Applied Mathematics E-Notes, 25(2025), 134-148 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On $\mathcal{I}$-Convergence Of Sequences In Neutrosophic 2-Normed Spaces*

Rahul Mondal†, Nesar Hossain‡

Received 7 January 2024

Abstract

Building on recent advancements, this paper introduces the concepts of I and I^*-convergence of sequences in neutrosophic 2-normed spaces, offering a novel generalization of statistical convergence in summability theory. We explore key properties and unveil the intricate relationship between these two innovative concepts. Also, we introduce and delve into the concept of I and I^*-Cauchy sequences for sequences, demonstrating the pivotal role of condition (AP) in understanding their interrelationship. Additionally, we establish that every N2-NS is complete in connection with an ideal within this specific framework.

1 Introduction

Zadeh [43] is the first prominent pioneering of the introduction of fuzzy set theory as an extension of classical set theory. Since its inception, it has been continually refined and integrated across various fields of engineering and science, including population dynamics [4], control of chaos [10], computer programming [12], nonlinear dynamical systems [18], fuzzy physics [27] etc. An intriguing extension of fuzzy sets, introduced by Atanassov [1], is known as intuitionistic fuzzy sets, which enhance the traditional fuzzy sets by incorporating a non-membership function alongside the membership function. Over time, the concept of fuzzy set has been fascinatingly expanded into new and innovative notions, often referred to as interval valued fuzzy sets [42], interval valued intuitionistic fuzzy set [2], vague sets [3] and the evolution of fuzzy sets has sparked the growth of numerous concepts in mathematical analysis. As a comprehensive generalization of these concepts, Smarandache [36] defined a new idea named as neutrosophic set by introducing the indeterminacy function to the intuitionistic fuzzy sets, i.e., an element of a neutrosophic set is characterized by a triplet: the truth-membership function, the indeterminacy-membership function, and the falsity-membership function. In a neutrosophic set, each element of the universe is defined by its specific degrees of these notions. The concept of fuzzy normed spaces, introduced by Felbin [9] in 1992, evolved over the years with Saadati and Park's [37] introduction of intuitionistic fuzzy normed spaces in 2006, followed by Karakus et al.'s exploration of statistical convergence [22] within these spaces in 2008, and Kumar et al.'s [23] in 2009 generalization to ideal convergence. Later on, Bera and Mahapatra explored the notion of neutrosophic soft linear space [5] and neutrosophic soft normed linear space [6]. Recently, Kirişci and Şimşek [24] introduced neutrosophic normed linear spaces and delved into the concept of statistical convergence, sparking further research into different types of sequence convergence within these spaces. For additional insights, see [7, 17, 25, 26]. In 2023, Murtaza et al. [29] introduced the groundbreaking concept of neutrosophic 2-normed linear space, a significant extension of neutrosophic normed space, and explored its statistical convergence and statistical completeness.

The concept of statistical convergence in sequences was first introduced independently by Fast [8], Steinhaus [33], and Schoenberg [34], expanding upon the traditional notion of ordinary sequence convergence. Later, a significant advancement in statistical convergence arose with the introduction of $\mathcal{I}$-convergence of

*Mathematics Subject Classifications: 40A35, 03E72, 46S40.

†Department of Mathematics, Vivekananda Satavarshiki Mahavidyalaya, Manikpara, Jhargram -721513, West Bengal, India
‡Department of Mathematics, The University of Burdwan, Burdwan - 713104, West Bengal, India

sequences by Kostyrko et al. [20]. This concept is based on the notion of an ideal $\mathcal{I}$, a collection of subsets of the natural numbers. Since its introduction, this pivotal concept has been explored and expanded in various directions by numerous researchers. [13, 14, 15, 16, 19, 28, 31, 32, 39, 40, 41].

Research on sequence convergence in neutrosophic 2-normed linear spaces is still in its early stages, with limited progress made thus far. However, the studies conducted to date reveal a compelling similarity in the behavior of sequence convergence within these spaces. So, keeping potential applicability of the concept of sequence convergence in mind, within this specific framework, We have introduced the concept of $\mathcal{I}$-convergence of sequences, extending the existing ideas of statistical convergence. We have explored several key properties of this newly introduced concept. Furthermore, we have defined the concept of $\mathcal{I}$-Cauchy sequences and demonstrated their equivalence to $\mathcal{I}$-convergent sequences within neutrosophic 2-normed spaces. Also, we have analyzed the equivalency of $\mathcal{I}^*$-convergence with comparing another pair of sequences such that one of them is ordinary convergent and collection of those $n \in \mathbb{N}$ such that another sequence not coinciding with zero element belongs to an ideal with respect to neutrosophic 2-norm.

2 Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ indicate the set of natural numbers and the set of reals respectively. First we recall some basic definitions and notations.

Definition 1 ([20]) *A family $\mathcal{I}$ of subsets of a non empty set $\mathcal{X}$ is said to be an ideal in $\mathcal{X}$ if the following conditions hold:*

1. $\emptyset \in \mathcal{I}$;
2. $\mathcal{A}, \mathcal{B} \in \mathcal{I}$ *implies* $\mathcal{A} \cup \mathcal{B} \in \mathcal{I}$;
3. $\mathcal{A} \in \mathcal{I}$ *and* $\mathcal{B} \subset \mathcal{A}$ *implies* $\mathcal{B} \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non trivial if $\mathcal{X} \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$.

Definition 2 ([20]) *A non trivial ideal $\mathcal{I} \subset 2^{\mathcal{X}}$ is called admissible if $\{\{x\} : x \in \mathcal{X}\} \subset \mathcal{I}$.*

Definition 3 ([20]) *A non empty family $\mathcal{F}$ of subsets of a non empty set $\mathcal{X}$ is called a filter in $\mathcal{X}$ if the following properties hold:*

1. $\emptyset \notin \mathcal{F}$;
2. $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ *implies* $\mathcal{A} \cap \mathcal{B} \in \mathcal{F}$;
3. $\mathcal{A} \in \mathcal{F}$ *and* $\mathcal{A} \subset \mathcal{B}$ *implies* $\mathcal{B} \in \mathcal{F}$.

If $\mathcal{I} \subset 2^{\mathcal{X}}$ is a non trivial ideal then the class $\mathcal{F}(\mathcal{I}) = \{\mathcal{X} \setminus \mathcal{A} : \mathcal{A} \in \mathcal{I}\}$ is a filter on $\mathcal{X}$ which is called filter associated with the ideal $\mathcal{I}$ [20].

Definition 4 ([20]) *An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{\mathcal{B}_1, \mathcal{B}_2, \ldots\}$ such that the symmetric difference $\mathcal{A}_i \triangle \mathcal{B}_i$ is finite for each $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathcal{B}_i \in \mathcal{I}$.*

Definition 5 *Let $\mathcal{K} \subset \mathbb{N}$. Then the natural density of $\mathcal{K}$, denoted by $\delta(\mathcal{K})$, is defined as*

$$\delta(\mathcal{K}) = \lim_{n \to \infty} \frac{1}{n} |\{k \leq n : k \in \mathcal{K}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

Definition 6 ([35]) *A binary operation* $\square : [0,1] \times [0,1] \to [0,1]$ *is named to be a continuous* t*-norm if the following conditions hold.*

1. $\square$ *is associative and commutative;*
2. $\square$ *is continuous;*
3. $x \square 1 = x$ *for all* $x \in [0,1]$*;*
4. $x \square y \leq z \square w$ *whenever* $x \leq z$ *and* $y \leq w$ *for each* $x, y, z, w \in [0,1]$*.*

Definition 7 ([35]) *A binary operation* $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ *is named to be a continuous* t*-conorm if the following conditions are satisfied.*

1. $\diamondsuit$ *is associative and commutative;*
2. $\diamondsuit$ *is continuous;*
3. $x \diamondsuit 0 = x$ *for all* $x \in [0,1]$*;*
4. $x \diamondsuit y \leq z \diamondsuit w$ *whenever* $x \leq z$ *and* $y \leq w$ *for each* $x, y, z, w \in [0,1]$*.*

Example 1 ([21]) *The following are examples of* t*-norms:*

1. $x \square y = min\{x, y\}$*;*
2. $x \square y = x.y$*;*
3. $x \square y = max\{x + y - 1, 0\}$*. This* t*-norm is known as Lukasiewicz* t*-norm.*

Example 2 ([21]) *The following are examples of* t*-conorms:*

1. $x \diamondsuit y = max\{x, y\}$*;*
2. $x \diamondsuit y = x + y - x.y$*;*
3. $x \diamondsuit y = min\{x + y, 1\}$*. This is known as Lukasiewicz* t*-conorm.*

Lemma 1 ([37]) *If* $\square$ *is a continuous* t*-norm,* $\diamondsuit$ *is a continuous* t*-conorm,* $r_i \in (0,1)$ *and* $1 \leq i \leq 7$*, then the following statements hold:*

1. *If* $r_1 > r_2$*, there are* $r_3, r_4 \in (0,1)$ *such that* $r_1 \square r_3 \geq r_2$ *and* $r_1 \geq r_2 \diamondsuit r_4$*;*
2. *If* $r_5 \in (0,1)$*, there are* $r_6, r_7 \in (0,1)$ *such that* $r_6 \square r_6 \geq r_5$ *and* $r_5 \geq r_7 \diamondsuit r_7$*.*

Now we recall the notions of 2-normed space and neutrosophic 2-normed space.

