Applied Mathematics E-Notes, 25(2025), 73-87 © ISSN 1607-2510
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Existence Of Extremal Solutions For A Class Of Functional Fractional Differential Equations With Maxima*

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Received 11 November 2023

Abstract

The main goal of this study is to construct extremal solutions for a class of fractional differential equations with maxima. We also give some examples to show what our results mean.

1 Introduction

The purpose of this work is to study the existence and uniqueness of solutions for the following problem

$$\begin{cases} ({}^{C}\mathfrak{D}_{0^+}^{\alpha}u)(t) = f(t, u(t), \max\limits_{s\in[t-r,t]} u(s)), \ t \in J = [0, T], \\ u(t) = \varphi(t), \ t \in [-r, 0], \end{cases} \tag{1}$$

where ${}^{C}\mathfrak{D}_{0^+}^{\alpha}$ is the Caputo fractional derivative of order α with $0 < \alpha \leq 1$, $T > 0$ and $r > 0$, $f : J\times\mathbb{R}\times C \to \mathbb{R}$ with $C = C([-r, T], \mathbb{R})$ and $\varphi : [-r, 0] \to \mathbb{R}$ are continuous.

It was automatic control that first used differential equations with maxima and differential inequalities with maxima. They were used to study systems with saturation and the stability of equations with retarded argument (see [27] and [19, Chapter 4 Section 5]). We also want to point out that differential equations with maxima show up in a lot of different fields, like psychology, the dynamic model for happiness, the theory of lateral inhibition, optimal control, and the economy (see [4], [5], [7], [9], [11], [18], [20], [21], [23], [25, Introduction] and [31]).

On the other hand, several authors have studied differential equations with maxima, fractional differential equations with maxima, and fractional integral equations with maxima using Banach's fixed point theorem, Leray-Schauder topological degree theory, weakly Picard operator theory, fixed point theorems associated with the measure of weak noncompactness, the upper and lower solutions method coupled with a monotone iterative technique, and numerical methods (see [1], [2], [3], [6], [7], [10], [12], [15], [16], [17], [26], [29], [30], [32] and [33] and the references cited therein).

A number of authors (see [6], [7] and [17]) are known for using the monotone iterative method and the method of upper and lower solutions to show that there are solutions for first-order differential equations with maxima. The work aims to demonstrate its successful application to problems of type (1), assuming an increase in nonlinearity f with respect to its third variable. Furthermore, we provide sufficient conditions that guarantee that problem (1) has a unique solution.

The structure of the present paper is as follows: In Section 2, we give some definitions and preliminary results. Section 3 presents and proves the main results, while Section 4 illustrates their application through some examples.

*Mathematics Subject Classifications: 34A08, 34K37.

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2 Definitions and Preliminary Results

This section provides definitions and preliminary results for the rest of the article.

Definition 1 (See [28, Chapter 1 page 33]) *Let $0 < q \leq 1$ and $h \in L^1(J, \mathbb{R})$. The Riemann-Liouville integral of order q of h is defined by*

$$\left(I_{0^+}^q h\right)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds, \text{ for a.e. } t \in J,$$

where Γ is the Gamma Euler function defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt,$$

where $x \in \mathbb{R}$ with $x > 0$.

Remark 1 *If $q = 0$, we put by definition $\left(I_{0^+}^0 h\right)(t) = h(t)$.*

Notation 1 *By $AC(J, \mathbb{R})$ we denote the set of real functions f which are absolutely continuous on J.*

Definition 2 (See [13, Chapter 3 page 50] or [22, Chapter 2 page 91]) *Let $0 < q \leq 1$ and $h \in AC(J, \mathbb{R})$. The Caputo fractional derivative of order q of h is defined by*

$$\left({}^{C}\mathfrak{D}_{0^+}^q h\right)(t) = \left({}^{RL}\mathfrak{D}_{0^+}^q \left(h - h(0)\right)\right)(t), \text{ for a.e. } t \in J,$$

where ${}^{RL}\mathfrak{D}_{0^+}^q$ is the Riemann-Liouville fractional derivative defined by

$$\begin{aligned} \left({}^{RL}\mathfrak{D}_{0^+}^q h\right)(t) &= \frac{d}{dt}\left(I_{0^+}^{1-q} h\right)(t) \\ &= \begin{cases} \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} h(s)\, ds \text{ if } 0 < q < 1, \\ h'(t) \text{ if } q = 1. \end{cases} \end{aligned}$$

Now, we consider the following Cauchy problem

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^q u\right)(t) = -Lu(t) + \widetilde{g}(t), \ t \in J, \\ u(0) = \xi, \end{cases} \tag{2}$$

where $0 < q \leq 1$, $L \geq 0$, $\xi \in \mathbb{R}$ and $\widetilde{g} : J \to \mathbb{R}$ is continuous.

