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Existence And Multiplicity Of Classical Solutions For Periodic Initial Value Problem Of Generalized Benjamin-Bona-Mahony Equation*

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Abstract

In this paper, we investigate an initial value problem for existence and multiplicity of global classical periodic solutions for a generalized Benjamin-Bona-Mahony equation. For this, a new topological approach which uses the abstract theory of the sum of two operators is used for investigations of existence of at least one and at least two classical solutions.

1 Introduction

In this paper, we investigate the following periodic initial value problem for generalized Benjamin-Bona-Mahony equation

$$\begin{aligned} u_t - u_{xxt} - \nu u_{xx} + f(u)_x &= g(u) + h(t,x), \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}, \\ u(t, x+\omega) &= u(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned} \tag{1}$$

where

(H1) $\omega > 0$, $\nu \in \mathbb{R}$, $f \in \mathcal{C}^1(\mathbb{R})$,

$$|f'(x)| \le \sum_{j=1}^{p} a_j |x|^{m_j}, \quad x \in \mathbb{R},$$

$a_j, m_j \ge 0$, $j \in \{1, \ldots, p\}$, $p \in \mathbb{N}^*$, $g \in \mathcal{C}(\mathbb{R})$,

$$|g(x)| \le \sum_{j=1}^{q} b_j |x|^{n_j}, \quad x \in \mathbb{R},$$

$b_j, n_j \ge 0$, $j \in \{1, \ldots, q\}$, $q \in \mathbb{N}^*$, $h \in \mathcal{C}([0,\infty) \times \mathbb{R})$,

$$h(t, x+\omega) = h(t,x), \ |h(t,x)| \le B, \ (t,x) \in [0,\infty) \times \mathbb{R},$$

B is a positive constant so that

$$|\nu|, a_j, b_l \le B, j \in \{1, \ldots, p\}, \ l \in \{1, \ldots, q\},$$

$u_0 \in \mathcal{C}^2(\mathbb{R})$,

$$u_0(x+\omega) = u_0(x), \ 0 \le u_0(x) \le B, \ x \in \mathbb{R}.$$

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Many mathematicians studied existence, uniqueness and the asymptotic behavior of solutions for such kind of equations. In [10] the generalized Benjamin-Bona-Mahony equation is studied from the point of view of dynamical systems. In [9], a spectral method is used to analyze the generalized Benjamin-Bona-Mahony equations. The existence and uniqueness of global smooth solution of these equations are proved. The large time error estimation between the spectral approximate solution and the exact solution is also obtained. In [12] a generalized variational principle and abundant exact solutions are obtained for a new fractal modified Benjamin-Bona-Mahony equation. The case $f(u)_x = \nu u + uu_x$ which corresponds to Benjamin-Bona-Mahony-Burgers equation is well studied, see for example [5, 11, 13, 15] and the references therein.

The aim of this paper is to investigate the initial value problem (1) for existence and nonuniqueness of global classical solutions. To obtain this goal, we use the method of fixed point, see e.g. [1], [3], [4] and [7]. This basic idea can be used for investigations for existence of global classical solutions for many of the interesting equations of mathematical physics. So, let X be the space of all continuous functions u on $[0,\infty)\times\mathbb{R}$ such that $u(t,x+\omega)=u(t,x)$, $(t,x)\in[0,\infty)\times\mathbb{R}$, and u_t, u_x, u_{xx}, u_{xxt} exist and are continuous on $[0,\infty)\times\mathbb{R}$. The space X will be endowed with the norm

$$\begin{aligned} \|u\| &= \max\Bigg\{\sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u_t(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u_x(t,x)|, \\ &\qquad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u_{xx}(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}}|u_{xxt}(t,x)\Bigg\}, \end{aligned}$$

provided it exists. Our main results are as follows.

Theorem 1 *Suppose that $(H1)$ holds. Then the problem (1) has at least one solution in X.*

Theorem 2 *Suppose that $(H1)$ holds. Then the problem (1) has at least two nonnegative solutions in X.*

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3, we prove Theorem 1. In Section 4, we prove Theorem 2. In Section 5, we give an example to illustrate our main results.

