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Reconstruction And Best Approximate Inversion Formulas For The Sturm-Liouville-Stockwell Transform*

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Abstract

We define and study the Stockwell transform $\mathscr{S}_g$ associated with the Sturm-Liouville operator $L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}$, where A is a nonnegative function satisfying certain conditions; and prove a Plancherel theorem and an inversion formula. We define a reconstruction function $f_{a,b}$, and we prove a Calderón's reproducing inversion formula for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$. We introduce and study the extremal function $f^*_{\eta,k} := (\eta I + \mathscr{S}^*_g \mathscr{S}_g)^{-1}\mathscr{S}^*_g(k)$, and we deduce best approximate inversion formulas for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$.

1 Introduction

The Stockwell transform, also known as the windowed Fourier transform [3], or the Gabor transform [24] is defined first by using translation, convolution and modulation operators of a single Gaussian to represent one dimensional signal. The time-frequency resolution is usually associated with the Stockwell transform, and recently this transform has become the focus of many works [7, 9, 24]. Another fundamental tool in time-frequency analysis is the Sturm-Liouville-Stockwell transform, which is the aim of the present study.

We consider the second-order differential operator defined on $(0,\infty)$ by

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where A is a nonnegative function satisfying certain conditions. This operator plays an important role in the analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of an operator of L type. The radial part of the Beltrami-Laplacian in a symmetric space is also of L type. Many aspects of such operators have been studied [2, 5, 23, 25, 26]. In particular, the two references [5, 23] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with L.

In this paper, we consider the Sturm-Liouville transform

$$\mathscr{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) A(x)\mathrm{d}x, \quad \lambda \geq 0,$$

where $\varphi_\lambda(x)$ is the Sturm-Liouville kernel given in Section 2 below. The Sturm-Liouville transform can be regarded as a generalization of the Fourier-Bessel transform [14], the Fourier-Jacobi transform [11] and the Fourier-Whittaker transform [21, 22]. Many results have already been proved for the Sturm-Liouville transform $\mathscr{F}$, for example, Bouattour and Trimèche [4] proved a Cowling-Price's theorem and Hardy's theorem, Daher et al. [6] established a Miyachi's theorem, Ma [12] proved a Heisenberg uncertainty principle

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and Soltani proved a local uncertainty principle [19] and studied the extremal functions on the Sturm-Liouville hypergroups [20],.... We associate to the Sturm-Liouville transform $\mathscr{F}$ the Lebesgue spaces $L^2(\mu)$ and $L^2(\nu)$ defined later in Section 2.

Let $f, g \in L^2(\nu)$. We define the convolution product $f\sharp g$ of f and g by

$$f\sharp g(\lambda) := \mathscr{F}(\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g))(\lambda), \quad \lambda \geq 0.$$

Let $g \in L^2(\nu)$. The Sturm-Liouville-Stockwell transform is the mapping $\mathscr{S}_g$ defined for $f \in L^2(\nu)$ by

$$\mathscr{S}_g(f)(\lambda, y) := f\sharp g_y(\lambda), \quad \lambda, y \geq 0,$$

where g_y is the modulation of g by y defined by

$$g_y := \mathscr{F}\Big(\sqrt{\tau_y|\mathscr{F}^{-1}(g)|^2}\Big).$$

Here τ_y, $y \geq 0$ are the Sturm-Liouville translation operators [5, 23] defined in Section 2 below.

Let $g \in L^2(\nu)$ be a non-zero function, such that $\mathscr{F}^{-1}(g) \in L^\infty(\mu)$. For $f \in L^2(\nu)$ and $0 < a < b < \infty$, we define the reconstruction function $f_{a,b}$ associated to $\mathscr{S}_g$, by

$$f_{a,b}(\xi) := \frac{1}{\|g\|^2_{L^2(\nu)}} \int_a^b \mathscr{S}_g(f)(.,y)\sharp g_y(\xi)\mathrm{d}\mu(y), \quad \xi \geq 0.$$

Let $g \in L^2(\nu)$. For any $k \in L^2(\nu \otimes \mu)$ and for any $\eta > 0$, we define the extremal function $f^*_{\eta,k}$ associated to $\mathscr{S}_g$, by

$$f^*_{\eta,k}(\xi) := (\eta I + \mathscr{S}_g^*\mathscr{S}_g)^{-1}\mathscr{S}_g^*(k)(\xi), \quad \xi \geq 0,$$

where I is the unit operator and $\mathscr{S}_g^* : L^2(\nu \otimes \mu) \longrightarrow \mathscr{H}^s$ is the adjoint of $\mathscr{S}_g$, being $\mathscr{H}^s$ the Strum-Liouville-Sobolev space of order s.

We give many harmonic analysis results related to the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$. In particular, we establish a Plancherel theorem and an inversion formula. Next, we prove the following Calderón's reproducing formula. That is, the function $f_{a,b}$ belongs to $L^2(\nu)$ and satisfies

$$\lim_{\substack{a\to 0\\ b\to\infty}} \|f_{a,b} - f\|_{L^2(\nu)} = 0.$$

Finally, we give an approximate inversion formula for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$. Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$. Then, for $f \in \mathscr{H}^s$, we have

$$\lim_{\eta\to 0^+} \|f^*_{\eta,\mathscr{S}_g(f)} - f\|_{\mathscr{H}^s} = 0.$$

Also, we obtain the following pointwise approximate inversion formula

$$\lim_{\eta\to 0^+} f^*_{\eta,\mathscr{S}_g(f)}(\xi) = f(\xi), \quad \xi \geq 0.$$

The paper is organized as follows. In Section 2, we recall some results about the harmonic analysis associated to Sturm-Liouville operator on $(0, \infty)$ (the Sturm-Liouville transform $\mathscr{F}$, Sturm-Liouville translation operators $\tau_y, y \geq 0$, Sturm-Liouville convolution product $\sharp$,...). Section 3 is devoted to study the Sturm-Liouville-Stockwell $\mathscr{S}_g$, for which we give a Plancherel formula, an inversion formula and a Calderón's reproducing formula. Finally, in Section 4, we give best approximate inversion formulas for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$.

2 The Sturm-Liouville Harmonic Analysis

We consider the second-order differential operator L defined on $(0,\infty)$ by

$$L := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A satisfies the following conditions:

(i) A is increasing and $\lim_{x\to\infty} A(x) = \infty$.

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$.

(iii) There exists a constant $\delta > 0$, such that

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x}D(x), \qquad \text{if } \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + e^{-\delta x}D(x), \qquad \text{if } \rho = 0,$$

where D is an infinitely differentiable function on $(0,\infty)$, bounded and with bounded derivatives on all intervals $[x_0,\infty)$, for $x_0 > 0$.