Lemma 1 (See [22, Chapter 4 page 231]) *The problem (2) admits a unique solution u, which is given by*

$$u(t) = \xi E_q(-Lt^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-L(t-s)^q)\widetilde{g}(s) ds, \text{ for all } t \in J,$$

where E_q and $E_{q,q}$ are the Mittag-Leffler functions defined by

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(qn+1)}, \ x \in \mathbb{R},$$

and

$$E_{q,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(q(n+1))}, \ x \in \mathbb{R}.$$

Theorem 1 (See [8, Theorem 4.2]) *For $0 < q \leq 1$, E_q has no zeros on the real axis, $0 < E_q(x) < 1$ for $x < 0$, and $\frac{d}{dx}E_q(x) > 0$ for the whole real axis.*

Lemma 2 (See [24]) *For $0 < q \leq 1$, one has*

(i) $E_{q,q}(-x) = -q\frac{d}{dx}E_q(-x)$, *for all* $x \geq 0$,

(ii) $E_{q,q}(-x) > 0$, *for all* $x \geq 0$.

Lemma 3 *Assume that $u \in C(J,\mathbb{R})$ with ${}^C\mathfrak{D}^q_{0+}u \in C(J,\mathbb{R})$ satisfying*

$$\begin{cases} \left({}^C\mathfrak{D}^q_{0+}u\right)(t) + Lu(t) \leq 0, \text{ for all } t \in J, \\ u(0) \leq 0, \end{cases}$$

where $0 < q \leq 1$ and $L \geq 0$. Then $u(t) \leq 0$, for all $t \in J$.

Proof. The proof follows immediately from Lemma 1, Theorem 1, and (ii) of Lemma 2. ■

Lemma 4 (See [14, Corollary 2.4]) *Let $0 < q \leq 1$ and $b > 0$ and assume $u \in C([0,b],\mathbb{R})$ with ${}^C D^q_{0+}u \in C([0,b],\mathbb{R})$. Then there exists some c in $(0,b)$ such that*

$$\frac{u(b) - u(0)}{b^q} = \frac{\left({}^C\mathfrak{D}^q_{0+}u\right)(c)}{\Gamma(q+1)}.$$

We have the following result.

Lemma 5 *Assume that $u \in C([-r,T],\mathbb{R})$ with ${}^C\mathfrak{D}^q_{0+}u \in C(J,\mathbb{R})$ satisfying*

$$\begin{cases} \left({}^C\mathfrak{D}^q_{0+}u\right)(t) \leq -\widetilde{M_1}u(t) - \widetilde{N_1}\max\limits_{s\in[t-r,t]} u(s), \ t \in J, \\ u(t) \leq 0, \text{ for all } t \in [-r,0], \end{cases}$$

where $0 < q \leq 1$, $\widetilde{M_1} \leq 0$ and $\widetilde{N_1} \leq 0$. If

$$-\left(\widetilde{M_1} + \widetilde{N_1}\right)\frac{T^q}{\Gamma(1+q)} < 1,$$

then $u(t) \leq 0$, for all $t \in [-r,T]$.

Proof. Assume that there exists $t_1 \in (0,T]$ such that

$$u(t_1) > 0.$$

We put by definition

$$u(t_*) = \max_{t\in[-r,t_1]} u(t) > 0,$$

where $t_* \in (0,t_1]$.

We have

$$\left({}^C\mathfrak{D}^q_{0+}u\right)(t) \leq -\widetilde{M_1}u(t) - \widetilde{N_1}\max_{s\in[t-r,t]} u(s), \ t \in J.$$

Which implies

$$\left({}^C\mathfrak{D}^q_{0+}u\right)(t) \leq -\left(\widetilde{M_1} + \widetilde{N_1}\right)u(t_*), \ t \in [0,t_1].$$

From Lemma 4, there exists $\xi \in (0,t_*)$ such that

$$u(t_*) - u(0) = t_*^q\frac{\left({}^C\mathfrak{D}^q_{0+}u\right)(\xi)}{\Gamma(q+1)}.$$

Then we have

$$u(t_*) - u(0) \le -t_*^q \frac{\left(\widetilde{M_1} + \widetilde{N_1}\right) u(t_*)}{\Gamma(q+1)}.$$

This implies

$$u(t_*) \le -\frac{\left(\widetilde{M_1} + \widetilde{N_1}\right) u(t_*)}{\Gamma(1+q)} T^q.$$

