Applied Mathematics E-Notes, 25(2025), 45-56 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

On The Stabilization For The Wave Equation With Past History And Fractional Damping Controls*

Ahmed Adnane†, Abbes Benaissa‡, Smain Moulai Khatir§

Received 21 November 2023

Abstract

In this paper, we consider vibrating equation of wave type in one-dimensional bounded domain with complementary past history and fractional damping controls. We establish well-posedness and asymptotic stability results for the system under some conditions imposed into the relaxation functions.

1 Introduction

In this paper, we are concerned with the well-posedness, smoothness and asymptotic behavior of the solution of the following wave equation

$$(P)\qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \int_0^\infty g(s)u_{xx}(t-s)\,ds + \gamma\partial_t^{\alpha,\eta}u(x,t) = 0 & \text{in } (0,1)\times(0,+\infty),\\ u(0,t) = u(1,t) = 0 & \text{on } (0,+\infty),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{on } (0,1),\end{cases}$$

where g is a positive non-increasing function defined on $\mathbb{R}_+, \gamma > 0$. u_0 and u_1 are given initial data. The infinite integral term in (P) and $\gamma\partial_t^{\alpha,\eta}u$ represent, respectively, the past history (infinite memory) and the fractional damping. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha \le 1)$, with respect to the time variable (see [6]). It is defined as follows

$$\partial_t^{\alpha,\eta}u(t) = \begin{cases} u_t(t) & \text{for } \alpha = 1,\ \eta \ge 0,\\ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{du}{ds}(s)\,ds, & \text{for } 0 < \alpha < 1,\ \eta \ge 0.\end{cases}$$

During the last few years, many people have been interested in the question of stability of wave equation with various kinds of (internal or boundary) dampings. To focus on our motivation, let us mention here only some known results related to the stabilization with finite or infinite memory controls (for further results of stabilization, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

In the absence of an internal fractional damping (i.e $\gamma = 0$) and g satisfies

$$\exists\delta_1,\delta_2 > 0 : -\delta_1 g(s) \le g'(s) \le -\delta_2 g(s), \quad \forall s \in \mathbb{R}_+,$$

the authors of [5] proved that (P) is exponentially stable.

If g satisfies

$$\exists\delta > 0, \exists p \in]1, 3/2[: g'(s) \le -\delta_2 g^p(s), \quad \forall s \in \mathbb{R}_+,$$

it was proved in [9] that (P) is polynomially stable.

*Mathematics Subject Classifications: 35B40, 47D03, 74D05.

†Laboratory of Analysis and Control of PDEs, Djillali Liabes University, P. O. Box 89, Sidi Bel Abbes 22000, Algeria
‡Laboratory of Analysis and Control of PDEs, Djillali Liabes University, P. O. Box 89, Sidi Bel Abbes 22000, Algeria
§Laboratory of Analysis and Control of PDEs, Djillali Liabes University, P. O. Box 89, Sidi Bel Abbes 22000, Algeria

In the presence of fractional damping, the linear wave equation with the generalized Caputo fractional derivatives has been considered by Ammari et al. in [2]. More precisely, they studied the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \gamma \partial_t^{\alpha,\eta} u(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega. \end{cases}$$

The authors proved well-posedness and polynomial asymptotic stability as $t^{-2/(1-\alpha)}$ (for $\eta > 0$). To the best of our knowledge, a linear wave equation with infinite memory and internal fractional damping has not been studied yet.

Our goals in this paper are: Investigating the effect of each control on the asymptotic behavior of the solutions of (P) and on the decay rate of its energy and giving an explicit and general characterization of the decay rate depending on the growth of g and α.

The paper is organized as follows. In section 2, we give preliminary results and we reformulate the problem (P) into an augmented system by coupling the viscoelastic wave equation with a suitable diffusion equation. Then, we convert the system into an evolution equation in an appropriate Hilbert space and prove the well-posedness of our problem by semigroup theory. In section 3, we study asymptotic stability of above model and we establish for a smooth solution an uniform decay estimate for $\eta \neq 0$. Furthermore, when $\eta = 0$, we prove that the energy of the system decays polynomially to zero like as $1/t$.

2 Preliminary Results and Well-Posedness

The integral term represents a history term with kernel g satisfying the following hypothesis:

$$(H) \qquad \begin{cases} g : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a non-increasing differentiable function such that } \lim_{s\to 0^+} g(s) \\ \text{exists and there exists } c > 0 \text{ such that} \\ \qquad\qquad g'(s) \leq -cg(s). \\ \text{Furthermore, we assume that } 1 - g_0 > 0, \text{ where } g_0 = \int_0^{+\infty} g(s)\,ds. \end{cases}$$

In this section we reformulate (P) into an augmented system. For that, we need the following proposition.