2 Preliminary Results

Below, assume that E is a real Banach space. Now, we will recall the definitions of compact and completely continuous operators in Banach spaces.

Definition 1 *Let $K : M\subset E\to E$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of E. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

To prove our first existence result we will use the following fixed point theorem. For its proof, we refer the reader to [6].

Theorem 3 *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with $0\in U$. Consider two operators T and S, where*

$$Tx=\varepsilon x,\ x\in\overline{U},$$

for $\varepsilon>0$ and $S:\overline{U}\to E$ be such that

(i) *$I-S:\overline{U}\to Y$ continuous, compact and*

(ii) *$\{x\in\overline{U}: x=\lambda(I-S)x, \quad x\in\partial U\}=\emptyset$, for any $\lambda\in\left(0,\frac{1}{\varepsilon}\right)$.*

Then there exists $x^*\in\overline{U}$ such that $Tx^*+Sx^*=x^*$.

Definition 2 *Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X, \ \ \textit{for any} \ \ x, y \in X.$$

Now, we will recall the definition of a cone in a Banach space.

Definition 3 *A closed, convex set $\mathcal{P}$ in a Banach space E is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

The next fixed point theorem will allow us to prove existence of at least two nonnegative global classical solutions of the IVP (1). For its proof, we refer the reader to [2] and [14].

Theorem 4 *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to E$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon > 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \ \ \textit{and} \ \ x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \ \ \textit{and} \ \ x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

3 Proof of Theorem 1

In order to prove Theorem 1, we give firstly an integral representation of solutions to the problem (1). After that we prove some a priori type estimates to these solutions. We end the proof by using results on fixed points for the sum of operators. So, for $u \in X$ and $(t, x) \in [0, \infty) \times \mathbb{R}$, let us define the operator S_1 by

$$\begin{aligned} S_1 u(t,x) &= u(t,x) - \frac{e^{-\omega}}{1 - e^{-\omega}} \int_0^\omega e^s \Bigg(u_0(s) + u_x(t,s) \\ &\quad + \int_0^t \Bigg(u_{xxt}(\tau,s) + \nu u_{xx}(\tau,s) - f'(u(\tau,s))u_x(\tau,s) + g(u(\tau,s)) + h(\tau,s) \Bigg) d\tau \Bigg) ds. \end{aligned}$$

Proposition 1 *Suppose that $(H1)$ holds. If $u \in X$ satisfies the equation*

$$S_1 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{2}$$

then u is a solution to the problem (1).

Proof. By (2) and for $(t,x) \in [0,\infty) \times \mathbb{R}$, we find

$$\begin{aligned} u(t,x) &= \frac{e^{-\omega}}{1-e^{-\omega}} \int_0^\omega e^s \Big(u_0(s) + u_x(t,s) \\ &+ \int_0^t \Big(u_{xxt}(\tau,s) + \nu u_{xx}(\tau,s) - f'(u(\tau,s)) u_x(\tau,s) + g(u(\tau,s)) + h(\tau,s) \Big) d\tau \Big) ds. \end{aligned} \tag{3}$$

It follows that

$$\begin{aligned} u_x(t,x) &= -u(t,x) + u_x(t,x) + u_0(x) + \int_0^t \Big(u_{xxt}(\tau,x) + \nu u_{xx}(\tau,x) - f'(u(\tau,x)) u_x(\tau,x) \\ &+ g(u(\tau,x)) + h(\tau,x) \Big) d\tau, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

whereupon

$$u(t,x) = u_0(x) + \int_0^t \Big(u_{xxt}(\tau,x) + \nu u_{xx}(\tau,x) - f'(u(\tau,x)) u_x(\tau,x) + g(u(\tau,x)) + h(\tau,x) \Big) d\tau, \ (t,x) \in [0,\infty) \times \mathbb{R}.$$