Since $u(t_*) > 0$, we get

$$1 \le -\left(\widetilde{M_1} + \widetilde{N_1}\right) \frac{T^q}{\Gamma(1+q)}.$$

Which is a contradiction with the assumption

$$-\left(\widetilde{M_1} + \widetilde{N_1}\right) \frac{T^q}{\Gamma(1+q)} < 1,$$

and consequently, we deduce that

$$u(t) \le 0, \text{ for all } t \in [-r, T].$$

■

Lemma 6 (See [30, Theorem 1 page 44]) *If the functions $u : [c,d] \to \mathbb{R}$ and $v : [c,d] \to \mathbb{R}$ are continuous on the segment $[c,d]$, then we have*

$$\left| \max_{t\in[c,d]} u(t) - \max_{t\in[c,d]} v(t) \right| \le \max_{t\in[c,d]} |u(t) - v(t)|.$$

3 Main Results

In this section, we give some definitions, state, and prove our results.

Definition 3 *We say that $\underline{u}$ is a lower solution of (1) if*

i) $\underline{u} \in C([-r,T], \mathbb{R})$ *with* ${}^C\mathfrak{D}_{0+}^{\alpha} \underline{u} \in C(J, \mathbb{R})$.

ii) $\begin{cases} ({}^C\mathfrak{D}_{0+}^{\alpha} \underline{u})(t) \le f(t, \underline{u}(t), \max\limits_{s\in[t-r,t]} \underline{u}(s)), \ t \in J, \\ \underline{u}(t) \le \varphi(t), \ t \in [-r, 0]. \end{cases}$

Definition 4 *We say that $\overline{u}$ is an upper solution of (1) if*

i) $\overline{u} \in C([-r,T], \mathbb{R})$ *with* ${}^C\mathfrak{D}_{0+}^{\alpha} \overline{u} \in C(J, \mathbb{R})$.

ii) $\begin{cases} ({}^C\mathfrak{D}_{0+}^{\alpha} \overline{u})(t) \ge f(t, \overline{u}(t), \max\limits_{s\in[t-r,t]} \overline{u}(s)), \ t \in J, \\ \overline{u}(t) \ge \varphi(t), \ t \in [-r, 0]. \end{cases}$

Definition 5 *We say that u is a solution of (1) if $u \in C([-r,T], \mathbb{R})$ with ${}^C\mathfrak{D}_{0+}^{\alpha} u \in C(J, \mathbb{R})$ and satisfies (1).*

We have the following result.

Theorem 2 *Let $\underline{u}$ and $\overline{u}$ be lower and upper solutions respectively for problem (1) such that $\underline{u} \le \overline{u}$ in $[-r, T]$ and assume that there exist a constant $M \ge 0$ satisfying*

(H1) $f(t,x_1,y) - f(t,x_2,y) \geq -M(x_1 - x_2)$, *for all* $t \in J$, $\underline{u}(t) \leq x_2 \leq x_1 \leq \overline{u}(t)$ *and* $\max\limits_{s\in[t-r,t]} \underline{u}(s) \leq y \leq \max\limits_{s\in[t-r,t]} \overline{u}(s)$.

(H2) *The function* $y \mapsto f(t,x,y)$ *is increasing for all* $t \in J$, $\underline{u}(t) \leq x \leq \overline{u}(t)$ *and* $\max\limits_{s\in[t-r,t]} \underline{u}(s) \leq y \leq \max\limits_{s\in[t-r,t]} \overline{u}(s)$.

Then the problem (1) has a minimal solution u_* *and a maximal solution* u^* *such that for every solution* u *of (1) with* $\underline{u} \leq u \leq \overline{u}$ *in* $[-r,T]$, *we have*

$$\underline{u} \leq u_* \leq u \leq u^* \leq \overline{u} \text{ in } [-r,T].$$

Proof. We take $\underline{u}_0 = \underline{u}$, and we define the sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} \left({}^C\mathfrak{D}^\alpha_{0+}\underline{u}_{n+1}\right)(t) + M\underline{u}_{n+1}(t) = f_n(t)\,,\ t \in J, \\ \underline{u}_{n+1}(t) = \varphi(t)\,,\ t \in [-r,0]\,, \end{cases} \tag{3}$$

where

$$f_n(t) = f(t, \underline{u}_n(t)\,, \max_{s\in[t-r,t]} \underline{u}_n(s)) + M\underline{u}_n(t)\,.$$