Proposition 1 (see [8]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \; 0 < \alpha < 1. \tag{1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2 + \eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \; \eta \geq 0, \; t > 0, \tag{2}$$

$$\phi(\xi,0) = 0, \tag{3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi, \tag{4}$$

where $U \in C^0([0,+\infty))$, is given by

$$O = I^{1-\alpha,\eta} U, \tag{5}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\,d\tau.$$

We shall need the following lemma in all Sections.

Lemma 1 (see [1]) *If $\lambda \in D_\eta = \mathbb{C} \backslash]-\infty, -\eta]$, then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\,d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha-1}.$$

We introduce, as in [5], the new variables

$$\nu(x,t,s) = u(x,t) - u(x,t-s).$$

This functional satisfies

$$\begin{cases} \partial_t \nu + \partial_s \nu - u_t = 0 \text{ in }]0,1[\times \mathbb{R}_+ \times \mathbb{R}_+, \\ \nu(0,t,s) = \nu(1,t,s) = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \nu(x,t,0) = 0 \text{ in }]0,1[\times \mathbb{R}_+. \end{cases} \tag{6}$$

In order to convert our problem to a system of first-order ordinary differential equations, we note the following:

$$\nu^0(x,s) = \nu(x,0,s),$$

$$U = (u, u_t, \nu, \phi)$$

and

$$U^0 = (u_0, u_1, \nu^0, 0).$$

Then (P) is equivalent to the following abstract system:

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \tag{7}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \nu \\ \phi \end{pmatrix} = \begin{pmatrix} v \\ \left(1 - \int_0^\infty g(s)\,ds\right) u_{xx} + \int_0^\infty g(s)\nu_{xx}\,ds - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)d\xi \\ v - \partial_s \nu \\ -(\xi^2+\eta)\phi + v(x)\mu(\xi) \end{pmatrix}, \tag{8}$$

where $\zeta = (\pi)^{-1}\sin(\alpha\pi)\gamma$.

We introduce the following phase space (the energy space):

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H^* \times L^2((0,1)\times(-\infty,+\infty)), \tag{9}$$

where

$$H^* = \left\{ f : \mathbb{R}_+ \to H_0^1(0,1), \int_0^1 \int_0^\infty g(s)|f_x(s)|^2\,ds\,dx < +\infty \right\}. \tag{10}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{U \in \mathcal{H} \backslash \mathcal{A}U \in \mathcal{H}, \nu(x,t,0) = 0, |\xi|\phi \in L^2((0,1)\times(-\infty,+\infty))\}.$$

$\mathcal{H}$ is a Hilbert spaces equipped with the inner product defined by

$$\begin{aligned} \langle U, \tilde{U}\rangle_{\mathcal{H}} \;&=\; \int_\Omega \left(v\bar{\tilde{v}} + \left(1 - \int_0^\infty g(s)\,ds\right) u_x \bar{\tilde{u}}_x \right) dx + \int_0^1 \int_0^{+\infty} g(s)\nu_x(s)\bar{\tilde{\nu}}_x(s)\,ds\,dx \\ &\quad + \zeta \int_0^1 \int_{-\infty}^{+\infty} \phi\bar{\tilde{\phi}}\,d\xi dx. \end{aligned}$$

Now, the domain of $D(\mathcal{A})$ is dense in $\mathcal{H}$ and a simple computation implies that, for $U \in D(\mathcal{A})$,

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\Re \int_0^1 g(s) \int_0^{+\infty} \partial_s |\partial_x \nu|^2\,ds\,dx - \zeta \int_0^1 \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(x,\xi)|^2\,d\xi\,dx.$$

Integration by parts, using $(H1)$ and the boundary conditions in (6), yields

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^1 g'(s) \int_0^{+\infty} |\partial_x \nu|^2\,ds\,dx - \zeta \int_0^1 \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(x,\xi)|^2\,d\xi\,dx \tag{11}$$

and then, because the kernel g is non-increasing,

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq 0.$$

This implies that $\mathcal{A}$ is a dissipative operator. Next, we prove that $\lambda I - \mathcal{A}$ is surjective. Let $F = (f_1, f_2, f_3, f_3) \in \mathcal{H}$. We prove the existence of $U = (u, v, \nu, \phi) \in D(\mathcal{A})$ solution of the equation

$$(\lambda I - \mathcal{A})U = F. \tag{12}$$

Equation (12) is equivalent to

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - \left(1 - \int_0^\infty g(s)\, ds\right) u_{xx} - \int_0^\infty g(s)\nu_{xx}\, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)\, d\xi = f_2, \\ \lambda \nu - v + \partial_s \nu = f_3, \\ \lambda \phi + (\xi^2 + \eta)\phi - v(x)\mu(\xi) = f_4. \end{cases} \tag{13}$$