This operator was studied in [5, 23], and the following results have been established:
(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} Lu = -(\lambda^2+\rho^2)u, \\ u(0) = 1, \quad u'(0) = 0, \end{cases}$$

admits a unique solution, denoted by φ_λ, with the following properties:
• for $x \geq 0$, the function $\lambda \to \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
• for $\lambda \in \mathbb{C}$, the function $x \to \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$.

(II) For nonzero $\lambda \in \mathbb{C}$, the equation

$$Lu = -(\lambda^2+\rho^2)u,$$

has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}}V(x,\lambda),$$

with

$$\lim_{x\to\infty} V(x,\lambda) = 1.$$

Consequently, there exists a function (spectral function) $\lambda \longrightarrow c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \geq 0,$$

for nonzero $\lambda \in \mathbb{C}$. Moreover, there exist positive constants k_1, k_2, k, such that

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$

for all λ such that $\mathrm{Im}\lambda \leq 0$ and $|\lambda| \geq k$.

Lemma 1 (See [4, 10]) *The Sturm-Liouville function $\varphi_\lambda(x)$; $\lambda, x \geq 0$, possesses the following properties.*

(i) $|\varphi_\lambda(x)| \leq 1$, *for* $\rho \geq 0$.

(ii) $|\varphi_\lambda(x)| \leq \varphi_0(x) \leq C(1+x)e^{-\rho x}$, *for* $\rho > 0$, *where* C *is a positive constant.*

(iii) $\varphi_0(x) = 1$, *for* $\rho = 0$.

We denote by

• μ the measure defined on $[0,\infty)$ by $\mathrm{d}\mu(x) := A(x)\mathrm{d}x$; and by $L^p(\mu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0,\infty)$, such that

$$\|f\|_{L^p(\mu)} := \left[\int_0^\infty |f(x)|^p \mathrm{d}\mu(x)\right]^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mu)} := \text{ ess } \sup_{x\in[0,\infty)} |f(x)| < \infty;$$

• ν the measure defined on $[0,\infty)$ by $\mathrm{d}\nu(\lambda) := \dfrac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2}$; and by $L^p(\nu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0,\infty)$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm-Liouville transform is the Fourier transform associated with the operator L and is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) \mathrm{d}\mu(x), \quad \lambda \geq 0.$$

Some of the properties of the Sturm-Liouville transform $\mathscr{F}$ are collected bellow (see [5, 23, 25]).

Theorem 1 *(i)* $L^1 - L^\infty$*-boundedness. For all* $f \in L^1(\mu)$, $\mathscr{F}(f) \in L^\infty(\nu)$ *and*

$$\|\mathscr{F}(f)\|_{L^\infty(\nu)} \leq \|f\|_{L^1(\mu)}.$$

(ii) Plancherel theorem. The Sturm-Liouville transform $\mathscr{F}$ *extends uniquely to an isometric isomorphism of* $L^2(\mu)$ *onto* $L^2(\nu)$. *In particular,*

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(iii) Inversion theorem. Let $f \in L^1(\mu)$, *such that* $\mathscr{F}(f) \in L^1(\nu)$. *Then*

$$f(x) = \int_0^\infty \varphi_\lambda(x)\mathscr{F}(f)(\lambda) d\nu(\lambda), \quad a.e. \ x \in [0,\infty).$$

The Sturm-Liouville kernel φ_λ satisfies the product formula [5, 23]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z) w(x,y,z) \mathrm{d}\mu(z) \quad \text{for } x, y \geq 0, \tag{1}$$

where $w(x,y,.)$ is a measurable positive function on $[0,\infty)$, with support in $[|x-y|, x+y]$, satisfying

$$\int_0^\infty w(x,y,z)\mathrm{d}\mu(z) = 1, \tag{2}$$

$$w(x,y,z) = w(y,x,z) \quad \text{for } z \geq 0, \tag{3}$$

$$w(x,y,z) = w(x,z,y) \quad \text{for } z > 0. \tag{4}$$

We now define the generalized translation operator induced by (1). For $f \in L^1(\mu)$, the linear operator

$$\tau_y f(x) := \int_0^\infty f(z) w(x,y,z) \mathrm{d}\mu(z), \quad x, y \geq 0,$$

will be called Sturm-Liouville translation.

As a first remark, we note that the relations (2), (3) and (4) mean that

$$\tau_y f(x) = \tau_x f(y), \quad x, y \geq 0,$$

and

$$\int_0^\infty \tau_y f(x) \mathrm{d}\mu(x) = \int_0^\infty f(x) \mathrm{d}\mu(x), \quad f \in L^1(\mu). \tag{5}$$

Theorem 2 *For all $y \geq 0$ and $f \in L^p(\mu)$, $p \in [1, \infty]$, we have*

$$\|\tau_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}, \quad f \in L^p(\mu).$$

Proof. If $p = 1, \infty$, the result follows from (2), (3) and (4). Assume therefore that $p \in (1, \infty)$ and let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. We write

$$|f(z)| w(x,y,z) = |f(z)| [w(x,y,z)]^{1/p} [w(x,y,z)]^{1/p'}.$$

Applying Hölder's inequality and (2), we obtain

$$|\tau_y f(x)|^p \leq \int_0^\infty |f(z)|^p w(x,y,z) \mathrm{d}\mu(z), \quad x, y \geq 0.$$

This gives the result. ■

The Sturm-Liouville translation operator is connected with the Sturm-Liouville transform $\mathscr{F}$ via the following formula.

Theorem 3 *For $f \in L^2(\mu)$ and $y \geq 0$, we have*

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y) \mathscr{F}(f)(\lambda), \quad \lambda \geq 0. \tag{6}$$

Proof. Let $f \in L^1 \cap L^2(\mu)$. Then

$$\begin{aligned} \mathscr{F}(\tau_y f)(\lambda) &= \int_0^\infty \tau_y f(x) \varphi_\lambda(x) \mathrm{d}\mu(x) \\ &= \int_0^\infty \left[\int_0^\infty f(z) w(x,y,z) \mathrm{d}\mu(z) \right] \varphi_\lambda(x) \mathrm{d}\mu(x). \end{aligned}$$

By using Fubini's theorem, (3) and (4) we obtain

$$\mathscr{F}(\tau_y f)(\lambda) = \int_0^\infty f(z) \left[\int_0^\infty \varphi_\lambda(x) w(z,y,x) \mathrm{d}\mu(x) \right] \mathrm{d}\mu(z).$$

And by (1) we deduce that

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y) \mathscr{F}(f)(\lambda).$$