Analogously, we take $\overline{u}_0 = \overline{u}$ and we define the sequence of functions $(\overline{u}_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} \left({}^C\mathfrak{D}^\alpha_{0+}\overline{u}_{n+1}\right)(t) + M\overline{u}_{n+1}(t) = \widetilde{f}_n(t)\,,\ t \in J, \\ \overline{u}_{n+1}(t) = \varphi(t)\,,\ t \in [-r,0]\,, \end{cases} \tag{4}$$

where

$$\widetilde{f}_n(t) = f(t, \overline{u}_n(t)\,, \max_{s\in[t-r,t]} \overline{u}_n(s)) + M\overline{u}_n(t)\,.$$

First we note that from Lemma 1, it follows that the sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ and $(\overline{u}_n)_{n\in\mathbb{N}}$ are well defined and for all $n \in \mathbb{N}$, we have

$$\underline{u}_{n+1}(t) = \begin{cases} \varphi(t)\,, \text{ if } t \in [-r,0]\,, \\ \varphi(0)E_\alpha(-Mt^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-M(t-s)^\alpha) f_n(s)\,ds, \text{ if } t \in J, \end{cases}$$

and

$$\overline{u}_{n+1}(t) = \begin{cases} \varphi(t)\,, \text{ if } t \in [-r,0]\,, \\ \varphi(0)E_\alpha(-Mt^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-M(t-s)^\alpha) \widetilde{f}_n(s)\,ds, \text{ if } t \in J. \end{cases}$$

Step 1: For all $n \in \mathbb{N}$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } [-r,T]\,.$$

Let

$$w_0(t) := \underline{u}_0(t) - \underline{u}_1(t)\,, t \in [-r,T]\,.$$

By (3) and using the definition of lower solution, we have

$$\begin{cases} \left({}^C\mathfrak{D}^\alpha_{0+}w_0\right)(t) + Mw_0(t) \leq 0, t \in J, \\ w_0(0) \leq 0. \end{cases}$$

Then from Lemma 3, we get

$$w_0(t) \leq 0, \text{ for all } t \in [0,T]\,,$$

and since

$$w_0(t) \leq 0, \text{ for all } t \in [-r,0]\,,$$

we obtain

$$w_0\left(t\right) \leq 0 \text{ for all } t \in \left[-r, T\right].$$

That is

$$\underline{u}_0 \leq \underline{u}_1 \text{ in } \left[-r, T\right]. \tag{5}$$

Similarly, we can prove that

$$\overline{u}_1 \leq \overline{u}_0 \text{ in } \left[-r, T\right]. \tag{6}$$

Now, we put by definition

$$p_1\left(t\right) = \underline{u}_1\left(t\right) - \overline{u}_1\left(t\right),\ t \in \left[-r, T\right].$$

Combining (3) and (4), we obtain

$$\left({}^C\mathfrak{D}_{0+}^{\alpha} p_1\right)\left(t\right) + Mp_1\left(t\right) = f_0\left(t\right) - \widetilde{f}_0\left(t\right), \text{ for all } t \in J.$$

Since $\underline{u}_0 = \underline{u} \leq \overline{u} = \overline{u}_0$ in $[-r, T]$ and using the hypothesis (H1) , we deduce that

$$\left({}^C\mathfrak{D}_{0+}^{\alpha} p_1\right)\left(t\right) + Mp_1\left(t\right) \leq 0,\ t \in J. \tag{7}$$

On the other hand, we have

$$p_1(t) = 0,\ \text{for all } t \in \left[-r, 0\right].$$

That is

$$p_1(0) = p_1(t) = 0,\ \text{for all } t \in \left[-r, 0\right]. \tag{8}$$

By the previous equality and (7), we have

$$\begin{cases} \left({}^C\mathfrak{D}_{0+}^{\alpha} p_1\right)\left(t\right) + Mp_1\left(t\right) \leq 0,\ t \in J, \\ p_1(0) = 0. \end{cases}$$

Then from Lemma 3, we get

$$p_1\left(t\right) \leq 0,\ \text{for all } t \in \left[0, T\right],$$

and since

$$p_1\left(t\right) = 0,\ \text{for all } t \in \left[-r, 0\right],$$

we obtain

$$p_1\left(t\right) \leq 0,\ \text{for all } t \in \left[-r, T\right].$$

That is

$$\underline{u}_1 \leq \overline{u}_1 \text{ in } \left[-r, T\right], \tag{9}$$

and then by (5), (6) and (9), we obtain

$$\underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_0 \text{ in } \left[-r, T\right].$$

Assume for fixed $n \geq 1$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } \left[-r, T\right],$$

and we show that

$$\underline{u}_{n+1} \leq \underline{u}_{n+2} \leq \overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } \left[-r, T\right].$$

We put by definition

$$w_{n+1}\left(t\right) := \underline{u}_{n+1}\left(t\right) - \underline{u}_{n+2}\left(t\right),\ t \in \left[-r, T\right].$$

