).

[7] K. Ahmad, M. Adil Khan, S. Khan and Y.-M. Chu, Improved bounds for the Jensen gap with applications in information theory, Sci. Asia., 48(2022).

[8] M. K. Bakula, J. Pečarić and M. Ribičić, Companian inequalities to Jensen's inequality for m-convex and (α, m)-convex functions, J. Inequal. Pure Appl. Math., 7(2006).

[9] S.-B. Chen, S. Rashid, M. A. Noor, Z. Hammouch and Y.-M. Chu, New fractional approaches for n-polynomial P-convexity with applications in special function theory, Adv. Difference Equ., 2020(2020), 31 pp.

[10] H.-H. Chu, H. Kalsoom, S. Rashid, M. Idrees, F. Safdar, Y.-M. Chu and D. Baleanu, Quantum analogs of Ostrowski type inequalities for Raina's function correlated with coordinated generalized Φ-convex functions, Symmetry, 12(2020).

[11] M. J. Cloud, B. C. Drachman and L. P. Lebedev, Inequalities with Applications to Engineering, Springer: Cham Heidelberg New York Dordrecht London, 2014.

[12] S. S. Dragomir, M. Adil Khan and A. Abathun, Refinement of the Jensen integral inequality, Open Math., 14(2016), 221–228.

[13] S. S. Dragomir and C. E. M. Pearce, Jensen's inequality for quasi-convex functions, Numer. Algebra, Control. Optim., 2(2012), 279–291.

[14] L. Horváth, D. Pečarić and J. Pečarić, Estimations of f- and Rényi divergences by using a cyclic refinement of the Jensen's inequality, Bull. Malays. Math. Sci. Soc., 42(2019), 933–946.

[15] A. Iqbal, M. Adil Khan, M. Suleman and Y.-M. Chu, The right Riemann-Liouville fractional Hermite-Hadamard type inequalities derived from Green's function, AIP Adv., 10(2020).

[16] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Adv., 8(2018).

[17] H. Kalsoom, S. Rashid, M. Idrees, F. Safdar, S. Akram, D. Baleanu and Y.-M. Chu, Post quantum integral inequalities of Hermite-Hadamard-type associated with co-ordinated higherorder generalized strongly pre-invex and quasi-pre-invex mappings, Symmetry, 12(2020), 443.

[18] S. Khan, M. Adil Khan and Y.-M. Chu, Converses of the Jensen inequality derived from the Green functions with applications in information theory, Math. Method. Appl. Sci., 43(2020), 2577–2587.

[19] S. Khan, M. Adil Khan and Y.-M. Chu, New converses of Jensen inequality via Green functions with applications, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 114(2020), 14 pp.

[20] K. A. Khan, A. Ditta, A. Nosheen and K. M. Awan, Rostin Matendo Mabela, Ostrowski type inequalities for s-convex functions via q-integrals, J. Funct. Spaces, 2022(2022), 8 pp.

[21] M. B. Khan, M. A. Noor, L. Abdullah and Y.-M. Chu, Some new classes of preinvex fuzzy-interval-valued functions and inequalities, Int. J. Comput. Intell. Syst., 14(2021), 1403–1418.

[22] M. B. Khan, M. A. Noor, K. I. Noor and Y.-M. Chu, Higher-order strongly preinvex fuzzy mappings and fuzzy mixed variational-like inequalities, Int. J. Comput. Intell. Syst., 14(2021), 1856–1870.

[23] Y. Khurshid, M. Adil Khan and Y.-M. Chu, Ostrowski type inequalities involving conformable integrals via preinvex functions, AIP Adv., 102020.

[24] V. Lakshmikantham and A. S. Vatsala, Theory of differential and integral inequalities with initial time difference and applications, Analytic and geometric inequalities and applications, Math. Appl., 478, Kluwer Acad. Publ., Dordrecht, (1999), 191–203.

[25] J. G. Liao and A. Berg, Sharpening Jensen's inequality, Amer. Statist., 73(2019), 278–281.Statist. Probab. Lett., 151(2019), 79–83.

[26] Q. Lin, Jensen inequality for superlinear expectations, Statist. Probab. Lett., 151(2019), 79–83.

[27] N. Lovričević, Đ. Pečarić and J. Pečarić, Zipf-Mandelbrot law, f-divergences and the Jensen-type interpolating inequalities, J. Inequal. Appl., 2018(2018), 20 pp.

[28] M. V. Mihai and C. P. Niculescu, A simple proof of the Jensen-type inequality of Fink and Jodeit, Mediterr. J. Math., 13(2016), 119–126.

[29] H. R. Moradi, M. E. Omidvar, M. Adil Khan and K. Nikodem, Around Jensen's inequality for strongly convex functions, Aequationes Math., 92(2018), 25–37.

[30] M. S. Moslehian and M. Kian, Jensen type inequalities for Q-class functions, Bull. Aust. Math. Soc., 85(2012), 128–142.

[31] Đ. Pečarić, J. Pečarić and M. Rodić, About the sharpness of the Jensen inequality, J. Inequal. Appl., 2018(2018), 12 pp.

[32] S. Rashid, S. Sultana, Y. Karaca, A. Khalid and Y.-M. Chu, Some further extensions considering discrete proportional fractional operators, Fractals, 30(2022).

[33] G. Sana, M. B. Khan, M. A. Noor, P. O. Mohammed and Y.-M. Chu, Harmonically convex fuzzy-intervalvalued functions and fuzzy-interval Riemann-Liouville fractional integral inequalities, Int. J. Comput. Intell. Syst., 14(2021), 1809–1822.

[34] M. Z. Sarikaya, A. Saglam and H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, Int. J. Open Probl. Comput. Sci. Math., 5(2012), 1–14.

[35] Z. M. M. M. Sayed, M. Adil Khan, S. Khan and J. Pečarić, Refinement of the classical Jensen inequality using finite sequences, Hacet. J. Math. Stat., 53(2024), 608–627.

[36] Y. Sayyari, H. Barsam and A. R. Sattarzadeh, On new refinement of the Jensen inequality using uniformly convex functions with applications, Appl. Anal., 102(2023), 5215–5223.

[37] M.-K. Wang, M.-Y. Hong, Y.-F. Xu, Z.-H. Shen, Y.-M. Chu, Inequalities for generalized trigonometric and hyperbolic functions with one parameter, J. Math. Inequal., 14(2020), 1–21.

[38] M.-K. Wang, W. Zhang and Y.-M. Chu, Monotonicity, convexity and inequalities involving the generalized elliptic integrals, Acta Math. Sci. Ser. B, 39(2019), 1440–1450.

[39] T.-H. Zhao, Z.-Y. He and Y.-M. Chu, On some refinements for inequalities involving zero-balanced hypergeometric function, AIMS Math., 5(2020), 6479–6495.

[40] T.-H. Zhao, L. Shi and Y.-M. Chu, Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 114(2020), 14 pp.

[41] T.-H. Zhao, M.-K. Wang, Y.-M. Chu, A sharp double inequality involving generalized complete elliptic integral of the first kind, AIMS Math., 5(2020), 4512–4528.

[42] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018(2018), 15 pp.

[43] T.-H. Zhao, B.-C. Zhou, M.-K. Wang and Y.-M. Chu, On approximating the quasi-arithmetic mean, J. Inequal. Appl., 2019(2019), 12 pp.

[44] Z. Zong, F. Hu, C. Yin and H. Wu, On Jensen's inequality, Hölder's inequality, and Minkowski's inequality for dynamically consistent nonlinear evaluations, J. Inequal. Appl., 2015(2015), 18 pp.