The first equation of (13) gives

$$v = \lambda u - f_1 \in H_0^1(0,1). \tag{14}$$

The last equation of (13) gives

$$\phi = \frac{f_4(x,\xi) + \mu(\xi)v(x)}{\xi^2 + \eta + \lambda} = \frac{f_4(x,\xi)}{\xi^2 + \eta + \lambda} + \frac{\lambda u(x)\mu(\xi)}{\xi^2 + \eta + \lambda} - \frac{f_1(x)\mu(\xi)}{\xi^2 + \eta + \lambda}. \tag{15}$$

The third equation of (13) and (14) give

$$\partial_s \nu + \lambda \nu = \lambda u - f_1 + f_3.$$

By integrating this differential equation and using the fact that $\nu(x, 0) = 0$, we get

$$\nu = \frac{1}{\lambda}(1 - e^{-\lambda s})(\lambda u - f_1) + \int_0^s e^{\lambda(\tau - s)} f_3(\tau)\, d\tau. \tag{16}$$

Inserting (14) into $(13)_2$, we get

$$\lambda^2 u - \left(1 - \int_0^\infty g(s)\, ds\right) u_{xx} - \int_0^\infty g(s)\nu_{xx}\, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi) d\xi = \lambda f_1 + f_2. \tag{17}$$

Inserting (15) and (16) into (17), we get

$$\begin{aligned} & (\lambda^2 + \gamma\lambda(\lambda+\eta)^{\alpha-1})u - \left(1 - \int_0^\infty g(s)e^{-\lambda s}\, ds\right) u_{xx} \\ = \; & \gamma(\lambda+\eta)^{\alpha-1} f_1 + \lambda f_1 + f_2 - \zeta \int_{-\infty}^{+\infty} \frac{f_4(x,\xi)\mu(\xi)}{\xi^2 + \eta + \lambda} d\xi - \frac{1}{\lambda} \int_0^\infty g(s)(1 - e^{-\lambda s})\, ds f_{1xx} \\ & + \int_0^\infty g(s) \int_0^s e^{\lambda(\tau - s)} f_{3xx}(\tau)\, d\tau\, ds. \end{aligned} \tag{18}$$

Solving equation (18) is equivalent to finding $u \in D(\mathcal{A})$ such that

$$a(u, w) = L(w) \quad \forall w \in H_0^1(0,1), \tag{19}$$

where the sesquilinear form $a : [H_0^1(0,1) \times H_0^1(0,1)] \to \mathbb{C}$ and the antilinear form $L : H_0^1(0,1) \to \mathbb{C}$ are defined by

$$a(u, w) = \int_0^1 ((\lambda^2 + \gamma\lambda(\lambda+\eta)^{\alpha-1})u\overline{w} + \left(1 - \int_0^\infty g(s)e^{-\lambda s}\, ds\right) u_x \overline{w}_x)\, dx,$$

$$L(w) = \int_0^1 (\gamma(\lambda+\eta)^{\alpha-1} f_1 + \lambda f_1 + f_2)\overline{w}\,dx - \zeta \int_0^1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_4(x,\xi)\,d\xi \overline{w}\,dx$$
$$+\frac{1}{\lambda}\int_0^\infty g(s)(1-e^{-\lambda s})\,ds \int_0^1 f_{1x}\overline{w}_x\,dx - \int_0^\infty g(s)\int_0^s e^{\lambda(\tau-s)}\int_0^1 f_{3x}(\tau)\overline{w}_x\,dx\,d\tau\,ds.$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H_0^1(0,1)$ problem (19) admits a unique solution $u \in H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (18) that $u \in H^2(0,1)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida theorem, we have the following result.

Theorem 1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (7) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (7) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

3 Stability of the System

3.1 Strong Stability of the System

In this subsection, we use a general criteria of Arendt-Batty [3] and Lyubich-Vu [7] to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (P) in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result in this part is the following theorem.

Theorem 2 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$, i.e, for all $U^0 \in \mathcal{H}$, the solution of (7) satisfies*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U^0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 2: First we will prove that the operator $\mathcal{A}$ has no eigenvalues on the imaginary axis. Then, we will establish that $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof for Theorem 2 relies on the subsequent lemmas.

Lemma 2 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Lemma 3 *We have*

$$\sigma(\mathcal{A}) \cap \{i\lambda,\ \lambda \in \mathbb{R},\ \lambda \neq 0\} = \emptyset.$$

Let us first prove Lemma 2.