We put $t = 0$ and we find $u(0,x) = u_0(x),\ x \in \mathbb{R}$. Now, we differentiate the last equation with respect to t and we arrive at the equation

$$u_t(t,x) = u_{xxt}(t,x) + \nu u_{xx}(t,x) - f'(u(t,x)) u_x(t,x) + g(u(t,x)) + h(t,x), \ (t,x) \in [0,\infty) \times \mathbb{R},$$

i.e., u satisfies the problem (1). This completes the proof. ■

Let

$$B_1 = \max \left\{ \left(1 + \frac{2\omega}{1-e^{-\omega}}\right) B, \frac{\omega}{1-e^{-\omega}} \left(B + B^2 + \sum_{j=1}^p B^{m_j+2} + \sum_{j=1}^q B^{n_j+1} + B \right) \right\}.$$

Lemma 1 *Suppose that $(H1)$ holds. If $u \in X$, $\|u\| \le B$, then*

$$|S_1 u(t,x)| \le B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. Suppose that $(H1)$ is satisfied and let $u \in X$, with $\|u\| \le B$. Then, for $(t,x) \in [0,\infty) \times \mathbb{R}$, we have

$$|f'(u(t,x))| \le \sum_{j=1}^p B^{m_j+1}, \qquad |g(u(t,x))| \le \sum_{j=1}^q B^{n_j+1},$$

and

$$\begin{aligned} |S_1 u(t,x)| &= \Bigg| u(t,x) - \frac{e^{-\omega}}{1-e^{-\omega}} \int_0^\omega e^s \Big(u_0(s) + u_x(t,s) \\ &+ \int_0^t \Big(u_{xxt}(\tau,s) + \nu u_{xx}(\tau,s) - f'(u(\tau,s)) u_x(\tau,s) + g(u(\tau,s)) + h(\tau,s) \Big) d\tau \Big) ds \Bigg| \\ &\le |u(t,x)| + \frac{e^{-\omega}}{1-e^{-\omega}} \int_0^\omega e^s \Big(u_0(s) + |u_x(t,s)| \\ &+ \int_0^t \Big(|u_{xxt}(\tau,s)| + |\nu||u_{xx}(\tau,s)| + |f'(u(\tau,s))||u_x(\tau,s)| \\ &+ |g(u(\tau,s))| + |h(\tau,s)| \Big) d\tau \Big) ds \\ &\le B + \frac{\omega}{1-e^{-\omega}} \Big(B + B + t \Big(B + B^2 + \sum_{j=1}^p B^{m_j+2} + \sum_{j=1}^q B^{n_j+1} + B \Big) \Big) \\ &\le (1+t) B_1. \end{aligned}$$

This completes the proof. ■

Let $g_1 \in \mathcal{C}([0,\infty)\times\mathbb{R})$ be a function such that $g_1 > 0$ on $(0,\infty)\times(\mathbb{R}\backslash\{x=0\})$ with

$$g_1(0,x) \quad = \quad g_1(t,0) = 0, \quad t\in[0,\infty), \quad x\in\mathbb{R},$$

and such that

$$4(1+t+t^2)\left(1+|x|+x^2\right)\int_0^t \left|\int_0^x g_1(t_1,s)ds\right| dt_1 \le A, \quad (t,x)\in[0,\infty)\times\mathbb{R}, \tag{4}$$

for some positive constant A. In the last section, we will give an example of such function g_1. For $u \in X$, define the operator

$$S_2u(t,x) = \int_0^t \int_0^x (t-t_1)(x-s)^2 g_1(t_1,s) S_1u(t_1,s)dsdt_1, (t,x)\in[0,\infty)\times\mathbb{R}. \tag{5}$$

Lemma 2 *Suppose that* $(H1)$ *holds. If* $u\in X$ *and* $\|u\|\le B$*, then*

$$\|S_2u\| \le AB_1.$$

Proof. Suppose that $(H1)$ is satisfied and let $u\in X$, with $\|u\|\le B$.