Since $L^1 \cap L^2(\mu)$ is dense in $L^2(\mu)$, the formula (6) remains valid for $f \in L^2(\mu)$. ■

Example 1 (The Bessel case ($\rho = 0$)**, see [1, 14])** *In this case* $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$ *and* L *is the Bessel operator denoted by* Δ_α*:*

$$\Delta_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}.$$

The Sturm-liouville kernel $\varphi_\lambda(x)$ *is the spherical Bessel function* $j_\alpha(\lambda x)$*. In particular* $\varphi_0(x) = j_\alpha(0) = 1$*. The two measures* μ *and* ν *will be denoted respectively by* μ_α *and* ν_α *and are given by*

$$d\mu_\alpha(x) = x^{2\alpha+1}dx \quad \textit{and} \quad d\nu_\alpha(\lambda) = \frac{\lambda^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\lambda.$$

The Sturm-Liouville transform is the Fourier-Bessel transform denoted by $\mathscr{F}_\alpha$*:*

$$\mathscr{F}_\alpha(f)(\lambda) = \int_0^\infty j_\alpha(\lambda x)f(x)d\mu_\alpha(x), \quad \lambda \geq 0.$$

The spherical Bessel function j_α *satisfies the product formula:*

$$j_\alpha(\lambda x)j_\alpha(\lambda y) = \int_0^\infty j_\alpha(\lambda z)w_\alpha(x,y,z)d\mu_\alpha(z), \quad \lambda, x, y \geq 0,$$

where $w_\alpha(x,y,.)$ *is the kernel given by*

$$w_\alpha(x,y,z) = a_\alpha\frac{[(x+y)^2 - z^2]^{\alpha-\frac{1}{2}}[z^2-(x-y)^2]^{\alpha-\frac{1}{2}}}{2^{2\alpha-1}(xyz)^{2\alpha}}\chi_{(|x-y|,x+y)}(z),$$

$$a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})},$$

where $\chi_{(|x-y|,x+y)}$ *is the characteristic function of the interval* $(|x-y|, x+y)$*. And the Bessel translation operator is given by*

$$\tau_y^\alpha f(x) := \int_0^\infty f(z)w_\alpha(x,y,z)d\mu_\alpha(z), \quad x, y \geq 0.$$

Example 2 (The Jacobi case ($\rho > 0$)**, see [8, 15])** *In this case* $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$*,* $\alpha > \beta \geq -1/2$ *and* $\rho = \alpha+\beta+1$*. The Sturm-Liouville operator* L *is the Jacobi operator denoted by* $\Delta_{\alpha,\beta}$*:*

$$\Delta_{\alpha,\beta} = \frac{d^2}{dx^2} + [(2\alpha+1)\coth(x) + (2\beta+1)\tanh(x)]\frac{d}{dx}.$$

The Sturm-liouville kernel $\varphi_\lambda(x)$ *is the Jacobi function denoted by* $\phi_\lambda^{(\alpha,\beta)}(x)$*:*

$$\phi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho+i\lambda), \alpha+1, -\sinh^2(x)),$$

where ${}_2F_1(a,b,c,z)$ *is the hypergeometric function. In particular*

$$\phi_0^{(\alpha,\beta)}(x) = {}_2F_1(\frac{1}{2}\rho, \frac{1}{2}\rho, \alpha+1, -\sinh^2(x)).$$

The two measures μ *and* ν *will be denoted respectively by* $\mu_{\alpha,\beta}$ *and* $\nu_{\alpha,\beta}$ *and are given by*

$$d\mu_{\alpha,\beta}(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)dx \quad \textit{and} \quad d\nu_{\alpha,\beta}(\lambda) = \frac{d\lambda}{2\pi|c_{\alpha,\beta}(\lambda)|^2},$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

The Sturm-Liouville transform is the Fourier-Jacobi transform denoted by $\mathscr{F}_{\alpha,\beta}$:

$$\mathscr{F}_{\alpha,\beta}(f)(\lambda)=\int_0^\infty \phi_\lambda^{(\alpha,\beta)}(x)f(x)d\mu_{\alpha,\beta}(x),\quad \lambda\geq 0.$$

The Jacobi function $\phi_\lambda^{(\alpha,\beta)}$ satisfies the product formula:

$$\phi_\lambda^{(\alpha,\beta)}(x)\phi_\lambda^{(\alpha,\beta)}(y)=\int_0^\infty \phi_\lambda^{(\alpha,\beta)}(z)w_{\alpha,\beta}(x,y,z)d\mu_{\alpha,\beta}(z),\quad \lambda,x,y\geq 0,$$

where $w_{\alpha,\beta}(x,y,.)$ is the kernel given by

$$\begin{aligned} w_{\alpha,\beta}(x,y,z) &= a_\alpha\frac{[\cosh(x)\cosh(y)\cosh(z)]^{-(\alpha+\beta+1}}{[\sinh(x)\sinh(y)\sinh(z)]^{2\alpha}}(1-B^2)^{\alpha-\frac12}\\ &\times\ {}_2F_1(\alpha+\beta,\alpha-\beta,\alpha+\frac12,\frac12(1-B))\chi_{(|x-y|,x+y)}(z),\end{aligned}$$

where

$$B=\frac{\cosh^2(x)+\cosh^2(y)+\cosh^2(z)-1}{2\cosh(x)\cosh(y)\cosh(z)}.$$

And the Jacobi translation operator is given by

$$\tau_y^{(\alpha,\beta)}f(x):=\int_0^\infty f(z)w_{\alpha,\beta}(x,y,z)d\mu_{\alpha,\beta}(z),\quad x,y\geq 0.$$

3 The Sturm-Liouville-Stockwell Transform

Let $f,g\in L^2(\nu)$. We define the convolution product $f\sharp g$ of f and g by

$$f\sharp g(\lambda):=\mathscr{F}(\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g))(\lambda).\tag{7}$$

In the same way as in ([13, page 238]), we obtain the following assertions.

Lemma 2 *(i) For $f\in L^1(\nu)$ and $g\in L^2(\nu)$, the function $f\sharp g$ belongs to $L^2(\nu)$, and*

$$\mathscr{F}^{-1}(f\sharp g)=\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g).$$

(ii) Let $f,g\in L^2(\nu)$. Then $f\sharp g$ belongs to $L^2(\nu)$ if and only if $\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g)$ belongs to $L^2(\mu)$, and

$$\mathscr{F}^{-1}(f\sharp g)=\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g),\quad \textit{in the } L^2(\mu)-\textit{case}.$$

(iii) Let $f,g\in L^2(\nu)$. Then

$$\int_0^\infty |f\sharp g(\lambda)|^2\,d\nu(\lambda)=\int_0^\infty |\mathscr{F}^{-1}(f)(x)|^2|\mathscr{F}^{-1}(g)(x)|^2\,d\mu(x),$$

where both sides are finite or infinite.