Proof of Lemma 2. From (8) we get that $(u, v, \nu, \phi)^T \in Ker(\mathcal{A}) \subset D(\mathcal{A})$ if and only if

$$\begin{cases} -v = 0, \\ -\left(1 - \int_0^\infty g(s)\,ds\right) u_{xx} - \int_0^\infty g(s)\nu_{xx}\,ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)d\xi = 0, \\ -v + \partial_s \nu = 0, \\ (\xi^2+\eta)\phi - v(x)\mu(\xi) = 0. \end{cases} \tag{20}$$

This implies that $v = 0, \phi = 0$ and

$$\int_0^1 g'(s) \int_0^{+\infty} |\partial_x \nu|^2\,ds\,dx = 0.$$

Due to hypothesis (H), it follows that

$$\int_0^1 g(s)\int_0^{+\infty}|\partial_x\nu|^2\,ds\,dx=0.$$

This implies that

$$\nu=0.$$

Then, we have

$$u=cx+c'.$$

As $u(0)=u(1)=0$, we deduce that $u=0$. Thus $U=0$.

Let us suppose that there is $\lambda\in\mathbb{R}$, $\lambda\neq 0$ and $U\neq 0$, such that $\mathcal{A}U=i\lambda U$. Then, we get

$$\begin{cases} i\lambda u-v=0,\\ i\lambda v-\left(1-\int_0^\infty g(s)\,ds\right)u_{xx}-\int_0^\infty g(s)\nu_{xx}\,ds+\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)d\xi=0,\\ i\lambda\nu-v+\partial_s\nu=0,\\ i\lambda\phi+(\xi^2+\eta)\phi-v(x)\mu(\xi)=0.\end{cases}\tag{21}$$

Then, from (11) we have

$$\phi\equiv 0\tag{22}$$

and

$$\partial_x\nu=0.\tag{23}$$

From $(21)_4$ and $(21)_1$, we have

$$u=v=0.\tag{24}$$

Hence, from $(21)_3$ we obtain

$$\nu=0.\tag{25}$$

Thus $U=0$. This concludes the proof of Lemma 2. ∎

Now, we prove Lemma 3.

Proof of Lemma 3. We will prove that the operator $i\lambda I-\mathcal{A}$ is surjective for $\lambda\neq 0$. For this purpose, let $F=(f_1,f_2,f_3,f_4)^T\in\mathcal{H}$, we seek $U=(u,v,\nu,\phi)^T\in D(\mathcal{A})$ solution of solution of the following equation

$$(i\lambda-\mathcal{A})U=F.\tag{26}$$

Equivalently, we have the following system

$$\begin{cases} i\lambda u-v=f_1,\\ i\lambda v-\left(1-\int_0^\infty g(s)\,ds\right)u_{xx}-\int_0^\infty g(s)\nu_{xx}\,ds+\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)d\xi=f_2,\\ i\lambda\nu-v+\partial_s\nu=f_3,\\ i\lambda\phi+(\xi^2+\eta)\phi-v(x)\mu(\xi)=f_4.\end{cases}\tag{27}$$

The function u satisfies the following equation

$$\begin{aligned} &(-\lambda^2+i\gamma\lambda(i\lambda+\eta)^{\alpha-1})u-\left(1-\int_0^\infty g(s)e^{-i\lambda s}\,ds\right)u_{xx}\\ =\;&\gamma(i\lambda+\eta)^{\alpha-1}f_1+i\lambda f_1+f_2-\zeta\int_{-\infty}^{+\infty}\frac{f_4(x,\xi)\mu(\xi)}{\xi^2+\eta+i\lambda}d\xi\\ &-\frac{1}{i\lambda}\int_0^\infty g(s)(1-e^{-i\lambda s})\,ds f_{1xx}+\int_0^\infty g(s)\int_0^s e^{i\lambda(\tau-s)}f_{3xx}(\tau)\,d\tau\,ds.\end{aligned}\tag{28}$$