Definition 8 ([11]) *Let* $\mathcal{Y}$ *be a real vector space of dimension* d*, where* $2 \leq d < \infty$*. A* 2*-norm on* $\mathcal{Y}$ *is a function* $\|.,.\| : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ *which satisfies the following conditions:*

1. $\|x, y\| = 0$ *if and only if* x *and* y *are linearly dependent in* $\mathcal{Y}$*;*
2. $\|x, y\| = \|y, x\|$ *for all* x, y *in* $\mathcal{Y}$*;*
3. $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ *for all* α *in* $\mathbb{R}$ *and for all* x, y *in* $\mathcal{Y}$*;*
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all* x, y, z *in* $\mathcal{Y}$*.*

Example 3 ([38]) *Let* $\mathcal{Y} = \mathbb{R}^2$*. Define* $\|\cdot,\cdot\|$ *on* $\mathbb{R}^2$ *by* $\|x, y\| = |x_1 y_2 - x_2 y_1|$*, where* $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$*. Then* $(\mathcal{Y}, \|\cdot,\cdot\|)$ *is a* 2*-normed space.*

Definition 9 ([29]) *Let $\mathcal{W}$ be a vector space and*

$$\mathcal{N}_2 = \{\langle (e,f), \mathcal{R}(e,f), \mathcal{S}(e,f), \mathcal{T}(e,f)\rangle : (e,f) \in \mathcal{W}\times\mathcal{W}\}$$

be a 2-normed space such that $\mathcal{N}_2 : \mathcal{W}\times\mathcal{W}\times\mathbb{R}^+ \to [0,1]$. Suppose $\square$ and $\diamondsuit$ are continuous t-norm and t-conorm respectively. Then the four tuple $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \diamondsuit)$ is named to be neutrosophic 2-normed space (N2-NS) if the following conditions hold for all $e, f, g \in \mathcal{X}$, $\eta, \zeta > 0$ and $\beta \neq 0$.

1. $0 \le \mathcal{R}(e,f;\eta) \le 1$, $0 \le \mathcal{S}(e,f;\eta) \le 1$ *and* $0 \le \mathcal{T}(e,f;\eta) \le 1$ *for every* $\eta > 0$;
2. $\mathcal{R}(e,f;\eta) + \mathcal{S}(e,f;\eta) + \mathcal{T}(e,f;\eta) \le 3$;
3. $\mathcal{R}(e,f;\eta) = 1$ *iff e and f are linearly dependent;*
4. $\mathcal{R}(\beta e,f;\eta) = \mathcal{R}(e,f;\frac{\eta}{|\beta|})$ *for each* $\beta \neq 0$;
5. $\mathcal{R}(e,f;\eta)\square\mathcal{R}(e,g;\zeta) \le \mathcal{R}(e,f+g;\eta+\zeta)$;
6. $\mathcal{R}(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a non-increasing function that runs continuously;*
7. $\lim_{\eta\to\infty} \mathcal{R}(e,f;\eta) = 1$;
8. $\mathcal{R}(e,f;\eta) = \mathcal{R}(f,e;\eta)$;
9. $\mathcal{S}(e,f;\eta) = 0$ *iff e and f are linearly dependent;*
10. $\mathcal{S}(\beta e,f;\eta) = \mathcal{S}(e,f;\frac{\eta}{|\beta|})$ *for each* $\beta \neq 0$;
11. $\mathcal{S}(e,f;\eta)\diamondsuit\mathcal{S}(e,g;\zeta) \ge \mathcal{S}(e,f+g;\eta+\zeta)$;
12. $\mathcal{S}(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a non-increasing function that runs continuously;*
13. $\lim_{\eta\to\infty} \mathcal{S}(e,f;\eta) = 0$;
14. $\mathcal{S}(e,f;\eta) = \mathcal{S}(f,e;\eta)$;
15. $\mathcal{T}(e,f;\eta) = 0$ *iff e and f are linearly dependent;*
16. $\mathcal{T}(\beta e,f;\eta) = \mathcal{T}(e,f;\frac{\eta}{|\beta|})$ *for each* $\beta \neq 0$;
17. $\mathcal{T}(e,f;\eta)\diamondsuit\mathcal{T}(e,g;\zeta) \ge \mathcal{T}(e,f+g;\eta+\zeta)$;
18. $\mathcal{T}(e,f;\cdot) : (0,\infty) \to [0,1]$ *is a non-increasing function that runs continuously;*
19. $\lim_{\eta\to\infty} \mathcal{T}(e,f;\eta) = 0$;
20. $\mathcal{T}(e,f;\eta) = \mathcal{T}(f,e;\eta)$;
21. *If* $\eta \le 0$, $\mathcal{R}(e,f;\eta) = 0$, $\mathcal{S}(e,f;\eta) = 1$, $\mathcal{T}(e,f;\eta) = 1$.

In this case, $\mathcal{N}_2 = (\mathcal{R}, \mathcal{S}, \mathcal{T})$ is called neutrosophic 2-norm on $\mathcal{W}$.

Definition 10 ([29]) *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \diamondsuit)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is named to be convergent if there exists a $n_0 \in \mathbb{N}$ and $l_0 \in \mathcal{W}$ such that*

$$\mathcal{R}(l_n - l_0, z;\eta) > 1-\varepsilon, \quad \mathcal{S}(l_n - l_0, z;\eta) < \varepsilon \quad \text{and} \quad \mathcal{T}(l_n - l_0, z;\eta) < \varepsilon$$

for all $n \ge n_0$ and $z \in \mathcal{X}$ which can be said that

$$\lim_{n\to\infty} \mathcal{R}(l_n - l_0, z;\eta) = 1, \quad \lim_{n\to\infty} \mathcal{S}(l_n - l_0, z;\eta) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{T}(l_n - l_0, z;\eta) = 0.$$

In this case we write $\mathcal{N}_2 - \lim_{n\to\infty} l_n = l_0$ or $l_n \xrightarrow{\mathcal{N}_2} l_0$ and l_0 is called $\mathcal{N}_2$-limit of $\{l_n\}_{n\in\mathbb{N}}$.

Definition 11 ([29]) *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_k\}_{k\in\mathbb{N}}$ is said to be statistically convergent to ξ if the natural density of the set*

$$\mathcal{A}(\varepsilon,\eta) = \{k \le n : \mathcal{R}(l_k - \xi, z; \eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_k - \xi, z;\eta) \ge \varepsilon \text{ and } \mathcal{T}(l_k - \xi, z; \eta) \ge \varepsilon\}$$

is zero for every $z \in \mathcal{X}$ i.e. $\delta(\mathcal{A}(\varepsilon,\eta)) = 0$.

Definition 12 ([29]) *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Choose $\varepsilon \in (0,1)$ and $\eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is named to be Cauchy if there exists $m_0 \in \mathbb{N}$ such that*

$$\mathcal{R}(l_n - l_m, z; \eta) > 1-\varepsilon, \quad \mathcal{S}(l_n - l_m, z;\eta) < \varepsilon \quad \text{and} \quad \mathcal{T}(l_n - l_m, z;\eta) < \varepsilon$$

for all $n, m \ge m_0$ and $z \in \mathcal{X}$.