(i) The estimation of $|S_2u(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}:$

$$\begin{aligned}
|S_2u(t,x)| &= \left|\int_0^t\int_0^x (t-t_1)(x-s)^2 g_1(t_1,s)S_1u(t_1,s)dsdt_1\right| \\
&\le B_1\int_0^t\left|\int_0^x (t-t_1)(1+t_1)(x-s)^2 g_1(t_1,s)ds\right|dt_1 \\
&\le 4B_1(1+t+t^2)|x|^2\int_0^t\left|\int_0^x g_1(t_1,s)ds\right|dt_1 \\
&\le AB_1.
\end{aligned}$$

(ii) The estimation of $|\frac{\partial}{\partial t}S_2u(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}:$

$$\begin{aligned}
\left|\frac{\partial}{\partial t}S_2u(t,x)\right| &= \left|\int_0^t\int_0^x (x-s)^2 g_1(t_1,s)S_1u(t_1,s)dsdt_1\right| \\
&\le \int_0^t\left|\int_0^x (x-s)^2 g_1(t_1,s)|S_1u(t_1,s)|ds\right|dt_1 \\
&\le B_1\int_0^t\left|\int_0^x (1+t_1)(x-s)^2 g_1(t_1,s)ds\right|dt_1 \\
&\le 4B_1(1+t)|x|^2\int_0^t\left|\int_0^x g_1(t_1,s)ds\right|dt_1 \\
&\le AB_1.
\end{aligned}$$

(iii) The estimation of $|\frac{\partial}{\partial x}S_2u(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}:$

$$\begin{aligned}
\left|\frac{\partial}{\partial x}S_2u(t,x)\right| &= \left|2\int_0^t\int_0^x (t-t_1)(x-s) g_1(t_1,s)S_1u(t_1,s)dsdt_1\right| \\
&\le 2\int_0^t\left|\int_0^x (t-t_1)|x-s| g_1(t_1,s)|S_1u(t_1,s)|ds\right|dt_1 \\
&\le 2B_1\int_0^t\left|\int_0^x (t-t_1)(1+t_1)|x-s| g_1(t_1,s)ds\right|dt_1 \\
&\le 4B_1(1+t+t^2)|x|\int_0^t\left|\int_0^x g_1(t_1,s)ds\right|dt_1 \\
&\le AB_1.
\end{aligned}$$

(iv) The estimation of $|\frac{\partial^2}{\partial x^2} S_2 u(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x^2} S_2 u(t,x)\right| &= \left|2\int_0^t \int_0^x (t-t_1) g_1(t_1,s) S_1 u(t_1,s) ds dt_1\right| \\
&\leq 2\int_0^t \left|\int_0^x (t-t_1) g_1(t_1,s) |S_1 u(t_1,s)| ds\right| dt_1 \\
&\leq 2B_1 \int_0^t \left|\int_0^x (t-t_1)(1+t_1) g_1(t_1,s) ds\right| dt_1 \\
&\leq 2B_1(1+t+t^2) \int_0^t \left|\int_0^x g_1(t_1,s) ds\right| dt_1 \\
&\leq AB_1.
\end{aligned}$$

(v) The estimation of $|\frac{\partial^3}{\partial x^2 \partial t} S_2 u(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
\left|\frac{\partial^3}{\partial x^2 \partial t} S_2 u(t,x)\right| &= \left|2\int_0^t \int_0^x g_1(t_1,s) S_1 u(t_1,s) ds dt_1\right| \\
&\leq 2\int_0^t \left|\int_0^x g_1(t_1,s) |S_1 u(t_1,s)| ds\right| dt_1 \\
&\leq 2B_1 \int_0^t \left|\int_0^x (1+t_1) g_1(t_1,s) ds\right| dt_1 \\
&\leq 2B_1(1+t) \int_0^t \left|\int_0^x g_1(t_1,s) ds\right| dt_1 \\
&\leq AB_1.
\end{aligned}$$

This completes the proof. ■

Lemma 3 *Suppose that* $(H1)$ *holds. If* $u \in X$ *satisfies the equation*

$$S_2 u(t,x) = C, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{6}$$

where C *is a constant, then* u *is a solution to the problem* (1).