Then

$$\begin{aligned}
&\int_0^1 ((-\lambda^2 + i\gamma\lambda(i\lambda+\eta)^{\alpha-1})u\overline{w} + \left(1 - \int_0^\infty g(s)e^{-i\lambda s}\,ds\right)u_x\overline{w}_x)\,dx \\
= \;& \int_0^1 (\gamma(i\lambda+\eta)^{\alpha-1}f_1 + i\lambda f_1 + f_2)\overline{w}\,dx - \zeta\int_0^1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_4(x,\xi)\,d\xi\overline{w}\,dx \\
&+\frac{1}{i\lambda}\int_0^\infty g(s)(1-e^{-i\lambda s})\,ds\int_0^1 f_{1x}\overline{w}_x\,dx - \int_0^\infty g(s)\int_0^s e^{i\lambda(\tau-s)}\int_0^1 f_{3x}(\tau)\overline{w}_x\,dx\,d\tau\,ds
\end{aligned}$$

for all $w \in H_0^1(0,1)$. We can rewrite (28) as

$$\mathcal{B}(u,w) = l(w), \quad \forall w \in H_0^1(0,1), \tag{29}$$

where

$$\mathcal{B}(u,w) = \mathcal{B}_1(u,w) + \mathcal{B}_2(u,w)$$

with

$$(*) \qquad \begin{cases}\mathcal{B}_1(u,w) = \int_0^1((i\gamma\lambda(i\lambda+\eta)^{\alpha-1})u\overline{w} + \left(1-\int_0^\infty g(s)e^{-i\lambda s}\,ds\right)u_x\overline{w}_x)\,dx, \\ \mathcal{B}_2(u,w) = -\int_0^1 \lambda^2 u\overline{w}\,dx,\end{cases}$$

and

$$\begin{aligned}
l(w) \;=\; & \int_0^1 (\gamma(i\lambda+\eta)^{\alpha-1}f_1 + i\lambda f_1 + f_2)\overline{w}\,dx - \zeta\int_0^1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_4(x,\xi)\,d\xi\overline{w}\,dx \\
&+\frac{1}{i\lambda}\int_0^\infty g(s)(1-e^{-i\lambda s})\,ds\int_0^1 f_{1x}\overline{w}_x\,dx - \int_0^\infty g(s)\int_0^s e^{i\lambda(\tau-s)}\int_0^1 f_{3x}(\tau)\overline{w}_x\,dx\,d\tau\,ds.
\end{aligned}$$

Let $H^{-1}(0,1)$ be the dual space of $H_0^1(0,1)$. Let us define the following operators

$$(**) \qquad \begin{array}{ll} B: H_0^1(0,1) \to H^{-1}(0,1) & B_i: H_0^1(0,1) \to H^{-1}(0,1) \quad i\in\{1,2\} \\ \qquad u \mapsto Bu & \qquad u \mapsto B_i u \end{array}$$

such that

$$(***) \qquad \begin{array}{l}(Bu)w = \mathcal{B}(u,w),\ \forall w \in H_0^1(0,1), \\ (B_iu)w = \mathcal{B}_i(u,w),\ \forall w\in H_0^1(0,1),\ i\in\{1,2\}.\end{array}$$

We need to prove that the operator B is an isomorphism. For this aim, we divide the proof into three steps:
Step 1. In this step, we want to prove that the operator B_1 is an isomorphism. For this aim, it is easy to see that $\mathcal{B}_1$ is sesquilinear, continuous form on $H_0^1(0,1)$. Furthermore

$$\begin{aligned}
\Re\mathcal{B}_1(u,u) \;&=\; \left(1-\int_0^\infty g(s)\cos\lambda s\,ds\right)\|u_x\|_2^2 + \gamma\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right)\|u\|^2 \\
&\geq \left(1-\int_0^\infty g(s)\,ds\right)\|u_x\|_2^2,
\end{aligned}$$

where we have used the fact that

$$\gamma\lambda\Re\left(i(i\lambda+\eta)^{\alpha-1}\right) = \zeta\lambda^2\int_{-\infty}^{+\infty}\frac{\mu(\xi)^2}{\lambda^2+(\eta+\xi^2)^2}\,d\xi > 0.$$

Thus $\mathcal{B}_1$ is coercive. Then, from $(**)$ and Lax-Milgram theorem, the operator B_1 is an isomorphism.

Step 2. In this step, we want to prove that the operator B_2 is compact. For this aim, from $(*)$ and $(***)$, we have

$$|\mathcal{B}_2(u,w)| \leq c\|u\|_{L^2(0,1)}\|w\|_{L^2(0,1)},$$

and consequently, using the compact embedding from $H_0^1(0,1)$ to $L^2(0,1)$ we deduce that B_2 is a compact operator. Therefore, from the above steps, we obtain that the operator $B = B_1 + B_2$ is a Fredholm operator of index zero. Now, following Fredholm alternative, we still need to prove that the operator B is injective to obtain that the operator B is an isomorphism.