Definition 13 ([29]) *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$, $\varepsilon > 0$ and $\eta > 0$. Then, $\{l_k\}_{k\in\mathbb{N}}$ is named to be statistical Cauchy if there exists $n_0 \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \mathcal{R}(l_k - l_{n_0}, z;\eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_k - l_{n_0}, z;\eta) \ge \varepsilon \text{ and } \mathcal{T}(l_k - l_{n_0}, z;\eta) \ge \varepsilon\}| = 0$$

for every $z \in \mathcal{X}$ or equivalently the natural density of the set

$$\mathcal{A}(\varepsilon,\eta) = \{k \le n : \mathcal{R}(l_k - l_{n_0}, z;\eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_k - l_{n_0}, z;\eta) \ge \varepsilon \text{ and } \mathcal{T}(l_k - l_{n_0}, z;\eta) \ge \varepsilon\}$$

is zero, i.e., $\delta(\mathcal{A}(\varepsilon,\eta)) = 0$.

3 Main Results

Throughout this section $\mathcal{I}$ stands for an admissible ideal of $\mathbb{N}$. $\mathcal{Y}$, $\mathcal{N}_2$ and $\mathcal{X}$ stand for 2-norm, neutrosophic 2-norm and neutrosophic 2-normed space respectively unless otherwise stated. First we introduce the notion of $\mathcal{I}$-convergence in neutrosophic 2-normed spaces.

Definition 14 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is named to be $\mathcal{I}$-convergent to $\xi \in \mathcal{W}$ if for each $\varepsilon \in (0,1)$ and $\eta > 0$ the set*

$$\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z;\eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_n - \xi, z;\eta) \ge \varepsilon \text{ and } \mathcal{T}(l_n - \xi, z;\eta) \ge \varepsilon\} \in \mathcal{I}$$

for every $z \in \mathcal{X}$. In this case, we write $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$ or $l_n \xrightarrow{\mathcal{I}(\mathcal{N}_2)} \xi$ and ξ is called $\mathcal{I}(\mathcal{N}_2)$-limit of $\{l_n\}_{n\in\mathbb{N}}$.

We now present a demonstration of this concept by the following example.

Example 4 *Let $\mathcal{W} = \mathbb{R}^2$ and $(\mathbb{R}^2, \|, \cdot, \|)$ be a 2-normed space with 2-norm defined as in Example 3. We take t-norm and t-conorm as $a\Box b = ab$ and $a\Diamond b = \min\{a+b, 1\}$ for $a, b \in [0,1]$. Let $\mathcal{I}$ be a class of subsets of $\mathbb{N}$ such that natural density of each subsets is zero. Then, $\mathcal{I}$ becomes a non trivial admissible ideal on $\mathbb{N}$. Now choose $\varepsilon \in (0,1)$ and $x, y \in \mathcal{W}, \eta > 0$ with $\eta > \|x, y\|$. Now we consider*

$$\mathcal{R}(x, y;\eta) = \frac{\eta}{\eta + \|x,y\|}, \quad \mathcal{S}(x,y;\eta) = \frac{\|x,y\|}{\eta + \|x,y\|}, \quad \mathcal{T}(x,y;\eta) = \frac{\|x,y\|}{\eta}.$$

Then $\mathcal{N}_2 = (\mathcal{R}, \mathcal{S}, \mathcal{T})$ is a neutrosophic 2-norm on $\mathcal{W}$ and the four tuple $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$ becomes a neutrosophic 2-normed space. Now we define a sequence $\{l_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ by

$$l_n = \begin{cases} (1,0) & \text{if } n = m^2 (m \in \mathbb{N}), \\ (0,0) & \text{otherwise.} \end{cases}$$

Now for $z \in \mathcal{X}$ and $\xi = (0,0)$, we have

$$\begin{aligned}\mathcal{A}(\varepsilon,\eta) &= \{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) \leq 1-\varepsilon \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \varepsilon, \mathcal{T}(l_n - \xi, z; \eta) \geq \varepsilon\} \\ &= \{n \in \mathbb{N} : n = m^2\}.\end{aligned}$$

Since $\delta(\{n \in \mathbb{N} : n = m^2\}) = 0$, $\mathcal{A}(\varepsilon,\eta) \in \mathcal{I}$.

From Definition 14, we can easily prove the following lemma. So, we omit details.

Theorem 1 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Then, for each $\varepsilon \in (0,1), \eta > 0$ and for every $z \in \mathcal{X}$ the following statements are equivalent:*

1. $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$;
2. $\{n \in \mathbb{N} : \mathcal{R}(l_n-\xi, z; \eta) \leq 1-\varepsilon\} \in \mathcal{I}$, $\{n \in \mathbb{N} : \mathcal{S}(l_n-\xi, z; \eta) \geq \varepsilon\} \in \mathcal{I}$ *and* $\{n \in \mathbb{N} : \mathcal{T}(l_n-\xi, z; \eta) \geq \varepsilon\} \in \mathcal{I}$;
3. $\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) > 1-\varepsilon$ *and* $\mathcal{S}(l_n - \xi, z; \eta) < \varepsilon, \mathcal{T}(l_n - \xi, z; \eta) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$;
4. $\{n \in \mathbb{N} : \mathcal{R}(l_n-\xi, z; \eta) > 1-\varepsilon\} \in \mathcal{F}(\mathcal{I})$, $\{n \in \mathbb{N} : \mathcal{S}(l_n-\xi, z; \eta) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$ *and* $\{n \in \mathbb{N} : \mathcal{T}(l_n-\xi, z; \eta) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$;
5. $\mathcal{I}(\mathcal{N}_2)-\lim_{n\to\infty} \mathcal{R}(l_n-\xi, z; \eta) = 1$, $\mathcal{I}(\mathcal{N}_2)-\lim_{n\to\infty} \mathcal{S}(l_n-\xi, z; \eta) = 0$ *and* $\mathcal{I}(\mathcal{N}_2)-\lim_{n\to\infty} \mathcal{T}(l_n-\xi, z; \eta) = 0$.

Limit of a sequence-convergence is very important tool in mathematical analysis. So, keeping this fact in mind, we exhibit the uniqueness of $\mathcal{I}$-limit of a sequence in relation to $\mathcal{N}_2$.

Theorem 2 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. If $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$, then $\mathcal{I}(\mathcal{N}_2)$-limit of $\{l_n\}_{n\in\mathbb{N}}$ is unique.*

Proof. If possible, let $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$ and $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \alpha$ with $\xi \neq \alpha$. For a given $\varepsilon \in (0,1)$, choose $\sigma \in (0,1)$ such that $(1-\sigma)\Box(1-\sigma) > 1-\varepsilon$ and $\sigma\Diamond\sigma < \varepsilon$. Then, for every $\eta > 0$ and $z \in \mathcal{X}$, the sets

$$\left\{n \in \mathbb{N} : \mathcal{R}\left(l_n - \xi, z; \eta\right) \leq 1-\sigma \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \sigma \text{ and } \mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) \geq \sigma\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} : \mathcal{R}\left(l_n - \alpha, z; \eta\right) \leq 1-\sigma \text{ or } \mathcal{S}(l_n - \alpha, z; \eta) \geq \sigma \text{ and } \mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) \geq \sigma\right\} \in \mathcal{I}.$$

Consider the sets

$$\begin{aligned}
\mathcal{B}_{\mathcal{R}1}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \frac{\eta}{2}) \leq 1-\sigma\};\\
\mathcal{B}_{\mathcal{R}2}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{R}(l_n - \alpha, z; \frac{\eta}{2}) \leq 1-\sigma\};\\
\mathcal{B}_{\mathcal{S}1}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{S}(l_n - \xi, z; \frac{\eta}{2}) \geq \sigma\};\\
\mathcal{B}_{\mathcal{S}2}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{S}(l_n - \alpha, z; \frac{\eta}{2}) \geq \sigma\};\\
\mathcal{B}_{\mathcal{T}1}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) \geq \sigma\};\\
\mathcal{B}_{\mathcal{T}2}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{T}(l_n - \alpha, z; \frac{\eta}{2}) \geq \sigma\}.
\end{aligned}$$

Then, by Theorem 1, we have all of $\mathcal{B}_{\mathcal{R}1}(\sigma,\eta)$, $\mathcal{B}_{\mathcal{R}2}(\sigma,\eta)$, $\mathcal{B}_{\mathcal{S}1}(\sigma,\eta)$, $\mathcal{B}_{\mathcal{S}2}(\sigma,\eta)$, $\mathcal{B}_{\mathcal{T}1}(\sigma,\eta)$, $\mathcal{B}_{\mathcal{T}2}(\sigma,\eta)$ belong to $\mathcal{I}$. Let

$$\mathcal{B}_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) = [\mathcal{B}_{\mathcal{R}1}(\sigma,\eta) \cup \mathcal{B}_{\mathcal{R}2}(\sigma,\eta)] \cap [\mathcal{B}_{\mathcal{S}1}(\sigma,\eta) \cup \mathcal{B}_{\mathcal{S}2}(\sigma,\eta)] \cap [\mathcal{B}_{\mathcal{T}1}(\sigma,\eta) \cup \mathcal{B}_{\mathcal{T}2}(\sigma,\eta)].$$

Then, $\mathcal{B}_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) \in \mathcal{I}$. Hence, $\mathcal{B}^c_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) \in \mathcal{F}(\mathcal{I})$. So, let $k \in \mathcal{B}^c_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta)$. Then, there arise three possible cases:

1. $k \in \mathcal{B}^c_{\mathcal{R}1}(\sigma,\eta) \cap \mathcal{B}^c_{\mathcal{R}2}(\sigma,\eta)$.
2. $k \in \mathcal{B}^c_{\mathcal{S}1}(\sigma,\eta) \cap \mathcal{B}^c_{\mathcal{S}2}(\sigma,\eta)$.
3. $k \in \mathcal{B}^c_{\mathcal{T}1}(\sigma,\eta) \cap \mathcal{B}^c_{\mathcal{T}2}(\sigma,\eta)$.