By (3), we have

$$\begin{cases} \left({}^C\mathfrak{D}_{0+}^{\alpha} w_{n+1}\right)\left(t\right) + Mw_{n+1}\left(t\right) = g_n\left(t\right),\ t \in J, \\ w_{n+1}(0) = w_{n+1}(t) = 0,\ t \in \left[-r, 0\right], \end{cases}$$

where

$$g_n(t) = f_n(t) - f_{n+1}(t), \text{ for all } t \in J.$$

Since $\underline{u}_n \leq \underline{u}_{n+1}$ in $[-r, T]$ and using the hypotheses (H1) and (H2), we obtain

$$\begin{cases} \left({}^C\mathfrak{D}_{0^+}^{\alpha} w_{n+1}\right)(t) + M w_{n+1}(t) \leq 0, \ t \in J, \\ w_{n+1}(0) = 0. \end{cases}$$

Then from Lemma 3, it follows that

$$w_{n+1}(t) \leq 0, \text{ for all } t \in [0, T].$$

and since

$$w_{n+1}(t) = 0, \text{ for all } t \in [-r, 0],$$

we obtain

$$w_{n+1}(t) \leq 0, \text{ for all } t \in [-r, T].$$

That is

$$\underline{u}_{n+1}(t) \leq \underline{u}_{n+2}(t), \text{ for all } t \in [-r, T]. \tag{10}$$

Similarly, we can prove that

$$\overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } [-r, T], \tag{11}$$

and

$$\underline{u}_{n+2} \leq \overline{u}_{n+2} \text{ in } [-r, T]. \tag{12}$$

Combining (10), (11) and (12), we obtain

$$\underline{u}_{n+1} \leq \underline{u}_{n+2} \leq \overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } [-r, T],$$

and consequently for all $n \in \mathbb{N}$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } [-r, T].$$

The proof of **Step 1** is complete.

Step 2: The consequence $(\underline{u}_n)_{n\in\mathbb{N}}$ converges to a minimal solution of (1).
By Step 1, it follows that the sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ converges to u_*. First it is not difficult to see that

$$u_*(t) = \varphi(t), \text{ for all } t \in [-r, 0].$$

Given that the sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ is uniformly bounded on $[0, T]$ and it is not difficult to show that this sequence of functions is equicontinuous on $[0, T]$, then by the Arzéla-Ascoli theorem, there exists a subsequence $(\underline{u}_{n_j})_{n_j\in\mathbb{N}}$ of $(\underline{u}_n)_{n\in\mathbb{N}}$ which converges in $C(J, \mathbb{R})$ to a function $\widetilde{u}_*$, and since the sequence of functions $(\underline{u}_n)_{n\in\mathbb{N}}$ converges to u_*, we have $\widetilde{u}_* = u_*$ and the sequence $(\underline{u}_n)_{n\in\mathbb{N}}$ converges in $C(J, \mathbb{R})$ to u_*. Now let $n \in \mathbb{N}^*$ and $t \in J$, then we have

$$\underline{u}_{n+1}(t) = \varphi(0)E_{\alpha}(-Mt^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-M(t-s)^{\alpha}) f_n(s)\, ds.$$

Letting $n \to +\infty$, we obtain

$$f_n(s) \to f(s, u_*(s), \max_{s\in[t-r,t]} u_*(s)) + M u_*(s).$$

Since the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded, then the dominated convergence theorem of Lebesgue implies that

$$u_*(t) = \varphi(0)E_{\alpha}(-Mt^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-M(t-s)^{\alpha}) \widetilde{f}(s)\, ds,$$

where

$$\widetilde{f}(s) = f(s, u_*(s), \max_{\tau\in[s-r,s]} u_*(\tau)) + Mu_*(s),$$

and from Lemma 1, we deduce that

$$\begin{cases} ({}^{C}\mathfrak{D}_{0^+}^{\alpha} u_*)(t) + Mu_*(t) = f(t, u_*(t), \max_{s\in[t-r,t]} u_*(s)) + Mu_*(t), \text{ if } t\in J, \\ u_*(t) = \varphi(t), \text{ if } t\in[-r,0]. \end{cases}$$

That is

$$\begin{cases} ({}^{C}\mathfrak{D}_{0^+}^{\alpha} u_*)(t) = f(t, u_*(t), \max_{s\in[t-r,t]} u_*(s)), \text{ if } t\in J, \\ u_*(t) = \varphi(t), \text{ if } t\in[-r,0]. \end{cases}$$

Thus, u_* is a solution to the problem (1).