Step 3. Let $u \in ker(B)$, then

$$\mathcal{B}(u,w) = 0 \quad \forall w \in H_0^1(0,1). \tag{30}$$

In particular for $w = u$, it follows that

$$\lambda^2\|u\|^2_{L^2(0,1)} + i\gamma\lambda(i\lambda+\eta)^{\alpha-1}\|u\|^2_{L^2(0,1)} = \left(1 - \int_0^\infty g(s)e^{-i\lambda s}\, ds\right)\|u_x\|^2_{L^2(0,1)}.$$

Hence, we have that $i\lambda$ is an eigenvalue of the operator $\mathcal{A}$. Then, according to Lemma 2, we deduce that $u = 0$ and consequently $Ker(B) = \{0\}$. Finally, from Step 3 and Fredholm alternative, we deduce that the operator B is isomorphism. It is easy to see that the operator l is a antilinear and continuous form on $H_0^1(0,1)$. Consequently, (29) admits a unique solution $u \in H_0^1(0,1)$. By using the classical elliptic regularity, we deduce that $U \in D(\mathcal{A})$ is a unique solution of (26). Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$. ■

3.2 Exponential Stability (for $\eta \neq 0$)

In order to establish the exponential energy decay rate, we need the following theorem.

Theorem 3 ([10]) *Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Our main result is the following.

Theorem 4 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is exponentially stable, i.e. there exist positive constants $M \geq 1$ and $\omega > 0$ independent of U_0 such that*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq Me^{-\omega t}\|U_0\|^2_{D(\mathcal{A})} \quad \forall t \geq 0.$$

Proof. Given $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, let $U = (u, v, \nu, \phi)^T \in D(\mathcal{A})$ be the solution of the resolvent equation $(i\lambda I - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, i.e.,

$$\begin{cases} i\lambda u - v = f_1, \\ i\lambda v - \left(1 - \int_0^\infty g(s)\, ds\right) u_{xx} - \int_0^\infty g(s)\nu_{xx}\, ds + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)d\xi = f_2, \\ i\lambda\nu - v + \partial_s\nu = f_3, \\ i\lambda\phi + (\xi^2+\eta)\phi - v(x)\mu(\xi) = f_4. \end{cases} \tag{31}$$

Taking the real part of the inner product of $(i\lambda I - \mathcal{A})U$ with U in $\mathcal{H}$, we get

$$|Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}}| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{32}$$

Using (11), we deduce that

$$-\frac{1}{2}\int_0^1 g'(s)\int_0^{+\infty} |\partial_x\nu|^2\, ds\, dx + \zeta\int_0^1\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,x)|^2\, d\xi\, dx \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{33}$$

Using condition (H) into (33), we obtain

$$\int_0^1 g(s)\int_0^{+\infty} |\partial_x \nu|^2\,ds\,dx \le \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{34}$$

Multiplying $(31)_3$ by $\overline{u}$ in $L_g^2(\mathbb{R}_+, H_0^1)$, then using the fact that $\|u\|_g^2 = g_0\|u_x\|_2^2$, we get

$$\begin{aligned} g_0\|u_x\|_2^2 &= \int_0^1\int_0^{+\infty} g(s)\nu_x\overline{u}_x\,ds\,dx + \frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g(s)\nu_{sx}\overline{u}_x\,ds\,dx \\ &\quad -\frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g(s)(f_3-f_1)\overline{u}_{xx}\,ds\,dx. \end{aligned} \tag{35}$$

Using integration by parts, condition (H) and the fact that $\nu(x,0)=0$, we get

$$\frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g(s)\nu_{sx}\overline{u}_x\,ds\,dx = -\frac{1}{i\lambda}\int_0^1\int_0^{+\infty} g'(s)\nu_x\overline{u}_x\,ds\,dx.$$

Applying Hölder's inequality in $L^2(0,1)$ and $L^2(0,+\infty)$, then using (33) and that $\lim_{s\to 0}\sqrt{g(s)}$ exists, we obtain

$$\begin{aligned} \left|\frac{1}{\lambda}\int_0^1\int_0^{+\infty} g(s)\nu_{sx}\overline{u}_x\,ds\,dx\right| &\le \frac{\lim_{s\to 0}\sqrt{g(s)}}{|\lambda|}\left(\int_0^1\int_0^{+\infty} -g'(s)|\nu_x|^2\,ds\,dx\right)^{1/2}\|u_x\|_2 \\ &\le \frac{C}{|\lambda|}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{1/2}\|u_x\|_2. \end{aligned} \tag{36}$$

Using (34), we get

$$\begin{aligned} \left|\int_0^1\int_0^{+\infty} g(s)\nu_x\overline{u}_x\,ds\,dx\right| &\le g_0^{1/2}\left(\int_0^1\int_0^{+\infty} g(s)|\nu_x|^2\,ds\,dx\right)^{1/2}\|u_x\|_2 \\ &\le g_0^{1/2}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{1/2}\|u_x\|_2, \end{aligned} \tag{37}$$

$$\left|\frac{1}{\lambda}\int_0^1\int_0^{+\infty} g(s)(f_3-f_1)\overline{u}_{xx}\,ds\,dx\right| \le \frac{1}{|\lambda|}(g_0^{1/2}+g_0)\|F\|_{\mathcal{H}}\|u_x\|_2. \tag{38}$$