We assume that $g\in L^2(\nu)$ and $y\geq 0$. The modulation of g by y is the function

$$g_y:=\mathscr{F}\Big(\sqrt{\tau_y|\mathscr{F}^{-1}(g)|^2}\Big).$$

From (5) and Theorem 1(ii) we have

$$\|g_y\|_{L^2(\nu)}=\|g\|_{L^2(\nu)}.\tag{8}$$

Let $g\in L^2(\nu)$. The Sturm-Liouville-Stockwell transform is the mapping $\mathscr{S}_g$ defined for $f\in L^2(\nu)$ by

$$\mathscr{S}_g(f)(\lambda,y):=f\sharp g_y(\lambda),\quad \lambda,y\geq 0.\tag{9}$$

The Sturm-Liouville-Stockwell transform $\mathscr{S}_g$ possesses the following property.

Lemma 3 *Let $f, g \in L^2(\nu)$. Then*

$$\mathscr{S}_g(f)(\lambda, y) = \int_0^\infty \varphi_\lambda(x)\mathscr{F}^{-1}(f)(x)\sqrt{\tau_y|\mathscr{F}^{-1}(g)|^2(x)}\,d\mu(x).$$

Proof. From Theorem 1(iii) and (7) we have

$$\mathscr{S}_g(f)(\lambda, y) = \int_0^\infty \varphi_\lambda(x)\mathscr{F}^{-1}(f)(x)\mathscr{F}^{-1}(g_y)(x)\mathrm{d}\mu(x).$$

We obtain the result from the fact that

$$\mathscr{F}^{-1}(g_y)(x) = \sqrt{\tau_y|\mathscr{F}^{-1}(g)|^2(x)}. \tag{10}$$

The lemma is proved. ∎

Theorem 4 (Plancherel formula) *Let $g \in L^2(\nu)$ be a non-zero function. Then, for all $f \in L^2(\nu)$, we have*

$$\|\mathscr{S}_g(f)\|_{L^2(\nu\otimes\mu)} = \|g\|_{L^2(\nu)}\|f\|_{L^2(\nu)}.$$

Proof. From Lemma 2(iii) and (9), we obtain

$$\begin{aligned}\int_0^\infty\int_0^\infty |\mathscr{S}_g(f)(\lambda, y)|^2\mathrm{d}\nu(\lambda)\mathrm{d}\mu(y) &= \int_0^\infty\int_0^\infty |f\sharp g_y(\lambda)|^2\mathrm{d}\nu(\lambda)\mathrm{d}\mu(y)\\ &= \int_0^\infty\int_0^\infty |\mathscr{F}^{-1}(f)(x)|^2|\mathscr{F}^{-1}(g_y)(x)|^2\mathrm{d}\mu(x)\mathrm{d}\mu(y).\end{aligned}$$

Using Theorem 1(ii), (5), (10) and Fubini-Tonelli theorem, we deduce

$$\begin{aligned}\int_0^\infty\int_0^\infty |\mathscr{S}_g(f)(\lambda, y)|^2\mathrm{d}\nu(\lambda)\mathrm{d}\mu(y) &= \int_0^\infty\int_0^\infty |\mathscr{F}^{-1}(f)(x)|^2\tau_y|\mathscr{F}^{-1}(g)|^2(x)\mathrm{d}\mu(x)\mathrm{d}\mu(y)\\ &= \|g\|^2_{L^2(\nu)}\|f\|^2_{L^2(\nu)}.\end{aligned}$$

The theorem is proved. ∎

Theorem 5 (Inversion formula) *Let $g \in L^2(\nu)$ be a non-zero function. For all $f \in L^1 \cap L^2(\nu)$ such that $\mathscr{F}^{-1}(f) \in L^1(\mu)$, we have*

$$f(\xi) = \frac{1}{\|g\|^2_{L^2(\nu)}}\int_0^\infty \mathscr{S}_g(f)(., y)\sharp g_y(\xi)\,d\mu(y), \quad \xi \geq 0.$$

Proof. By Lemma 2(i), we have $\mathscr{S}_g(f)(., y) \in L^2(\nu)$. Then by (7), we obtain

$$\mathscr{S}_g(f)(., y)\sharp g_y(\xi) = \int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(\mathscr{S}_g(f)(., y))(x)\mathscr{F}^{-1}(g_y)(x)\mathrm{d}\mu(x).$$

But by Lemma 2(i) and (10), we have

$$\mathscr{F}^{-1}(\mathscr{S}_g(f)(., y))(x) = \mathscr{F}^{-1}(f)(x)\mathscr{F}^{-1}(g_y)(x) = \mathscr{F}^{-1}(f)(x)\sqrt{\tau_y|\mathscr{F}^{-1}(g)|^2(x)}.$$

Thus,

$$\mathscr{S}_g(f)(., y)\sharp g_y(\xi) = \int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(f)(x)\tau_y|\mathscr{F}^{-1}(g)|^2(x)\mathrm{d}\mu(x).$$

Therefore, by Fubini's theorem, Theorem 1(iii) and (5), we deduce that

$$\begin{aligned}\int_0^\infty \mathscr{S}_g(f)(.,y)\sharp g_y(\xi)\mathrm{d}\mu(y) &= \|g\|^2_{L^2(\nu)}\int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(f)(x)\mathrm{d}\mu(x)\\ &= \|g\|^2_{L^2(\nu)}f(\xi).\end{aligned}$$

This completes the proof of the theorem. ■

In the following we establish a reconstruction formula for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$. Let $0<a<b<\infty$ and let $f\in L^2(\nu)$, we define the reconstruction function $f_{a,b}$ associated with $\mathscr{S}_g$, by

$$f_{a,b}(\xi) := \frac{1}{\|g\|^2_{L^2(\nu)}}\int_a^b \mathscr{S}_g(f)(.,y)\sharp g_y(\xi)\mathrm{d}\mu(y), \quad \xi\geq 0.$$

Theorem 6 (Reconstruction formula) *Let $g\in L^2(\nu)$ be a non-zero function, such that $\mathscr{F}^{-1}(g)\in L^\infty(\mu)$. Then, for $f\in L^2(\nu)$, the function $f_{a,b}$ belongs to $L^2(\nu)$ and satisfies*

$$\lim_{\substack{a\to 0\\ b\to\infty}}\|f_{a,b}-f\|_{L^2(\nu)} = 0. \tag{11}$$

Proof. By Lemma 2(ii), $\mathscr{S}_g(f)(.,y)\in L^2(\nu)$, then by (7), we obtain

$$\mathscr{S}_g(f)(.,y)\sharp g_y(\xi) = \int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(\mathscr{S}_g(f)(.,y))(x)\mathscr{F}^{-1}(g_y)(x)\mathrm{d}\mu(x).$$