Now, we prove that if u is another solution of (1) such that $\underline{u} \le u \le \overline{u}$, then $u_* \le u$. Since u is an upper solution of (1), then by **Step 1**, we have

$$\forall n\in\mathbb{N},\ \underline{u}_n \le u.$$

Letting $n\to+\infty$, we obtain

$$u_* = \lim_{n\to+\infty} \underline{u}_n \le u.$$

Which mean that u_* is a minimal solution of problem (1). The proof of **Step 2** is complete. Similarly, we can prove that the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ converges to a maximal solution u^* of (1).

The proof of Theorem 2 is complete. ∎

To prove the uniqueness of solutions for the problem (1), it is necessary to impose additional conditions on f.

On the nonlinearity f, we shall impose the following additional conditions.

(H3) There exists $M_1 \le 0$ such that the function $x \longmapsto f(t,x,y) + M_1 x$ is decreasing for all $t\in J$, $\underline{u}(t) \le x \le \overline{u}(t)$ and $\max_{s\in[t-r,t]} \underline{u}(s) \le y \le \max_{s\in[t-r,t]} \overline{u}(s)$.

(H4) There exists $M_2 \le 0$ such that the function $y \longmapsto f(t,x,y) + M_2 y$ is decreasing for all $t\in J$, $\underline{u}(t) \le x \le \overline{u}(t)$ and $\max_{s\in[t-r,t]} \underline{u}(s) \le y \le \max_{s\in[t-r,t]} \overline{u}(s)$.

(H5) $-(M_1+M_2)\frac{T^{\alpha}}{\Gamma(1+\alpha)} < 1$.

We have the following result.

Theorem 3 *Assume that hypotheses (Hi) for $i = 1;...,5$ are satisfied and $\underline{u}$ and $\overline{u}$ be lower and upper solutions respectively for problem (1) such that $\underline{u} \le \overline{u}$ in $[-r,T]$. Then the problem (1) admits a unique solution u such that $\underline{u} \le u \le \overline{u}$ in $[-r,T]$.*

Proof. From Theorem 2, the problem (*1*) admits a minimal solution u_* and a maximal solution u^* such that

$$\underline{u} \le u_* \le u^* \le \overline{u} \text{ in } [-r,T].$$

We put by definition

$$\widehat{u}(t) = u^*(t) - u_*(t),\ t\in[-r,T].$$

We have

$$\widehat{u}(t) \ge 0 \text{ for all } t\in[-r,T]. \tag{13}$$

Now we are going to prove that

$$\widehat{u}(t) \le 0 \text{ for all } t\in[-r,T].$$

We have
$$\begin{cases} ({}^{C}\mathfrak{D}_{0+}^{\alpha}\widehat{u})(t) = f(t, u^{*}(t), \max\limits_{s\in[t-r,t]} u^{*}(s)) - f(t, u_{*}(t), \max\limits_{s\in[t-r,t]} u_{*}(s)),\ t\in J, \\ \widehat{u}(t) = 0,\ t\in[-r,0]. \end{cases}$$
Using the hypothesis (H3), we get
$$\begin{cases} ({}^{C}\mathfrak{D}_{0+}^{\alpha}\widehat{u})(t) + M_1\widehat{u}(t) \le f(t, u_{*}(t), \max\limits_{s\in[t-r,t]} u^{*}(s)) - f(t, u_{*}(t), \max\limits_{s\in[t-r,t]} u_{*}(s)),\ t\in J, \\ \widehat{u}(t) = 0,\ t\in[-r,0]. \end{cases}$$
Now from Lemma 6, we have
$$\max_{s\in[t-r,t]} \widehat{u}(s) = \max_{s\in[t-r,t]} |u^{*}(s) - u_{*}(s)| \ge \max_{s\in[t-r,t]} u^{*}(s) - \max_{s\in[t-r,t]} u_{*}(s),$$
and then by the hypothesis (H4), we obtain
$$\begin{cases} ({}^{C}\mathfrak{D}_{0+}^{\alpha}\widehat{u})(t) + M_1\widehat{u}(t) + M_2 \max\limits_{s\in[t-r,t]} \widehat{u}(s) \le 0,\ t\in J, \\ \widehat{u}(0) = \widehat{u}(t) = 0,\ t\in[-r,0]. \end{cases}$$
Now using the hypothesis (H5), then from Lemma 5, we deduce that
$$\widehat{u}(t) \le 0 \text{ for all } t\in[-r,T],$$
and then by the inequality (13), it follows that
$$\widehat{u}(t) = 0 \text{ for all } t\in[-r,T].$$
That is
$$u^{*}(t) = u_{*}(t) \text{ for all } t\in[-r,T].$$
Thus, it follows that the problem (1) has a unique solution. ■

4 Applications

In this section, we give some examples illustrating how our results are applied.