Using (35), (36), (37) and (38), we deduce that

$$\|u_x\|_2^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|_{\mathcal{H}}^2. \tag{39}$$

Multiplying $(31)_2$ by $\overline{u}$ and integrating over $(0,1)$ we get

$$\begin{aligned} &-\lambda^2\|u\|_2^2 + \left(1-\int_0^\infty g(s)\,ds\right)\|u_x\|_2^2 + \int_0^1\int_0^\infty g(s)\nu_x\overline{u}_x\,ds\,dx + \zeta\int_0^1 \overline{u}\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)d\xi\,dx \\ =&\int_0^1 (f_2+i\lambda f_1)\overline{u}\,dx. \end{aligned} \tag{40}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left|\int_0^1 \overline{u}\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi\,dx\right| &\le \left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac12}\|u\|_{L^2(0,1)}\left(\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(x,\xi)|^2\,dx\,d\xi\right)^{\frac12} \\ &\le \frac{1}{\sqrt{\zeta}}\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac12}\|u\|_{L^2(0,1)}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}})^{1/2}. \end{aligned} \tag{41}$$

Hence, from $(31)_1$, (40), (39) and (41), we obtain

$$\|v\|_{L^2(0,1)} \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C'\|F\|^2_{\mathcal{H}}. \tag{42}$$

Since $\eta > 0$, we have

$$\|\phi\|^2_{L^2((0,1)\times(-\infty,\infty))} \leq \frac{1}{\eta}\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(x,\xi)|^2\,d\xi\,dx \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{43}$$

Finally, by using (39), (42), (34) and (43), we get

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq C.$$

Applying Theorem 3, we obtain that

$$E(t) \leq Me^{-\omega t}\|U_0\|^2_{D(\mathcal{A})}.$$

■

3.3 Polynomial Stability (for $\eta = 0$)

3.3.1 Lack of Exponential Stability

Theorem 5 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(x\sin x\pi, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(u, v, \nu, \phi)^T$ the image of $(x\sin x\pi, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(x,\xi) = -|\xi|^{\frac{2\alpha-5}{2}}x\sin x\pi$. But, then $\phi \notin L^2((0,1)\times(-\infty,+\infty))$, since $\alpha \in]0,1[$. So $(u,v,\nu,\phi)^T \notin D(\mathcal{A})$. ■

By Theorem 5, 0 is a spectral point. Therefore it is convenient to have the following generalization of Theorem 3 at hand:

Theorem 6 ([4]) *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. Assume that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ and that there exist $\vartheta > 1$ and $\upsilon > 0$ such that*

$$\|(isI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \begin{cases} O(|s|^{-\vartheta}), & s \to 0, \\ O(|s|^{\upsilon}), & |s| \to \infty. \end{cases}$$

Then there exist constants $C, t_0 > 0$ such that for all $t \geq t_0$ and $U_0 \in D(\mathcal{A}) \cap R(\mathcal{A})$ we have

$$\|e^{\mathcal{A}t}U_0\|^2 \leq C\frac{1}{t^{\frac{2}{\varsigma}}}\|U_0\|^2_{D(\mathcal{A})\cap R(\mathcal{A})},$$

where $\varsigma = \max\{\vartheta, \upsilon\}$.

Our main result is the following.

Theorem 7 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A})\cap R(\mathcal{A})}.$$

Now, from $(31)_4$, we obtain

$$v(x)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_4(x,\xi). \tag{44}$$

By multiplying (44) by $(i\lambda + \xi^2 + \eta)^{-2}|\xi|$, we get

$$(i\lambda + \xi^2 + \eta)^{-2}v(x)\mu(\xi)|\xi| = (i\lambda + \xi^2 + \eta)^{-1}|\xi|\phi - (i\lambda + \xi^2 + \eta)^{-2}|\xi|f_4(x,\xi). \tag{45}$$