But by Lemma 2(ii) and (10), we have

$$\mathscr{F}^{-1}(\mathscr{S}_g(f)(.,y))(x) = \mathscr{F}^{-1}(f)(x)\mathscr{F}^{-1}(g_y)(x) = \mathscr{F}^{-1}(f)(x)\sqrt{\tau_y|\mathscr{F}^{-1}(g)|^2(x)}. \tag{12}$$

Thus,

$$\mathscr{S}_g(f)(.,y)\sharp g_y(\xi) = \int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(f)(x)\tau_y|\mathscr{F}^{-1}(g)|^2(x)\mathrm{d}\mu(x),$$

and

$$f_{a,b}(\xi) = \frac{1}{\|g\|^2_{L^2(\nu)}}\int_a^b\int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(f)(x)\tau_y|\mathscr{F}^{-1}(g)|^2(x)\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Then, by Fubini's theorem we get

$$f_{a,b}(\xi) = \int_0^\infty \varphi_\xi(x)\mathscr{F}^{-1}(f)(x)K_{a,b}(x)\mathrm{d}\mu(x), \tag{13}$$

where

$$K_{a,b}(x) = \frac{1}{\|g\|^2_{L^2(\nu)}}\int_a^b \tau_y|\mathscr{F}^{-1}(g)|^2(x)\mathrm{d}\mu(y).$$

From (5), it is easy to see that

$$\|K_{a,b}\|_{L^\infty(\mu)}\leq 1.$$

On the other hand, by Hölder's inequality, we deduce that

$$|K_{a,b}(x)|^2 \leq \frac{1}{\|g\|^4_{L^2(\nu)}}\left[\int_a^b \mathrm{d}\mu(y)\right]\left[\int_a^b |\tau_y|\mathscr{F}^{-1}(g)|^2(x)|^2\mathrm{d}\mu(y)\right].$$

Hence, by Theorem 2 we find

$$\begin{aligned}\|K_{a,b}\|^2_{L^2(\mu)} &\leq \frac{1}{\|g\|^4_{L^2(\nu)}}\left[\int_a^b \mathrm{d}\mu(y)\right]^2\left[\int_0^\infty |\mathscr{F}^{-1}(g)(x)|^4 \mathrm{d}\mu(x)\right]\\ &\leq \frac{\|\mathscr{F}^{-1}(g)\|^2_{L^\infty(\mu)}}{\|g\|^2_{L^2(\nu)}}\left[\int_a^b \mathrm{d}\mu(y)\right]^2.\end{aligned}$$

Thus $K_{a,b} \in L^\infty \cap L^2(\mu)$. Therefore and by (13), $f_{a,b} = \mathscr{F}(K_{a,b}\mathscr{F}^{-1}(f))$ and by Theorem 1(ii), $f_{a,b} \in L^2(\nu)$ and

$$\mathscr{F}^{-1}(f_{a,b}) = K_{a,b}\mathscr{F}^{-1}(f).$$

From this relation it follows that

$$\|f_{a,b} - f\|^2_{L^2(\nu)} = \int_0^\infty |\mathscr{F}^{-1}(f)(x)|^2(1 - K_{a,b}(x))^2 \mathrm{d}\mu(x).$$

But by (5) we have

$$\lim_{\substack{a\to 0\\ b\to\infty}} K_{a,b}(x) = 1, \qquad \text{for all } \; x \geq 0,$$

and

$$|\mathscr{F}^{-1}(f)(x)|^2(1 - K_{a,b}(x))^2 \leq |\mathscr{F}^{-1}(f)(x)|^2, \qquad \text{for all } \; x \geq 0.$$

So, the relation (11) follows from the dominated convergence theorem. ■

Example 3 (The Bessel case $(\rho = 0)$**, see [1])** *Let $f, g \in L^2(\mu_\alpha)$ the Bessel convolution of f and g is given by*

$$f\sharp_\alpha g(x) = \int_0^\infty f(t)\tau_x^\alpha g(t)\, d\mu_\alpha(t), \quad x \geq 0.$$

Let $g \in L^2(\mu_\alpha)$ and let $y \geq 0$. The modulation of g by y is the function:

$$g_y := \frac{1}{(2^\alpha\Gamma(\alpha+1))^2}\mathscr{F}_\alpha\Big(\sqrt{\tau_y^\alpha|\mathscr{F}_\alpha(g)|^2}\Big).$$

For $f, g \in L^2(\mu_\alpha)$, the Bessel-Stockwell transform is given by

$$\mathscr{S}_{\alpha,g}(f)(x,y) = f\sharp_\alpha g_y(x) = \int_0^\infty f(t)\tau_x^\alpha g_y(t)\, d\mu_\alpha(t), \quad x, y \geq 0.$$

The Bessel-Stockwell transform $\mathscr{S}_{\alpha,g}$ has the following properties:

$$\mathscr{S}_{\alpha,g}(f)(x,y) = \int_0^\infty j_\alpha(\lambda x)\mathscr{F}_\alpha(f)(\lambda)\sqrt{\tau_y^\alpha|\mathscr{F}_\alpha(g)|^2(\lambda)}\, d\nu_\alpha(\lambda),$$

and

$$\|\mathscr{S}_{\alpha,g}(f)\|_{L^2(\mu_\alpha\otimes\nu_\alpha)} = \|g\|_{L^2(\mu_\alpha)}\|f\|_{L^2(\mu_\alpha)}.$$

Let $0 < a < b < \infty$ and let $f \in L^2(\mu_\alpha)$, the reconstruction function $f_{a,b}$ associated with the Bessel-Stockwell transform $\mathscr{S}_{\alpha,g}$ is given by

$$f_{a,b}(z) := \frac{1}{\|g\|^2_{L^2(\mu_\alpha)}}\int_a^b \mathscr{S}_{\alpha,g}(f)(.,y)\sharp_\alpha g_y(z)\, d\nu_\alpha(y), \quad z \geq 0.$$

4 Approximate Inversion Formulas

In this section, by using the theory of the modified Sturm-Liouville transform $\mathscr{F}$, and building on the ideas of Saitoh [16, 17, 18], we give best approximate inversion formula for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$. Pointwise approximate inversion formulas for $\mathscr{S}_g$ are also discussed.