If $k \in \mathcal{B}^c_{\mathcal{R}1}(\sigma,\eta) \cap \mathcal{B}^c_{\mathcal{R}2}(\sigma,\eta)$, we have

$$\mathcal{R}(\xi-\alpha, z;\eta) \geq \mathcal{R}(l_k-\xi, z; \frac{\eta}{2}) \square \mathcal{R}(l_k-\alpha, z; \frac{\eta}{2}) > (1-\sigma)\square(1-\sigma) > 1-\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\mathcal{R}(\xi-\alpha,z;\eta) = 1$ which yields $\xi-\alpha = 0$ i.e. $\xi = \alpha$. If we take $k \in \mathcal{B}^c_{\mathcal{S}1}(\sigma,\eta) \cap \mathcal{B}^c_{\mathcal{S}2}(\sigma,\eta)$, we get

$$\mathcal{S}(\xi-\alpha, z;\eta) \leq \mathcal{S}(l_k-\xi, z; \frac{\eta}{2}) \diamondsuit \mathcal{S}(l_k-\alpha, z; \frac{\eta}{2}) < \sigma \diamondsuit \sigma < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\mathcal{S}(\xi-\alpha,z;\eta) = 0$ which yields $\xi-\alpha = 0$ i.e. $\xi = \alpha$. For $k \in \mathcal{B}^c_{\mathcal{T}1}(\sigma,\eta) \cap \mathcal{B}^c_{\mathcal{T}2}(\sigma,\eta)$,

$$\mathcal{T}(\xi-\alpha, z;\eta) \leq \mathcal{T}(l_k-\xi, z; \frac{\eta}{2}) \diamondsuit \mathcal{T}(l_k-\alpha, z; \frac{\eta}{2}) < \sigma \diamondsuit \sigma < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\mathcal{S}(\xi-\alpha,z;\eta) = 0$ which gives $\xi-\alpha = 0$ i.e. $\xi = \alpha$. This contradicts the fact $\xi \neq \alpha$. Therefore, $\mathcal{I}(\mathcal{N}_2)$-limit of $\{l_n\}_{n\in\mathbb{N}}$ is unique. ■

Now, we find out the relationship between ordinary convergence and $\mathcal{I}$-convergence with regard to $\mathcal{N}_2$.

Theorem 3 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \diamondsuit)$. If $l_n \xrightarrow{\mathcal{N}_2} l_0$, then $l_n \xrightarrow{\mathcal{I}(\mathcal{N}_2)} l_0$.*

Proof. Since $l_n \xrightarrow{\mathcal{N}_2} l_0$, for every $\varepsilon \in (0,1)$ and $\eta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{R}(l_n-l_0, z;\eta) > 1-\varepsilon$, $\mathcal{S}(l_n-l_0,z;\eta) < \varepsilon$ and $\mathcal{T}(l_n-l_0,z;\eta) < \varepsilon$ for all $n \geq n_0$ and $z \in \mathcal{X}$. This shows that

$$\{n \in \mathbb{N} : \mathcal{R}(l_n-l_0,z;\eta) \leq 1-\varepsilon \text{ or } \mathcal{S}(l_n-l_0,z;\eta) \geq \varepsilon, \mathcal{T}(l_n-l_0,z;\eta) \geq \varepsilon\} \subset \{1,2,\ldots,n_0-1\}.$$

Since $\mathcal{I}$ is an admissible ideal and $\{1,2,\ldots,n_0-1\} \in \mathcal{I}$,

$$\{n \in \mathbb{N} : \mathcal{R}(l_n-l_0,z;\eta) \leq 1-\varepsilon \text{ and } \mathcal{S}(l_n-l_0,z;\eta) \geq \varepsilon, \mathcal{T}(l_n-l_0,z;\eta) \geq \varepsilon\} \in \mathcal{I}.$$

Therefore, $l_n \xrightarrow{\mathcal{I}(\mathcal{N}_2)} l_0$. ■

Remark 1 *The converse of Theorem 3 is not true. If we take the same neutrosophic 2-normed space $\mathcal{X}$ defined as in Example 4 and a sequence $\{l_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ given by*

$$l_n = \begin{cases} (n,0) & \text{if } n = k^2 (k \in \mathbb{N}), \\ (0,0) & \text{otherwise.} \end{cases}$$

Then, $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}(\mathcal{N}_2)$-convergent to $(0,0) \in \mathcal{X}$, but it is not ordinary convergent with respect to $\mathcal{N}_2$ on $\mathcal{W}$.

We shall explore that $\mathcal{I}$-convergent sequences with respect to $\mathcal{N}_2$ obeys the algebras of $\mathcal{I}$-limits in the following theorem.

Theorem 4 *Let $\mathcal{W}$ be a real vector space. Let $\{l_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$ be two sequences in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \diamondsuit)$. Then, we have*

1. *If $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = l_0$, $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} p_n = p_0$ and $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n + p_n = l_0 + p_0$.*
2. *If $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = l_0$, $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} \beta l_n = \beta l_0$ for each $\beta \neq 0$.*

Proof. For a given $\varepsilon \in (0,1)$, choose $\sigma \in (0,1)$ such that $(1-\sigma)\Box(1-\sigma) > 1-\varepsilon$ and $\sigma\Diamond\sigma < \varepsilon$. Since $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = l_0$ and $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} p_n = p_0$, then for every $\eta > 0$ and $z \in \mathcal{X}$, the sets

$$\Big\{n \in \mathbb{N} : \mathcal{R}(l_n - l_0, z; \frac{\eta}{2}) \le 1-\sigma \text{ or } \mathcal{S}(l_n - l_0, z; \frac{\eta}{2}) \ge \sigma \text{ and } \mathcal{T}(l_n - l_0, z; \frac{\eta}{2}) \ge \sigma\Big\} \in \mathcal{I}$$

and

$$\Big\{n \in \mathbb{N} : \mathcal{R}(p_n - p_0, z; \frac{\eta}{2}) \le 1-\sigma \text{ or } \mathcal{S}(p_n - p_0, z; \frac{\eta}{2}) \ge \sigma \text{ and } \mathcal{T}(p_n - p_0, z; \frac{\eta}{2}) \ge \sigma\Big\} \in \mathcal{I}.$$

Let

$$\mathcal{A}_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) = [\mathcal{A}_{\mathcal{R}1}(\sigma,\eta) \cup \mathcal{A}_{\mathcal{R}2}(\sigma,\eta)] \cap [\mathcal{A}_{\mathcal{S}1}(\sigma,\eta) \cup \mathcal{A}_{\mathcal{S}2}(\sigma,\eta)] \cap [\mathcal{A}_{\mathcal{T}1}(\sigma,\eta) \cup \mathcal{A}_{\mathcal{T}2}(\sigma,\eta)]$$

where

$$\begin{aligned}
\mathcal{A}_{\mathcal{R}1}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{R}(l_n - l_0, z; \frac{\eta}{2}) \le 1-\sigma\};\\
\mathcal{A}_{\mathcal{R}2}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{R}(p_n - p_0, z; \frac{\eta}{2}) \le 1-\sigma\};\\
\mathcal{A}_{\mathcal{S}1}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{S}(l_n - l_0, z; \frac{\eta}{2}) \ge \sigma\};\\
\mathcal{A}_{\mathcal{S}2}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{S}(p_n - p_0, z; \frac{\eta}{2}) \ge \sigma\};\\
\mathcal{A}_{\mathcal{T}1}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{T}(l_n - l_0, z; \frac{\eta}{2}) \ge \sigma\};\\
\mathcal{A}_{\mathcal{T}2}(\sigma,\eta) &= \{n \in \mathbb{N} : \mathcal{T}(p_n - p_0, z; \frac{\eta}{2}) \ge \sigma\}.
\end{aligned}$$