4.1 Example 1

We consider the following problem
$$\begin{cases} ({}^{C}\mathfrak{D}_{0+}^{\alpha}u)(t) = f(t, u(t), \max\limits_{s\in[t-1,t]} u(s)),\ t\in[0,1], \\ u(s) = 0,\ s\in[-1,0], \end{cases} \tag{14}$$
where
$$f(t, u(t), \max_{s\in[t-1,t]} u(s)) = -au(t) + b\max_{s\in[t-1,t]} u(s) + h(t),$$
with a and b are positive real numbers such that $b \ge a$, $\dfrac{b}{\Gamma(1+\alpha)} < 1$, and $h : [0,1] \to \mathbb{R}_{+}^{*}$ continuous such that $\max\limits_{t\in[0,1]} h(t) \le b - a$.

We put by definition
$$\underline{u}(t) = 0 \text{ and } \overline{u}(t) = 1,\ \text{ for all } t\in[-1,1].$$
It is not difficult to show that $\underline{u}$ and $\overline{u}$ are the lower and upper solutions for the problem (14).

On the other hand, it is easy to observe that the function f satisfies the assumptions of Theorem 3 and consequently, it follows that the problem (14) admits a unique solution u, such that $0 \le u \le 1$.

Remark 2 *If we consider the following problem*

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{\alpha}u\right)(t) = b\max\limits_{s\in[t-1,t]} u(s) + b\frac{t^\alpha}{\Gamma(\alpha+1)}, \ t \in [0,1], \\ u(s) = 0, \ s \in [-1,0], \end{cases} \tag{15}$$

where b a positive real number. The problem (15) admits a unique solution u which is given by

$$u(t) = \begin{cases} 0, & \text{if } t \in [-1,0], \\ E_\alpha(bt^\alpha) - b\frac{t^\alpha}{\Gamma(\alpha+1)} - 1, & \text{if } t \in (0,1]. \end{cases}$$

It should be noted that the problem (15) is a generalization of the one presented in Example 1 of [32].

4.2 Example 2

We consider the following problem

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{\frac{1}{2}}u\right)(t) = f(t, u(t), \max\limits_{s\in[t-1,t]} u(s)), \ t \in \left[0, \frac{3}{4}\right], \\ u(t) = t, \ t \in [-1,0], \end{cases} \tag{16}$$

where

$$f(t, u(t), \max_{s\in[t-1,t]} u(s)) = -\sin(u(t)) + \frac{\max\limits_{s\in[t-1,t]} u(s)}{2} + \frac{\sin t}{2}.$$

We put by definition

$$\underline{u}(t) = -1, \text{ for all } t \in \left[-1, \frac{3}{4}\right],$$

and

$$\overline{u}(t) = t, \text{ for all } t \in \left[-1, \frac{3}{4}\right].$$

First, $\underline{u}$ is a lower solution for the problem (16) if we have

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{\frac{1}{2}}\underline{u}\right)(t) \leq -\sin(\underline{u}(t)) + \frac{\max\limits_{s\in[t-1,t]} \underline{u}(s) + \sin t}{2}, \ t \in \left[0, \frac{3}{4}\right], \\ \underline{u}(t) \leq t, \ t \in [-1,0]. \end{cases}$$

That is

$$\begin{cases} 0 \leq \sin 1 + \frac{-1+\sin t}{2}, & t \in \left[0, \frac{3}{4}\right], \\ -1 \leq t, & t \in [-1,0]. \end{cases}$$

Since

$$\sin 1 + \frac{-1+\sin t}{2} \geq \sin 1 - \frac{1}{2} = 0.34147 \geq 0, \text{ for all } t \in \left[0, \frac{3}{4}\right],$$

we obtain, $\underline{u}$ is a lower solution for the problem (16).

Now, $\overline{u}$ is an upper solution for the problem (16) if we have

$$\begin{cases} \left({}^{C}\mathfrak{D}_{0^+}^{\frac{1}{2}}\overline{u}\right)(t) \geq -\sin(\overline{u}(t)) + \frac{\max\limits_{s\in[t-1,t]} \overline{u}(s) + \sin t}{2}, \ t \in \left[0, \frac{3}{4}\right], \\ \overline{u}(t) \geq t, \ t \in [-1,0]. \end{cases}$$

That is

$$\begin{cases} \dfrac{\sqrt{t}}{\Gamma\left(\frac{3}{2}\right)} \geq \dfrac{t-\sin t}{2}, & t \in \left[0, \dfrac{3}{4}\right], \\ t \geq t, & t \in [-1, 0]. \end{cases}$$

Since

$$\phi_1(t) = \frac{\sqrt{t}}{\Gamma\left(\frac{3}{2}\right)} - \left(\frac{t-\sin t}{2}\right) \geq 0, \text{ for all } t \in \left[0, \frac{3}{4}\right],$$

we obtain, $\overline{u}$ is an upper solution for the problem (16).