Let $s \geq 0$. We define the Strum-Liouville-Sobolev space of order s, that will be denoted $\mathscr{H}^s$, as the set of all $f \in L^2(\nu)$ such that $(1+x^2)^{s/2}\varphi_0^{-1}(x)\mathscr{F}^{-1}(f) \in L^2(\mu)$. The space $\mathscr{H}^s$ is equipped with the norm

$$\|f\|_{\mathscr{H}^s} := \left[\int_0^\infty (1+x^2)^s \varphi_0^{-2}(x)|\mathscr{F}^{-1}(f)(x)|^2 \mathrm{d}\mu(x)\right]^{1/2}.$$

Lemma 4 *Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$. The space $\mathscr{H}^s$ satisfies the inclusion*

$$\mathscr{F}^{-1}(\mathscr{H}^s) \subset L^1 \cap L^2(\mu),$$

and has the reproducing kernel

$$K_s(\lambda, \xi) = \int_0^\infty \frac{\varphi_\lambda(x)\varphi_\xi(x)\varphi_0^2(x)}{(1+x^2)^s} d\mu(x), \quad \lambda, \xi \geq 0.$$

Proof. Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$, and let $f \in \mathscr{H}^s$. From ([10, Lemma 21]), we have
- For $\rho > 0$,

$$A(x) \sim e^{2\rho x}, \quad (x \longrightarrow \infty).$$

- For $\rho = 0$,

$$A(x) \sim x^{2\alpha+1}, \quad (x \longrightarrow \infty).$$

Therefore and by Lemma 1, the integral $\int_0^\infty \frac{\varphi_\lambda(x)\varphi_\xi(x)\varphi_0^2(x)}{(1+x^2)^s} \mathrm{d}\mu(x)$ is convergent. Then the inclusion follows from the inequality

$$\|\mathscr{F}^{-1}(f)\|_{L^1(\mu)} \leq C_s\|f\|_{\mathscr{H}^s},$$

where

$$C_s := \left[\int_0^\infty \frac{\varphi_0^2(x)}{(1+x^2)^s} \mathrm{d}\mu(x)\right]^{1/2}.$$

On the other hand, from Theorem 1(iii), we have

$$\mathscr{F}^{-1}(K_s(.,\xi))(x) = \frac{\varphi_\xi(x)\varphi_0^2(x)}{(1+x^2)^s}, \quad x \geq 0.$$

By Lemma 1, we get

$$\|K_s(.,\xi)\|_{\mathscr{H}^s} = \left[\int_0^\infty \frac{|\varphi_\xi(x)|^2\varphi_0^2(x)}{(1+x^2)^s} \mathrm{d}\mu(x)\right]^{1/2} \leq C_s < \infty.$$

Moreover,

$$\langle f, K_s(.,\xi)\rangle_{\mathscr{H}^s} = \int_0^\infty \mathscr{F}^{-1}(f)(x)\varphi_\xi(x)\mathrm{d}\mu(x) = f(\xi).$$

This completes the proof of the lemma. ■

Let $g \in L^2(\nu)$. By Theorem 4, the mapping $\mathscr{S}_g$ is bounded from $\mathscr{H}^s$ into $L^2(\nu \otimes \mu)$, and

$$\|\mathscr{S}_g(f)\|_{L^2(\nu\otimes\mu)} \leq \|g\|_{L^2(\nu)}\|f\|_{\mathscr{H}^s}.$$

Let $\eta > 0$. We denote by $\langle ., . \rangle_{\eta,\mathscr{H}^s}$ the inner product defined on the space $\mathscr{H}^s$ by

$$\langle f, h\rangle_{\eta,\mathscr{H}^s} := \eta\langle f, h\rangle_{\mathscr{H}^s} + \langle \mathscr{S}_g(f), \mathscr{S}_g(h)\rangle_{L^2(\nu\otimes\mu)}.$$

We assume that $g \in L^2(\nu)$. By Theorem 4, the inner product $\langle .,. \rangle_{\eta,\mathscr{H}^s}$ can be written

$$\langle f,h\rangle_{\eta,\mathscr{H}^s} = \eta\langle f,h\rangle_{\mathscr{H}^s} + \|g\|^2_{L^2(\nu)}\langle f,h\rangle_{L^2(\nu)}.$$

Let $\eta > 0$ and let $g \in L^2(\nu)$. Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$. The space $\mathscr{H}^s$ equipped with the norm $\|.\|_{\eta,\mathscr{H}^s}$ has the reproducing kernel

$$K_{s,\eta,g}(\lambda,\xi) = \int_0^\infty \frac{\varphi_\lambda(x)\varphi_\xi(x)\varphi_0^2(x)}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}}\mathrm{d}\mu(x), \quad \lambda,\xi \geq 0.$$

Therefore, we have the functional equation

$$(\eta I + \mathscr{S}_g^*\mathscr{S}_g)K_{s,\eta,g}(.,\xi) = K_s(.,\xi), \quad \xi \geq 0,$$

where I is the unit operator and $\mathscr{S}_g^* : L^2(\nu\otimes\mu) \longrightarrow \mathscr{H}^s$ is the adjoint of $\mathscr{S}_g$.

For any $k \in L^2(\nu\otimes\mu)$ and for any $\eta > 0$, we define the extremal function $f^*_{\eta,k}$ by

$$f^*_{\eta,k}(\xi) := (\eta I + \mathscr{S}_g^*\mathscr{S}_g)^{-1}\mathscr{S}_g^*(k)(\xi), \quad \xi \geq 0. \tag{14}$$

The function $f^*_{\eta,k}$ is the unique solution (see [16]) of the Tikhonov regularization problem

$$\inf_{f\in\mathscr{H}^s}\Big\{\eta\|f\|^2_{\mathscr{H}^s} + \|k - \mathscr{S}_g(f)\|^2_{L^2(\nu\otimes\mu)}\Big\}.$$

This function possesses the following integral representation.