Proof. We differentiate two times with respect to t and three times with respect to x, the equation (6) and we find

$$g_1(t,x) S_1 u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1 u(t,x) = 0, \quad (t,x) \in (0,\infty) \times (\mathbb{R} \backslash \{x=0\}).$$

Since $S_1 u(\cdot,\cdot) \in \mathcal{C}([0,\infty) \times \mathbb{R})$, then for $(t,x) \in [0,\infty) \times \mathbb{R}$, we get

$$\begin{aligned}
0 &= \lim_{t\to 0} S_1 u(t,x) \\
&= S_1 u(0,x) \\
&= \lim_{x\to 0} S_1 u(t,x) \\
&= S_1 u(t,0) \\
&= 0.
\end{aligned}$$

Thus,

$$S_1 u(t,x) = 0, \ (t,x) \in [0,\infty) \times \mathbb{R}.$$

In virtue of Proposition 1, we conclude that u is a solution to the problem (1). This completes the proof. ■

Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Set $Y = \overline{\widetilde{Y}}$ and

$$U \quad = \quad \{u \in Y : \|u\| < B\}.$$

For $u \in \overline{U}$, $(t,x) \in [0,\infty) \times \mathbb{R}$ and $\epsilon > 0$, define the operators T and S as follows

$$\begin{aligned} Tu(t,x) &= \epsilon u(t,x), \\ Su(t,x) &= u(t,x) - \epsilon u(t,x) - \epsilon S_2 u(t,x). \end{aligned}$$

where the operator S_2 is given by formula (5). For $u \in \overline{U}$, we have

$$\begin{aligned} \|(I-S)u\| &= \|\epsilon u + \epsilon S_2 u\| \\ &\leq \epsilon\|u\| + \epsilon\|S_2 u\| \\ &\leq \epsilon B + \epsilon A B_1. \end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there exists $u \in \partial U$ such that

$$u = \lambda(I-S)u$$

or equivalently

$$u = \lambda\epsilon\,(u + S_2 u), \tag{7}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $S_2 u(0,x) = 0$, we get

$$u(0,x) = \lambda\epsilon u(0,x), \quad x \in \mathbb{R},$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently,

$$\{u \in \overline{U} : u = \lambda_1(I-S)u,\ u \in \partial U\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 3, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t,x) &= Tu^*(t,x) + Su^*(t,x) \\ &= \epsilon u^*(t,x) + u^*(t,x) \\ &\quad -\epsilon u^*(t,x) - \epsilon S_2 u^*(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

whereupon

$$S_2 u^*(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then, in virtue of Lemma 3, it follows that u^* is a solution to the equation (1). This completes the proof of Theorem 1.

4 Proof of Theorem 2

Let X be the space used in the previous section and r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq B,\ AB_1 < \frac{L}{5}.$$

Here, B and A are the constants which appear in condition $(H1)$ and formula (4), respectively and B_1 is the constant of Lemma 1. Set

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$ and $(t,x) \in [0,\infty) \times \mathbb{R}$, define the operators T_1 and S_3 as follows

$$\begin{aligned} T_1 v(t,x) &= (1+m\epsilon)v(t,x) - \epsilon\frac{L}{10}, \\ S_3 v(t,x) &= -\epsilon S_2 v(t,x) - m\epsilon v(t,x) - \epsilon\frac{L}{10}, \end{aligned}$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (5). Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1). Now, let us define the following sets:

$\Omega = \mathcal{P},\ U_1 = \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\},\ U_2 = \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}$, and $U_3 = \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}$.

1. For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1+m\epsilon)\|v_1 - v_2\|,$$

   whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

2. For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned} \|S_3 v\| &\leq \epsilon\|S_2 v\| + m\epsilon\|v\| + \epsilon\frac{L}{10} \\ &\leq \epsilon\left(AB_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

   Therefore $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

3. Let $v_1 \in \overline{\mathcal{P}_{R_1}}$. Set

$$v_2 = v_1 + \frac{1}{m}S_2 v_1 + \frac{L}{5m}.$$

   Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[0,\infty[\times\mathbb{R}$. We have $v_2 \geq 0$ on $[0,\infty[\times\mathbb{R}$. Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2 v_1 - \epsilon\frac{L}{10} - \epsilon\frac{L}{10}$$

   or equivalently,

$$\begin{aligned} (I - T_1)v_2 &= -\epsilon m v_2 + \epsilon\frac{L}{10} \\ &= S_3 v_1. \end{aligned}$$

   Consequently $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $v_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $v \in \partial\mathcal{P}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

   Then

$$-\epsilon S_2 v - m\epsilon v - \epsilon\frac{L}{10} = -m\epsilon(v - \lambda v_0) + \epsilon\frac{L}{10}$$

   or equivalently,

$$-S_2 v = \lambda m v_0 + \frac{L}{5}.$$

   Hence,

$$\|S_2 v\| = \left\|\lambda m v_0 + \frac{L}{5}\right\| \geq \frac{L}{5}.$$

   This is a contradiction.