By Theorem 1, we get $\mathcal{A}_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) \in \mathcal{I}$. Then, $\mathcal{A}^c_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) \in \mathcal{F}(\mathcal{I})$ and hence $\mathcal{A}^c_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta) \ne \emptyset$. So, let $n \in \mathcal{A}^c_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta)$. Then, we have

$$\begin{aligned}
\mathcal{R}(l_n + p_n - (l_0 + p_0), z; \eta) &\ge \mathcal{R}(l_n - l_0, z; \frac{\eta}{2})\Box\mathcal{R}(p_n - p_0, z; \frac{\eta}{2})\\
&> (1-\sigma)\Box(1-\sigma) > 1-\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(l_n + p_n - (l_0 + p_0), z; \eta) &\le \mathcal{S}(l_n - l_0, z; \frac{\eta}{2})\Diamond\mathcal{S}(p_n - p_0, z; \frac{\eta}{2})\\
&< \sigma\Diamond\sigma < \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}(l_n + p_n - (l_0 + p_0), z; \eta) &\le \mathcal{T}(l_n - l_0, z; \frac{\eta}{2})\Diamond\mathcal{T}(p_n - p_0, z; \frac{\eta}{2})\\
&< \sigma\Diamond\sigma < \varepsilon.
\end{aligned}$$

This shows that

$$\{n \in \mathbb{N} : \mathcal{R}(l_n + p_n - (l_0 + p_0), z; \eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_n + p_n - (l_0 + p_0), z; \eta) \ge \varepsilon$$

and

$$\mathcal{T}(l_n + p_n - (l_0 + p_0), z; \eta) \ge \varepsilon\} \subset \mathcal{A}_{\mathcal{R},\mathcal{S},\mathcal{T}}(\sigma,\eta).$$

Consequently

$$\begin{aligned}
&\{n \in \mathbb{N} : \mathcal{R}(l_n + p_n - (l_0 + p_0), z; \eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_n + p_n - (l_0 + p_0), z; \eta) \ge \varepsilon\\
&\text{ and } \mathcal{T}(l_n + p_n - (l_0 + p_0), z; \eta) \ge \varepsilon\} \in \mathcal{I}.
\end{aligned}$$

Therefore, $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n + p_n = l_0 + p_0$.

Since $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = l_0$, then for every $\varepsilon \in (0,1), \eta > 0$ and $z \in \mathcal{X}$ the set

$$\left\{ n \in \mathbb{N} : \mathcal{R}(l_n - l_0, z; \frac{\eta}{|\beta|}) \leq 1-\varepsilon \text{ or } \mathcal{S}(l_n - l_0, z; \frac{\eta}{|\beta|}) \geq \varepsilon \text{ and } \mathcal{T}(l_n - l_0, z; \frac{\eta}{|\beta|}) \geq \varepsilon \right\} \in \mathcal{I}$$

i.e.

$$\{n \in \mathbb{N} : \mathcal{R}(\beta l_n - \beta l_0, z; \eta) \leq 1-\varepsilon \text{ or } \mathcal{S}(\beta l_n - \beta l_0, z; \eta) \geq \varepsilon \text{ and } \mathcal{T}(\beta l_n - \beta l_0, z; \eta) \geq \varepsilon\} \in \mathcal{I}.$$

Therefore, $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} \beta l_n = \beta l_0$. This completes the proof. ■

Now, we proceed with the notion of $\mathcal{I}^*$-convergence in a neutrosophic 2-normed space $\mathcal{X}$.

Definition 15 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is named to be $\mathcal{I}^*$-convergent to $\xi \in \mathcal{X}$ with regards to $\mathcal{N}_2$ if there exists a set $\mathcal{K} = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\mathcal{K} \in \mathcal{F}(\mathcal{I})$ and $\mathcal{N}_2 - \lim_{k\to\infty} l_{m_k} = \xi$. In this case, we write $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$ or $l_n \xrightarrow{\mathcal{I}^*(\mathcal{N}_2)} \xi$ and ξ is called $\mathcal{I}^*(\mathcal{N}_2)$-limit of $\{l_n\}_{n\in\mathbb{N}}$.*

We establish the intricate relationship between the two innovative concepts of $\mathcal{I}$ and $\mathcal{I}^*$-convergence using the condition (AP) in neutrosophic 2-normed spaces.

Theorem 5 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. If $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$, then $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$.*

Proof. Since $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$, there exists a set $\mathcal{K} = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\mathcal{K} \in \mathcal{F}(\mathcal{I})$ and $\mathcal{N}_2 - \lim_{k\to\infty} l_{m_k} = \xi$ i.e., for every $\varepsilon \in (0,1), \eta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{R}(l_{m_k} - \xi, z; \eta) > 1-\varepsilon, \mathcal{S}(l_{m_k} - \xi, z; \eta) < \varepsilon$ and $\mathcal{T}(l_{m_k} - \xi, z; \eta) < \varepsilon$ for all $k \geq n_0$ and $z \in \mathcal{X}$. So,

$$\begin{aligned}&\{m_k \in \mathbb{N} : \mathcal{R}(l_{m_k} - \xi, z; \eta) \leq 1-\varepsilon \text{ or } \mathcal{S}(l_{m_k} - \xi, z; \eta) \geq \varepsilon \text{ and } \mathcal{T}(l_{m_k} - \xi, z; \eta) \geq \varepsilon\} \\ &\subset \{m_1, m_2, \ldots, m_{n_0-1}\}.\end{aligned}$$

Now, let $\mathcal{H} = \mathbb{N} \setminus \mathcal{K}$. Then, we have

$$\begin{aligned}&\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) \leq 1-\varepsilon \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \varepsilon \text{ and } \mathcal{T}(l_n - \xi, z; \eta) \geq \varepsilon\} \\ &\subset \mathcal{H} \cup \{m_1, m_2, \ldots, m_{n_0-1}\}.\end{aligned}$$

Since $\mathcal{I}$ is an admissible ideal, therefore

$$\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) \leq 1-\varepsilon \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \varepsilon \text{ and } \mathcal{T}(l_n - \xi, z; \eta) \geq \varepsilon\} \in \mathcal{I}.$$

This shows that $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$. ■

Remark 2 *In general, the converse of Theorem 5 is not true which can be illustrated by the following example.*

Example 5 *We consider the neutrosophic 2-normed space defined as in Example 4. Let $\mathbb{N} = \bigcup_i \mathcal{D}_i$ be a decomposition of $\mathbb{N}$ such that for any $p \in \mathbb{N}$ each $\mathcal{D}_i$ contains infinitely many i's where $i \geq p$ and $\mathcal{D}_i \cap \mathcal{D}_p = \emptyset$ whenever $i \neq p$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ which intersects only a finite number of $\mathcal{D}_i'$s. Then $\mathcal{I}$ becomes a non trivial admissible ideal of $\mathbb{N}$. Now we define a sequence $\{l_k\}_{k\in\mathbb{N}} \in \mathcal{X}$ by $l_k = (\frac{1}{k}, 0)$ if $k \in \mathcal{D}_k$. Now for $z = (z_1, z_2)$ and $\xi = (0,0)$ we have*

$$\mathcal{R}(l_k - \xi, z; \eta) = \frac{\eta}{\eta + \|l_k, z\|} = \frac{\eta}{\eta + |z_2|\frac{1}{k}} \longrightarrow 1,$$

$$\mathcal{S}(l_k - \xi, z; \eta) = \frac{\|l_k, z\|}{\|l_k, z\| + \eta} = \frac{|z_2|\frac{1}{k}}{\eta + |z_2|\frac{1}{k}} \longrightarrow 0,$$

$$\mathcal{T}(l_k - \xi, z; \eta) = \frac{\|l_k, z\|}{\eta} = \frac{|z_2|\frac{1}{k}}{\eta} \longrightarrow 0 \text{ as } k \to \infty.$$

Since $\mathcal{I}$ is an admissible ideal, therefore $\mathcal{I}(\mathcal{N}_2) - \lim_{k\to\infty} l_k = \xi$.