Theorem 7 *Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$, and let $g \in L^2(\nu)$. Then for any $k \in L^2(\nu\otimes\mu)$ and for any $\eta > 0$, we have*

$$f^*_{\eta,k}(\xi) = \int_0^\infty\int_0^\infty \frac{\varphi_\xi(x)\varphi_0^2(x)\mathscr{F}^{-1}(k(.,t))(x)\sqrt{\tau_t|\mathscr{F}^{-1}(g)|^2(x)}}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}}d\mu(x)d\mu(t).$$

Proof. From Lemma 4 and relation (14), we have

$$\begin{aligned} f^*_{\eta,k}(\xi) &= \langle(\eta I + \mathscr{S}_g^*\mathscr{S}_g)^{-1}\mathscr{S}_g^*(k), K_s(.,\xi)\rangle_{\mathscr{H}^s} \\ &= \langle\mathscr{S}_g^*(k), (\eta I + \mathscr{S}_g^*\mathscr{S}_g)^{-1}K_s(.,\xi)\rangle_{\mathscr{H}^s} \\ &= \langle\mathscr{S}_g^*(k), K_{s,\eta,g}(.,\xi)\rangle_{\mathscr{H}^s}. \end{aligned}$$

Hence

$$f^*_{\eta,k}(\xi) = \langle k, \mathscr{S}_g(K_{s,\eta,g}(.,\xi))\rangle_{L^2(\nu\otimes\mu)}. \tag{15}$$

By Lemma 1, the function $x \longrightarrow \dfrac{\varphi_\xi(x)\varphi_0^2(x)}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}}$ belongs to $L^1 \cap L^2(\mu)$. Then from Theorem 1(ii), it follows that $K_{s,\eta,g}(.,\xi)$ belongs to $L^2(\nu)$, and

$$\mathscr{F}^{-1}(K_{s,\eta,g}(.,\xi))(x) = \frac{\varphi_\xi(x)\varphi_0^2(x)}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}}, \quad x \geq 0. \tag{16}$$

By Lemma 3 and (16), we have

$$\begin{aligned} \mathscr{S}_g(K_{s,\eta,g}(.,\xi))(\lambda,t) &= \int_0^\infty \varphi_\lambda(x)\mathscr{F}^{-1}(K_{s,\eta,g}(.,\xi))(x)\sqrt{\tau_t|\mathscr{F}^{-1}(g)|^2(x)}\mathrm{d}\mu(x) \\ &= \int_0^\infty \varphi_\lambda(x)\varphi_\xi(x)\frac{\varphi_0^2(x)\sqrt{\tau_t|\mathscr{F}^{-1}(g)|^2(x)}}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}}\mathrm{d}\mu(x). \end{aligned}$$

Therefore,

$$f^*_{\eta,k}(\xi)=\int_0^\infty\int_0^\infty\int_0^\infty k(\lambda,t)\varphi_\lambda(x)\varphi_\xi(x)\frac{\varphi_0^2(x)\sqrt{\tau_t|\mathscr{F}^{-1}(g)|^2(x)}}{\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}}\mathrm{d}\mu(x)\mathrm{d}\nu(\lambda)\mathrm{d}\mu(t).$$

Thus, by Fubini's theorem, we obtain the result. ■

Remark 1 *Let $s>3/2$ when $\rho>0$ and $s>\alpha+1$ when $\rho=0$, and let $g\in L^2(\nu)$. Then for any $k\in L^2(\nu\otimes\mu)$ and for any $\eta>0$, we have*

(i) $|f^*_{\eta,k}(\xi)|\leq\frac{C_s}{2\sqrt{\eta}}\|k\|_{L^2(\nu\otimes\mu)}$, $\xi\geq 0$,

(ii) $\|f^*_{\eta,k}\|_{\mathscr{H}^s}\leq\frac{1}{2\sqrt{\eta}}\|k\|_{L^2(\nu\otimes\mu)}$.

Proof. (i) Indeed, from Theorem 4 and (15), we have

$$\begin{aligned}|f^*_{\eta,k}(\xi)| &\leq \|k\|_{L^2(\nu\otimes\mu)}\|\mathscr{S}_g(K_{s,\eta,g}(.,\xi))\|_{L^2(\nu\otimes\mu)}\\ &\leq \|k\|_{L^2(\nu\otimes\mu)}\|g\|_{L^2(\nu)}\|K_{s,\eta,g}(.,\xi)\|_{L^2(\nu)}.\end{aligned}$$

From Theorem 1(ii) and (16), we deduce

$$\begin{aligned}|f^*_{\eta,k}(\xi)| &\leq \|k\|_{L^2(\nu\otimes\mu)}\|g\|_{L^2(\nu)}\|\mathscr{F}^{-1}(K_{s,\eta,g}(.,\xi))\|_{L^2(\mu)}\\ &\leq \|k\|_{L^2(\nu\otimes\mu)}\|g\|_{L^2(\nu)}\left[\int_0^\infty\frac{|\varphi_\xi(x)|^2\varphi_0^4(x)}{[\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}]^2}\mathrm{d}\mu(x)\right]^{1/2}.\end{aligned}$$

Since $\left[\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}\right]^2\geq 4\eta(1+x^2)^s\varphi_0^2(x)\|g\|^2_{L^2(\nu)}$, we obtain the required result.

(ii) The function

$$x\longrightarrow\frac{\varphi_0^2(x)}{\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}}\int_0^\infty\mathscr{F}^{-1}(k(.,t))(x)\sqrt{\tau_t|\mathscr{F}^{-1}(g)|^2(x)}\mathrm{d}\mu(t),$$

belongs to $L^1\cap L^2(\mu)$. Then by Theorem 7, it follows that $f^*_{\eta,k}$ belongs to $L^2(\nu)$, and

$$\mathscr{F}^{-1}(f^*_{\eta,k})(x)=\frac{\varphi_0^2(x)}{\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}}\int_0^\infty\mathscr{F}^{-1}(k(.,t))(x)\sqrt{\tau_t|\mathscr{F}^{-1}(g)|^2(x)}\mathrm{d}\mu(t).$$

Thus, by Hölder's inequality and (8) we have

$$|\mathscr{F}^{-1}(f^*_{\eta,k})(x)|^2\leq\frac{\varphi_0^4(x)\|g\|^2_{L^2(\nu)}}{[\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}]^2}\int_0^\infty|\mathscr{F}^{-1}(k(.,t))(x)|^2\mathrm{d}\mu(t).$$

Thus,

$$\begin{aligned}\|f^*_{\eta,k}\|^2_{\mathscr{H}^s} &\leq \int_0^\infty\frac{(1+x^2)^s\varphi_0^2(x)\|g\|^2_{L^2(\nu)}}{[\eta(1+x^2)^s+\varphi_0^2(x)\|g\|^2_{L^2(\nu)}]^2}\left[\int_0^\infty|\mathscr{F}^{-1}(k(.,t))(x)|^2\mathrm{d}\mu(t)\right]\mathrm{d}\mu(x)\\ &\leq \frac{1}{4\eta}\int_0^\infty\left[\int_0^\infty|\mathscr{F}^{-1}(k(.,t))(x)|^2\mathrm{d}\mu(t)\right]\mathrm{d}\mu(x)=\frac{1}{4\eta}\|k\|^2_{L^2(\nu\otimes\mu)},\end{aligned}$$

which ends the proof. ■

We establish approximate inversion formula for the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$.