Hence, by taking absolute values of both sides of (45), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ and applying Cauchy-Schwarz inequality, we obtain

$$\mathcal{S}|v(x)| \leq \sqrt{2}\mathcal{U}\left(\int_{-\infty}^{+\infty} \xi^2|\phi|^2\, d\xi\right)^{\frac{1}{2}} + 2\mathcal{V}\left(\int_{-\infty}^{+\infty} |f_4(x,\xi)|^2\, d\xi\right)^{\frac{1}{2}}, \tag{46}$$

where

$$\mathcal{S} = \left|\int_{-\infty}^{+\infty} (i\lambda+\xi^2+\eta)^{-2}|\xi|\mu(\xi)\, d\xi\right| = \frac{|1-2\alpha|}{4}\frac{\pi}{|\sin\frac{(2\alpha+3)}{4}\pi|}|i\lambda+\eta|^{\frac{(2\alpha-5)}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} (|\lambda|+\xi^2+\eta)^{-2}\, d\xi\right)^{\frac{1}{2}} = (\frac{\pi}{2})^{1/2}||\lambda|+\eta|^{-\frac{3}{4}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|\lambda|+\xi^2+\eta)^{-4}|\xi|^2\, d\xi\right)^{\frac{1}{2}} = \left(\frac{\pi}{16}||\lambda|+\eta|^{-\frac{5}{2}}\right)^{1/2}.$$

Thus, by using the inequality $2PQ \leq P^2+Q^2$, $P\geq 0$, $Q\geq 0$, again, we get

$$\mathcal{S}^2\int_0^1 |v(x)|^2\, dx \leq 2\mathcal{U}^2\left(\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi|^2\, d\xi\, dx\right) + 4\mathcal{V}^2\left(\int_0^1\int_{-\infty}^{+\infty}|f_4(x,\xi)|^2\, d\xi\, dx\right). \tag{47}$$

We deduce that For λ near 0, we have from (47)

$$\int_0^1 |v(x)|^2\, dx \leq c|\lambda|^{1-\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c|\lambda|^{-\alpha}\|F\|^2_{\mathcal{H}}, \tag{48}$$

$$\begin{aligned}
\|\phi\|^2 &\leq 2\int_0^1 |v(x)|^2\, dx\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{|i\lambda+\xi^2|^2}\, d\xi + 2\int_0^1\int_{-\infty}^{+\infty}\frac{|f_4(x,\xi)|^2}{|i\lambda+\xi^2|^2}\, d\xi\, dx\\
&\leq 4\int_0^1 |v(x)|^2\, dx\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{(|\lambda|+\xi^2)^2}\, d\xi + 4\int_0^1\int_{-\infty}^{+\infty}\frac{|f_4(x,\xi)|^2}{(|\lambda|+\xi^2)^2}\, d\xi\, dx\\
&\leq 4(1-\alpha)\frac{\pi}{\sin\alpha\pi}\|v\|^2_{L^2(0,1)}|\lambda|^{(\alpha-2)} + 4|\lambda|^{-2}\|f_4\|^2_{L^2((0,1)\times(-\infty,+\infty))}.
\end{aligned} \tag{49}$$

Then

$$\|\phi\|^2_{L^2((0,1)\times(-\infty,+\infty))} \leq c|\lambda|^{-1}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c|\lambda|^{-2}\|F\|^2_{\mathcal{H}}.$$

Finally, we deduce that

$$\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{H}} \leq \frac{C}{|\lambda|^2} \text{ as } \lambda\to 0.$$

Applying Theorem 6, we obtain that

$$E(t) \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A})\cap R(\mathcal{A})}.$$

Acknowledgments. The authors would like to thank very much the referee for their constructive comments and suggestions that helped to improve this article.

References

[1] Z. Achouri, N. Amroun and A. Benaissa, The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type, Math. Methods Appl. Sci., 40(2017), 3837–3854.

[2] K. Ammari, H. Fathi and L. Robbiano, Fractional-feedback stabilization for a class of evolution systems, J. Differential Equations, 268(2020), 5751–5791.

[3] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Amer. Math. Soc., 306(1988), 837–852.

[4] C. J. K. Batty, R. Chill and Y. Tomilov, Fine scales of decay of operator semigroups, J. Eur. Math. Soc., 18(2016), 853–929.

[5] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, Electron. J. Differential Equations, (2002), 14 pp.

[6] J. U. Choi and R. C. Maccamy, Fractional order Volterra equations with applications to elasticity, J. Math. Anal. Appl., 139(1989), 448–464.

[7] I. L. Yu and V. Q. Phóng, Asymptotic stability of linear differential equations in Banach spaces, Studia Math., 88(1988), 37–42.

[8] B. Mbodje, Wave energy decay under fractional derivative controls, IMA J. Math. Control Inform., 23(2006), 237–257.

[9] S. A. Messaoudi and B. Said-Houari, Uniform decay in a Timoshenko-type system with past history, J. Math. Anal. Appl., 360(2009), 459–475.

[10] J. Prüss, On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc., 284(1984), 847–857.