Now, if possible, let $\mathcal{I}^*(\mathcal{N}_2) - \lim_{k\to\infty} l_k = \xi$. Then, there exists a set $\mathcal{K} = \{k_1 < k_2 < \cdots < k_m < \cdots\} \subset \mathbb{N}$ such that $\mathcal{K} \in \mathcal{F}(\mathcal{I})$ and $\mathcal{N}_2 - \lim_{m\to\infty} l_{k_m} = \xi$. Since $\mathcal{K} \in \mathcal{F}(\mathcal{I})$, there is $\mathcal{M} \in \mathcal{I}$ such that $\mathbb{N} \setminus \mathcal{K} = \mathcal{M}$. Now by the construction of $\mathcal{I}$, there is $j \in \mathbb{N}$ such that $\mathcal{M} \subset \bigcup_{i=1}^{j} \mathcal{D}_i$. But then $\mathcal{D}_{j+1} \subset \mathcal{K}$ and therefore $l_{k_m} = (\frac{1}{j+1}, 0)$ for infinitely many $k_m \in \mathcal{K}$ which contradicts $\mathcal{N}_2 - \lim_{m\to\infty} l_{k_m} = \xi$. Therefore, $\{l_k\}_{k\in\mathbb{N}}$ is not $\mathcal{I}^*(\mathcal{N}_2)$-convergent to $\xi \in \mathcal{X}$.

Generally, a question arises that uder which condition does the converse of the Theorem 5 hold good? The following theorem establishes this important fact.

Theorem 6 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \Diamond)$. If $\mathcal{I}$ satisfies the condition (AP) and $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$, then $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$.*

Proof. Suppose that $\mathcal{I}$ satisfies the condition (AP) and $\mathcal{I}(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$. Then, for each $\varepsilon \in (0,1)$ and $\eta > 0$, the set

$$\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) \leq 1 - \varepsilon \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \varepsilon \text{ and } \mathcal{T}(l_n - \xi, z; \eta) \geq \varepsilon\} \in \mathcal{I}$$

for every $z \in \mathcal{X}$. Now, we define

$$\mathcal{A}_j = \bigg\{n \in \mathbb{N} : 1 - \frac{1}{j} \leq \mathcal{R}(l_n - \xi, z; \eta) < 1 - \frac{1}{1+j} \text{ or } \frac{1}{j+1} < \mathcal{S}(l_n - \xi, z; \eta) \leq \frac{1}{j} \text{ and } \frac{1}{j+1} < \mathcal{T}(l_n - \xi, z; \eta) \leq \frac{1}{j}\bigg\}.$$

Clearly $\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ is countable and pairwise disjoint and each $\mathcal{A}_j \in \mathcal{I}$. Since $\mathcal{I}$ satisfies the condition (AP), there exists a countable family $\{\mathcal{B}_1, \mathcal{B}_2, \ldots\}$ of subsets of $\mathbb{N}$ belonging to $\mathcal{I}$ and $\mathcal{A}_i \triangle \mathcal{B}_i$ is finite for each i and $\mathcal{H} = \bigcup_i \mathcal{B}_i \in \mathcal{I}$. Now from the associated filter of $\mathcal{I}$ there is $\mathcal{M} \in \mathcal{F}(\mathcal{I})$ such that $\mathcal{M} = \mathbb{N} \setminus \mathcal{H}$. It is sufficient to prove the theorem that the subsequence $\{l_n\}_{n\in\mathcal{M}}$ is convergent to ξ with respect to $\mathcal{N}_2$. Let $\vartheta \in (0,1)$, $\eta > 0$ and $z \in \mathcal{X}$. We choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \vartheta$. Then, clearly

$$\begin{aligned}
&\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) \leq 1 - \vartheta \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \vartheta \text{ and } \mathcal{T}(l_n - \xi, z; \eta) \geq \vartheta\} \\
&\subset \bigg\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \eta) \leq 1 - \frac{1}{n_0} \text{ or } \mathcal{S}(l_n - \xi, z; \eta) \geq \frac{1}{n_0} \text{ and } \mathcal{T}(l_n - \xi, z; \eta) \geq \frac{1}{n_0}\bigg\} \\
&\subset \bigcup_{i=1}^{n_0+1} \mathcal{A}_i.
\end{aligned}$$

Since $\mathcal{A}_i \triangle \mathcal{B}_i$, $i = 1, 2, \ldots, n_0 + 1$ are finite, there is $p_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{n_0+1} \mathcal{B}_i\right) \cap \{n \in \mathbb{N} : n \geq p_0\} = \left(\bigcup_{i=1}^{n_0+1} \mathcal{A}_i\right) \cap \{n \in \mathbb{N} : n \geq p_0\}. \tag{1}$$

Now, if $n \geq p_0$ and $n \in \mathcal{M}$, $n \notin \bigcup_{i=1}^{n_0+1} \mathcal{B}_i$. So, by equation (1), $n \notin \bigcup_{i=1}^{n_0+1} \mathcal{A}_i$. Therefore, for every $n \geq p_0$ and $n \in \mathcal{M}$ we get $\mathcal{R}(l_n - \xi, z; \eta) > 1 - \vartheta$, $\mathcal{S}(l_n - \xi, z; \eta) < \vartheta$ and $\mathcal{T}(l_n - \xi, z; \eta) < \vartheta$ for every $z \in \mathcal{X}$. Since $\vartheta \in (0,1)$ is arbitrary, we have $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$. This completes the proof. ■

Now, we discuss on the equivalency of $\mathcal{I}^*$-convergence with comparing another pair of sequences such that one of them is $\mathcal{N}_2$-convergent and collection of those $n \in \mathbb{N}$ such that another sequence not coinciding with zero element belongs to an ideal with regard to $\mathcal{N}_2$.

Theorem 7 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Then, the following statements are equivalent:*

1. $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$.

2. *There exist two sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that $l_n = t_n + w_n$, $t_n \xrightarrow{\mathcal{N}_2} \xi$ and $\{n \in \mathbb{N} : w_n \neq \theta\} \in \mathcal{I}$, θ being the zero element in $\mathcal{W}$.*

Proof. First suppose that (1) holds. Then there exists a set $\mathcal{K} = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\mathcal{K} \in \mathcal{F}(\mathcal{I})$ and $\mathcal{N}_2 - \lim_{k\to\infty} l_{m_k} = \xi$ i.e.,

$$\mathcal{R}(l_n - \xi, z; \eta) > 1 - \varepsilon,\ \mathcal{S}(l_n - \xi, z; \eta) < \varepsilon \text{ and } \mathcal{T}(l_n - \xi, z; \eta) < \varepsilon \tag{2}$$

whenever $n \in \mathcal{K}$. Now we define the sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ as follows:

$$t_n = \begin{cases} l_n, & \text{if } n \in \mathcal{K}, \\ \xi, & \text{if } n \in \mathcal{K}^c, \end{cases} \tag{3}$$

and $w_n = l_n - t_n \forall n \in \mathbb{N}$. Let $\varepsilon \in (0,1)$ and $\eta > 0$. Then for each $n \in \mathcal{K}^c$ and $z \in \mathcal{X}$ we have

$$\mathcal{R}(t_n - \xi, z; \eta) = \mathcal{R}(\theta, z; \eta) = 1 > 1 - \varepsilon,$$

$$\mathcal{S}(t_n - \xi, z; \eta) = \mathcal{S}(\theta, z; \eta) = 0 < \varepsilon$$

and

$$\mathcal{T}(t_n - \xi, z; \eta) = \mathcal{T}(\theta, z; \eta) = 0 < \varepsilon.$$

Therefore, using (3) and (2) we get $t_n \xrightarrow{\mathcal{N}_2} \xi$. From (3) we have

$$\begin{aligned} & \{n \in \mathbb{N} : t_n \neq l_n\} \subset \mathcal{K}^c \\ \implies & \{n \in \mathbb{N} : l_n - t_n \neq \theta\} \subset \mathcal{K}^c \\ \implies & \{n \in \mathbb{N} : w_n \neq \theta\} \subset \mathcal{K}^c. \end{aligned}$$

Therefore $\{n \in \mathbb{N} : w_n \neq \theta\} \in \mathcal{I}$.