Theorem 8 *Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$, and let $g \in L^2(\nu)$. For $f \in \mathscr{H}^s$, the function $f^*_{\eta,\mathscr{S}_g(f)}$ belongs to $\mathscr{H}^s$ and satisfies*

$$\lim_{\eta\to 0^+} \|f^*_{\eta,\mathscr{S}_g(f)} - f\|_{\mathscr{H}^s} = 0.$$

Proof. By (12) and Theorem 7, we have

$$f^*_{\eta,\mathscr{S}_g(f)}(\xi) = \|g\|^2_{L^2(\nu)} \int_0^\infty \varphi_\xi(x) \frac{\varphi_0^2(x)\mathscr{F}^{-1}(f)(x)}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}} \mathrm{d}\mu(x), \quad y \geq 0. \tag{17}$$

Since $\mathscr{F}^{-1}(f) \in L^1 \cap L^2(\mu)$, from Theorem 1(iii) and (17), we deduce that

$$f^*_{\eta,\mathscr{S}_g(f)}(\xi) - f(\xi) = -\int_0^\infty \varphi_\xi(x) \frac{\eta(1+x^2)^s\mathscr{F}^{-1}(f)(x)}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}} \mathrm{d}\mu(x). \tag{18}$$

Thus,

$$\mathscr{F}^{-1}(f^*_{\eta,\mathscr{S}_g(f)} - f)(x) = -\frac{\eta(1+x^2)^s\mathscr{F}^{-1}(f)(x)}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}}.$$

Consequently,

$$\|f^*_{\eta,\mathscr{S}_g(f)} - f\|^2_{\mathscr{H}^s} = \int_0^\infty \frac{\eta^2(1+x^2)^{3s}\varphi_0^{-2}(x)|\mathscr{F}^{-1}(f)(x)|^2}{[\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}]^2} \mathrm{d}\mu(x).$$

Using the dominated convergence theorem and the fact that

$$\frac{\eta^2(1+x^2)^{3s}\varphi_0^{-2}(x)|\mathscr{F}^{-1}(f)(x)|^2}{[\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}]^2} \leq (1+x^2)^s\varphi_0^{-2}(x)|\mathscr{F}^{-1}(f)(x)|^2,$$

we deduce the result. ■

Remark 2 *Let $s > 3/2$ when $\rho > 0$ and $s > \alpha + 1$ when $\rho = 0$, and let $g \in L^2(\nu)$, and let $f \in \mathscr{H}^s$. By (18), we have*

$$|f^*_{\eta,\mathscr{S}_g(f)}(\xi) - f(\xi)| \leq \int_0^\infty \frac{\eta(1+x^2)^s|\mathscr{F}^{-1}(f)(x)|}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}} d\mu(x).$$

Using the dominated convergence theorem and the fact that

$$\frac{\eta(1+x^2)^s|\mathscr{F}^{-1}(f)(x)|}{\eta(1+x^2)^s + \varphi_0^2(x)\|g\|^2_{L^2(\nu)}} \leq |\mathscr{F}^{-1}(f)(x)|,$$

we obtain the following pointwise approximate inversion formula

$$\lim_{\eta\to 0^+} f^*_{\eta,\mathscr{S}_g(f)}(\xi) = f(\xi), \quad \xi \geq 0.$$

Example 4 (The Bessel case $(\rho = 0)$**)** *Let $s > \alpha + 1$. The Bessel-Sobolev space of order s, that will be denoted $\mathscr{H}^s_\alpha$, is the set of all $f \in L^2(\mu_\alpha)$ such that*

$$\|f\|_{\mathscr{H}^s_\alpha} = \left[\int_0^\infty (1+\lambda^2)^s|\mathscr{F}_\alpha(f)(\lambda)|^2 \, d\nu_\alpha(\lambda)\right]^{1/2} < \infty.$$

The Bessel-Sobolev space $\mathscr{H}^s_\alpha$ has the reproducing kernel

$$K_s(x,y) = \int_0^\infty \frac{j_\alpha(\lambda x)j_\alpha(\lambda y)}{(1+\lambda^2)^s} d\nu_\alpha(\lambda), \quad x, y \geq 0.$$

Let $\eta > 0$ and let $g \in L^2(\mu_\alpha)$. If the Bessel-Sobolev space $\mathscr{H}_\alpha^s$ is equipped with the inner product

$$\langle f, h\rangle_{\eta,\mathscr{H}_\alpha^s} = \eta\langle f, h\rangle_{\mathscr{H}_\alpha^s} + \|g\|^2_{L^2(\mu_\alpha)}\langle f, h\rangle_{L^2(\mu_\alpha)},$$

then $\mathscr{H}_\alpha^s$ has the reproducing kernel

$$K_{s,\eta,g}(x, y) = \int_0^\infty \frac{j_\alpha(\lambda x)j_\alpha(\lambda y)}{\eta(1+\lambda^2)^s + \|g\|^2_{L^2(\mu_\alpha)}} d\nu_\alpha(\lambda), \quad x, y \geq 0.$$

Then, we have the functional equation

$$(\eta I + \mathscr{S}^*_{\alpha,g}\mathscr{S}_{\alpha,g})K_{s,\eta,g}(., y) = K_s(., y), \quad y \geq 0,$$

where $\mathscr{S}^*_{\alpha,g} : L^2(\mu_\alpha \otimes \nu_\alpha) \longrightarrow \mathscr{H}_\alpha^s$ is the adjoint of $\mathscr{S}_{\alpha,g}$.

For any $k \in L^2(\mu_\alpha \otimes \nu_\alpha)$ and for any $\eta > 0$, the extremal function $f^*_{\eta,k}$ is given by

$$f^*_{\eta,k}(y) := (\eta I + \mathscr{S}^*_{\alpha,g}\mathscr{S}_{\alpha,g})^{-1}\mathscr{S}^*_{\alpha,g}(k)(y), \quad y \geq 0.$$

This function is the unique solution of the problem

$$\inf_{f\in\mathscr{H}_\alpha^s}\left\{\eta\|f\|^2_{\mathscr{H}_\alpha^s} + \|k - \mathscr{S}_{\alpha,g}(f)\|^2_{L^2(\mu_\alpha\otimes\nu_\alpha)}\right\},$$

and has the following integral representation

$$f^*_{\eta,k}(y) = \int_0^\infty\int_0^\infty \frac{j_\alpha(\lambda y)\mathscr{F}_\alpha(k(., t))(\lambda)\sqrt{\tau_t^\alpha|\mathscr{F}_\alpha(g)|^2(\lambda)}}{\eta(1+\lambda^2)^s + \|g\|^2_{L^2(\mu_\alpha)}} d\nu_\alpha(\lambda)d\nu_\alpha(t).$$

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