5. Let $\epsilon_1 = \frac{2}{5m}$. Suppose that there exist a $v_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1+\epsilon_1$ such that
$$S_3 v_1 = (I - T_1)(\lambda_1 v_1). \tag{8}$$
Moreover,
$$-\epsilon S_2 v_1 - m\epsilon v_1 - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon v_1 + \epsilon\frac{L}{10},$$
or equivalently,
$$S_2 v_1 + \frac{L}{5} = (\lambda_1 - 1)m v_1.$$
From here,
$$2\frac{L}{5} > \left\|S_2 v_1 + \frac{L}{5}\right\| = (\lambda_1 - 1)m\|v_1\| = (\lambda_1 - 1)mL$$
and
$$\frac{2}{5m} + 1 > \lambda_1,$$
which is a contradiction.

Therefore all conditions of Theorem 4 hold. Hence, the problem (1) has at least two solutions u_1 and u_2 so that
$$\|u_1\| = L < \|u_2\| < R_1$$
or
$$r < \|u_1\| < L < \|u_2\| < R_1.$$

5 An Example

Below, we will illustrate our main results. Let $B = 1$ and
$$R_1 = \frac{9}{10}, \quad L = \frac{3}{5}, \quad r = \frac{2}{5}, \quad A = \frac{1}{10B_1},$$
$$p = q = \nu = \omega = 1, \quad m_1 = 4, \quad n_1 = 3, \quad a_1 = \frac{1}{2}, \quad b_1 = \frac{1}{20}.$$
Then
$$B_1 = \frac{5e}{e-1}$$
and
$$r < L < R_1 < B, \qquad AB_1 < \frac{L}{5}.$$
Take
$$h_1(s) = \log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$
Then
$$h_1'(s) = \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \quad \text{and} \quad l'(s) = \frac{11\sqrt{2}s^{10}(1 + s^{22})}{1 + s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$
Therefore
$$-\infty < \lim_{s\to\pm\infty}(1 + s + s^2)^3 h(s) < \infty, \quad \text{and} \quad -\infty < \lim_{s\to\pm\infty}(1 + s + s^2)^3 l(s) < \infty.$$

Hence, there exists a positive constant C_1 such that

$$(1+s+s^2)^3\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right)\leq C_1 \quad \text{for } s\in\mathbb{R}.$$

Note that $\lim_{s\to\pm 1} l(s)=\frac{\pi}{2}$ and by [8, pp. 707, Integral 79], we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s)=\frac{s^{10}}{(1+s^{44})(1+s+s^2)^2},\quad s\in\mathbb{R},\quad \text{and}\quad g_2(t,x)=Q(t)Q(x),\quad t\in[0,\infty),\quad x\in\mathbb{R}.$$

Then there exists a constant $C>0$ such that

$$4(1+t+t^2)\left(1+|x|+x^2\right)\int_0^t\left|\int_0^x g_2(\tau,z)dz\right|d\tau\leq C,\quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Let

$$g_1(t,x)=\frac{A}{C}g_2(t,x),\quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Then

$$4(1+t+t^2)\left(1+|x|+x^2\right)\int_0^t\left|\int_0^x g_1(\tau,z)dz\right|d\tau\leq A,\quad (t,x)\in[0,\infty)\times\mathbb{R},$$

Therefore for the problem

$$\begin{aligned}
u_t-u_{xxt}-u_{xx}+\tfrac{1}{10}(u^5)_x &= \tfrac{1}{20}u^3+\tfrac{1}{1+t^2(\sin(2\pi x))^2},\quad (t,x)\in(0,\infty)\times\mathbb{R},\\
u(0,x) &= \tfrac{1}{1+(\cos(2\pi x))^4+(\sin(2\pi x))^2}\quad x\in\mathbb{R},\\
u(t,x) &= u(t,x+1),\quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

all conditions of Theorem 1 and Theorem 2 are satisfied.

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