Assume that the condition (2) holds. Then, clearly the set $\{n \in \mathbb{N} : w_n = \theta\} \in \mathcal{F}(\mathcal{I})$ must be infinite. Let $\mathcal{K} = \{n \in \mathbb{N} : w_n = \theta\}$. Since $t_n \xrightarrow{\mathcal{N}_2} \xi$ and $t_n = l_n$ for $n \in \mathcal{K}$, $\mathcal{N}_2 - \lim_{n\in\mathcal{K}, n\to\infty} l_n = \xi$. Hence $\mathcal{I}^*(\mathcal{N}_2) - \lim_{n\to\infty} l_n = \xi$. This completes the proof. ■

Now, we define the notion of $\mathcal{I}$-Cauchy sequence and nurture it with convergence in ideal context within neutrosophic 2-norm.

Definition 16 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$ and $\varepsilon \in (0,1), \eta > 0$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is named to be $\mathcal{I}$-Cauchy with regard to $\mathcal{N}_2$ if there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that the set*

$$\mathcal{A}(\varepsilon, \eta) = \{n \in \mathbb{N} : \mathcal{R}(l_n - l_{n_0}, z; \eta) \leq 1 - \varepsilon \text{ or } \mathcal{S}(l_n - l_{n_0}, z; \eta) \geq \varepsilon \text{ and } \mathcal{T}(l_n - l_{n_0}, z; \eta) \geq \varepsilon\} \in \mathcal{I}$$

for every $z \in \mathcal{X}$.

Definition 17 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. Then, $\{l_n\}_{n\in\mathbb{N}}$ is named to be $\mathcal{I}^*$-Cauchy with regard to $\mathcal{N}_2$ if there exists a set $\mathcal{K} = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\mathcal{K} \in \mathcal{F}(\mathcal{I})$ and the subsequence $\{l_{m_k}\}$ is an ordinary Cauchy sequence with regard to $\mathcal{N}_2$.*

Theorem 8 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. If $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-Cauchy with regard to $\mathcal{N}_2$ then $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regard to $\mathcal{N}_2$.*

Theorem 9 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$. If $\mathcal{I}$ satisfies the condition (AP) and $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regards to $\mathcal{N}_2$ then $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-Cauchy with regard to $\mathcal{N}_2$.*

Theorem 8 and Theorem 9 can be proved in the line of Theorem 5 and Theorem 6 respectively. So, we omit details.

Now, we proceed with the investigations of relation ship between $\mathcal{I}$-Cauchy sequence and $\mathcal{I}$-convergence with respect to the neutrosophic 2-norm $\mathcal{N}_2$.

Theorem 10 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \diamondsuit)$. If $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent with regards to $\mathcal{N}_2$, then $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regards to $\mathcal{N}_2$.*

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to $\xi \in \mathcal{X}$ and $\varepsilon \in (0,1)$ be given. Choose $\sigma \in (0,1)$ such that $(1-\sigma)\square(1-\sigma) > 1-\varepsilon$ and $\sigma\diamondsuit\sigma < \varepsilon$. Then, for every $\eta > 0$ and $z \in \mathcal{X}$, the set

$$\mathcal{A}(\sigma,\eta) = \{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \frac{\eta}{2}) \le 1-\sigma \text{ or } \mathcal{S}(l_n - \xi, z; \frac{\eta}{2}) \ge \sigma \text{ and } \mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) \ge \sigma\} \in \mathcal{I}.$$

Then $\mathcal{A}^c(\sigma,\eta) \in \mathcal{F}(\mathcal{I})$. So $\mathcal{A}^c(\sigma,\eta) \neq \emptyset$. Then there is $n_0 \in \mathcal{A}^c(\sigma,\eta)$. Now, we define

$$\mathcal{B}(\varepsilon,\eta) = \{n \in \mathbb{N} : \mathcal{R}(l_n - l_{n_0}, z; \eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_n - l_{n_0}, z; \eta) \ge \varepsilon \text{ and } \mathcal{T}(l_n - l_{n_0}, z; \eta) \ge \varepsilon\}.$$

It is sufficient to prove the theorem that $\mathcal{B}(\varepsilon,\eta) \subset \mathcal{A}(\sigma,\eta)$. Let $k \in \mathcal{B}(\varepsilon,\eta)$. Then we get

$$\mathcal{R}(l_k - l_{n_0}, z; \eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_k - l_{n_0}, z; \eta) \ge \varepsilon \text{ and } \mathcal{T}(l_k - l_{n_0}, z; \eta) \ge \varepsilon.$$

<u>Case-i:</u> We consider $\mathcal{R}(l_k - l_{n_0}, z; \eta) \le 1-\varepsilon$. We show $\mathcal{R}(l_k - \xi, z; \frac{\eta}{2}) \le 1-\sigma$. If possible, let $\mathcal{R}(l_k - \xi, z; \frac{\eta}{2}) > 1-\sigma$. Then, we have

$$1-\varepsilon \ge \mathcal{R}(l_k - l_{n_0}, z; \eta) \ge \mathcal{R}(l_k - \xi, z; \frac{\eta}{2})\square\mathcal{R}(l_{n_0} - \xi, z; \frac{\eta}{2}) > (1-\sigma)\square(1-\sigma) > 1-\varepsilon,$$

which is not possible. So we have $\mathcal{R}(l_k - \xi, z; \frac{\eta}{2}) \le 1-\sigma$. Hence $k \in \mathcal{A}(\sigma,\eta)$, i.e. $\mathcal{B}(\varepsilon,\eta) \subset \mathcal{A}(\sigma,\eta)$.

<u>Case-ii:</u> We consider $\mathcal{S}(l_k - l_{n_0}, z; \eta) \ge \varepsilon$. We show $\mathcal{S}(l_k - \xi, z; \frac{\eta}{2}) \ge \sigma$. If possible, let $\mathcal{S}(l_k - \xi, z; \frac{\eta}{2}) < \sigma$. Then, we have

$$\varepsilon \le \mathcal{S}(l_k - l_{n_0}, z; \eta) \le \mathcal{S}(l_k - \xi, z; \frac{\eta}{2})\diamondsuit\mathcal{S}(l_{n_0} - \xi, z; \frac{\eta}{2}) < \sigma\diamondsuit\sigma < \varepsilon,$$

which is not possible. So we have $\mathcal{S}(l_k - \xi, z; \frac{\eta}{2}) \ge \sigma$. Hence $k \in \mathcal{A}(\sigma,\eta)$, i.e. $\mathcal{B}(\varepsilon,\eta) \subset \mathcal{A}(\sigma,\eta)$.

<u>Case-iii:</u> If we consider $\mathcal{T}(l_k - l_{n_0}, z; \eta) \ge \varepsilon$ then in the similar way as Case-ii we can show that $\mathcal{B}(\varepsilon,\eta) \subset \mathcal{A}(\sigma,\eta)$.

Hence, $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regard to $\mathcal{N}_2$. This completes the proof. ■

Theorem 11 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \square, \diamondsuit)$. If $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regard to $\mathcal{N}_2$, then $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent with regards to $\mathcal{N}_2$.*

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regards to $\mathcal{N}_2$ but not $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to $\xi \in \mathcal{X}$ with regards to $\mathcal{N}_2$. Then for $\varepsilon \in (0,1)$ and $\eta > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\mathcal{A}(\varepsilon,\eta) = \{n \in \mathbb{N} : \mathcal{R}(l_n - l_{n_0}, z; \eta) \le 1-\varepsilon \text{ or } \mathcal{S}(l_n - l_{n_0}, z; \eta) \ge \varepsilon \text{ and } \mathcal{T}(l_n - l_{n_0}, z; \eta) \ge \varepsilon\} \in \mathcal{I}$$

for every $z \in \mathcal{X}$. And,

$$\mathcal{B}(\varepsilon,\eta) = \Big\{n \in \mathbb{N} : \mathcal{R}(l_n - \xi, z; \frac{\eta}{2}) > 1-\varepsilon \text{ or } \mathcal{S}(l_n - \xi, z; \frac{\eta}{2}) < \varepsilon \text{ and } \mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) < \varepsilon\Big\} \in \mathcal{I}.$$

So, $\mathcal{B}^c(\varepsilon,\eta) \in \mathcal{F}(\mathcal{I})$. Since,

$$\mathcal{R}(l_n - l_{n_0}, z; \eta) \ge 2\mathcal{R}(l_n - \xi, z; \frac{\eta}{2}) > 1-\varepsilon$$

and

$$\mathcal{S}(l_n - l_{n_0}, z; \eta) \le 2\mathcal{S}(l_n - \xi, z; \frac{\eta}{2}) < \varepsilon,$$

we see that $\mathcal{T}(l_n - l_{n_0}, z; \eta) \leq 2\mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) < \varepsilon$ if

$$\mathcal{R}(l_n - \xi, z; \frac{\eta}{2}) > \frac{1-\varepsilon}{2},\ \mathcal{S}(l_n - \xi, z; \frac{\eta}{2}) < \frac{\varepsilon}{2} \text{ and } \mathcal{T}(l_n - \xi, z; \frac{\eta}{2}) < \frac{\varepsilon}{2}.$$

This implies $\mathcal{A}^c(\varepsilon, \eta) \in \mathcal{I}$ which leads to a contradiction because $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy with regard to $\mathcal{N}_2$. Therefore, $\{l_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent with regards to $\mathcal{N}_2$. This completes the proof. ■

Definition 18 *A N2-NS $\mathcal{X} = (\mathcal{W}, \mathcal{N}_2, \Box, \Diamond)$ is named to be $\mathcal{I}$-complete with regard to $\mathcal{N}_2$ if every $\mathcal{I}$-Cauchy sequence is $\mathcal{I}$-convergent with regards to $\mathcal{N}_2$.*

Remark 3 *In the light of Theorem 10 and 11, we see every neutrosophic 2-normed space is $\mathcal{I}$-complete.*

Conclusion and Future Developments

Incorporating the latest advancements, this paper has unveiled the innovative concepts of $\mathcal{I}(\mathcal{N}_2)$ and $\mathcal{I}^*(\mathcal{N}_2)$-convergence, presenting a groundbreaking generalization of statistical convergence within summability theory. We have delved into the key properties and revealed the intricate relationship between these two pioneering concepts using the condition (AP). In this work, we have explored the concept of $\mathcal{I}(\mathcal{N}_2)$ and $\mathcal{I}^*(\mathcal{N}_2)$-Cauchy sequences, highlighting the critical role of condition (AP) in understanding their interrelationship. Furthermore, we have established the completeness of every N2-NS within this ideal-driven framework. Research on sequence convergence in neutrosophic 2-normed linear spaces is still in its early stages, with limited progress made thus far. Building upon the insights gained from this research, future studies may extend this notion to encompass lacunary sequences and λ-density, further exploring its connections to multiple sequences within the framework of $\mathcal{N}_2$. This concept can also be applied to convergence-related challenges across various branches of science and engineering, offering valuable insights and solutions.

Acknowledgments. We extend our heartfelt gratitude and deep respect to the referees and reviewers for their valuable comments, which have significantly enhanced the quality of this paper. Also, the second author is grateful to The Council of Scientific and Industrial Research (CSIR), HRDG, India, for the grant of Senior Research Fellowship during the preparation of this paper.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1986), 87–96.

[2] K. T. Atanassov and G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 31(1989), 343–349.

[3] H. Bustince and P. Burillo, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems, 79(1996), 403–405.

[4] L. C. Barros, R. C. Bassanezi and P. A. Tonelli, Fuzzy modelling in population dynamics, Ecol. Model., 128(2000), 27–33.

[5] T. Bera and N. K. Mahapatra, On neutrosophic soft linear spaces, Fuzzy Inf. Eng., 9(2017), 299–324.

[6] T. Bera and N. K. Mahapatra, Neutrosophic soft normed linear space, Neutrosophic Sets and Systems, 23(2018), 52–71.

[7] S. Debnath, S. Debnath and C. Choudhury, On deferred statistical convergence of sequences in neutrosophic normed spaces, Sahand Commun. Math. Anal., 19(2022), 81–96.

[8] H. Fast, Sur la convergence statistique, Colloq. Math., 2(1951), 241–244.

[9] C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets Systems, 48(1992), 239–248.

[10] A. L. Fradkov and R. J. Evans, Control of chaos: Methods and applications in engineering, Annu. Rev. Control, 29(2005), 33–56.

[11] S. Gähler, 2-normed spaces, Math. Nachr., 28(1964), 1–43.

[12] R. Giles, A computer program for fuzzy reasoning, Fuzzy Sets Systems, 4(1980), 221–234.

[13] M. Gürdal and A. Şahiner, Extremal I-limit points of double sequences, Appl. Math. E-Notes, 8(2008), 131–137.

[14] M. Gürdal and I. Açık, On I-Cauchy sequences in 2-normed spaces, Math. Inequal. Appl., 11(2008), 349–354.

[15] M. Gürdal, A. Şahiner and I. Açık, Approximation theory in 2-Banach spaces, Nonlinear Anal. Theory Methods Appl., 71(2009), 1654–1661.

[16] M. Gürdal and M. B. Huban, On I-convergence of double sequences in the topology induced by random 2-norms, Mat. Vesnik, 66(2014), 73–83.

[17] C. Granados and A. Dhital, Statistical convergence of double sequences in neutrosophic normed spaces, Neutrosophic Sets and Systems, 42(2021), 333–344.

[18] L. Hong and J. Q. Sun, Bifurcations of fuzzy nonlinear dynamical systems, Commun. Nonlinear Sci. Numer. Simul., 11(2006), 1–12.

[19] B. Hazarika, On ideal convergent sequences in fuzzy normed linear spaces, Afr. Mat., 25(2014), 987–999.

[20] P. Kostyrko, T. Šalát and W. Wilczyński, I-convergence, Real Anal. Exchange, 26(2000), 669–685.

[21] E. P. Klement, R. Mesiar and E. Pap, Triangular norms. Position paper I: basic analytical and algebraic properties, Fuzzy Sets Systems, 143(2004), 5–26.

[22] S. Karakus, K. Demirci and O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos Solitons Fractals, 35(2008), 763–769.

[23] V. Kumar and K. Kumar, On ideal convergence of sequences in intuitionistic fuzzy normed spaces, Selçuk J. Appl. Math., 10(2009), 27–41.

[24] M. Kirişci and N. Şimşek, Neutrosophic normed spaces and statistical convergence, J. Anal., 28(2020), 1059–1073.

[25] Ö. Kişi, Ideal convergence of sequences in neutrosophic normed spaces, J. Intell. Fuzzy Syst., 41(2021), 2581–2590.

[26] V. A. Khan, M. D. Khan and M. Ahmad, Some new type of lacunary statistically convergent sequences in neutrosophic normed space, Neutrosophic Sets and Systems, 42(2021), 241–252.

[27] J. Madore, Fuzzy physics, Ann. Physics, 219(1992), 187–198.

[28] M. Mursaleen and A. Alotaibi, On $\mathcal{I}$-convergence in random 2-normed spaces, Math. Slovaca, 61(2011), 933-940.

[29] S. Murtaza, A. Sharma and V. Kumar, Neutrosophic 2-normed spaces and generalized summability, Neutrosophic Sets and Systems, 55(2023).

[30] A. Nabiev, S. Pehlivan and M. Gürdal, On $\mathcal{I}$-Cauchy sequences, Taiwanese J. Math., 11(2007), 569–576.

[31] A. Nabiev, E. Savaş and M. Gürdal, Statistically localized sequences in metric spaces, J. Appl. Anal. Comput., 9(2019), 739–746.

[32] M. H. M. Rashid and L. D. R. Kočinac, Ideal convergence in 2-fuzzy 2-normed spaces, Hacet. J. Math. Stat., 46(2017), 149–162.

[33] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2(1951), 73–74.

[34] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66(1959), 361–375.

[35] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math., 10(1960), 313–334.

[36] F. Smarandache, Neutrosophic set-a generalisation of the intuitionistic fuzzy sets, Int. J. Pure. Appl. Math., 24(2005), 287–297.

[37] R. Saadati and J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals, 27(2006), 331–344.

[38] A. Şahiner, M. Gürdal, S. Saltan and H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese J. Math., 11(2007), 1477–1484.

[39] A. Şahiner, M. Gürdal and T. Yiğit, Ideal convergence characterization of the completion of linear n-normed spaces, Comput. Math. Appl., 61(2011), 683–689.

[40] E. Savaş and M. Gürdal, Certain summability methods in intuitionistic fuzzy normed spaces, J. Intell. Fuzzy Systems, 27(2014), 1621–1629.

[41] E. Savaş and M. Gürdal, I-statistical convergence in probabilistic normed spaces, U.P.B. Sci. Bull., Series A, 77(2015), 195–204.

[42] I. B. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy Sets Systems, 20(1986), 191–210.

[43] L. A. Zadeh, Fuzzy sets, Inform. control, 8(1965), 338–353.