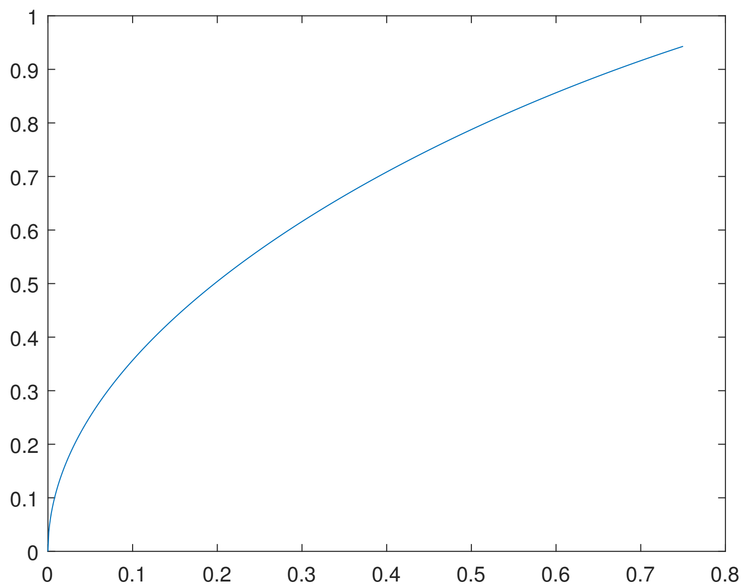

Figure 1: Graph of the function ϕ_1.

Now if we choose $M_1 = 0$ and $M_2 = -\dfrac{1}{2}$, we have

$$-\left(M_1 + M_2\right) \frac{T^\alpha}{\Gamma(1+\alpha)} = \frac{\sqrt{3}}{4\Gamma\left(1+\dfrac{1}{2}\right)} = 0.4886 < 1.$$

On the other hand, it's straightforward to see that the function f satisfies the other assumptions of Theorem 3, and consequently, it follows that the problem (16) admits a unique solution u such that $-1 \leq u \leq t$.

4.3 Example 3

We consider the following problem

$$\begin{cases} \left({}^C\mathfrak{D}_{0^+}^{\frac{3}{4}} u\right)(t) = f(t, u(t), \max\limits_{s\in[t-1,t]} u(s)), \ t \in \left[0, \frac{1}{2}\right], \\ u(t) = t, \ t \in [-1, 0], \end{cases} \tag{17}$$

where

$$f(t, u(t), \max_{s\in[t-1,t]} u(s)) = -u(t) + (t+1) \max_{s\in[t-1,t]} u(s) + \frac{t}{2} + 1.$$

We put by definition

$$\underline{u}(t) = t, \text{ for all } t \in \left[-1, \frac{1}{2}\right],$$

and

$$\overline{u}(t) = \begin{cases} t, & \text{if } t \in [-1, 0[\,, \\ 2t^{\frac{3}{4}}, & \text{if } t \in \left[0, \frac{1}{2}\right]. \end{cases}$$

First, $\underline{u}$ is a lower solution for problem (17), if we have

$$\begin{cases} \left({}^{C}\mathfrak{D}^{\frac{3}{4}}_{0^+}\underline{u}\right)(t) \leq f(t, \underline{u}\,(t)\,, \max\limits_{s\in[t-1,t]} \underline{u}\,(s)), \; t \in \left[0, \frac{1}{2}\right], \\ \underline{u}(t) \leq t, \; t \in [-1, 0]\,. \end{cases}$$

That is

$$\begin{cases} \dfrac{\sqrt[4]{t}}{\Gamma\left(\dfrac{5}{4}\right)} \leq t^2 + \dfrac{t}{2} + 1, & t \in \left[0, \dfrac{1}{2}\right], \\ t \leq t, & t \in [-1, 0]\,. \end{cases}$$

Since

$$\phi_2(t) = \frac{\sqrt[4]{t}}{\Gamma\left(\dfrac{5}{4}\right)} - t^2 - \frac{t}{2} - 1 \leq 0, \text{ for all } t \in \left[0, \frac{1}{2}\right],$$

we obtain $\underline{u}$ is a lower solution for the problem (17).

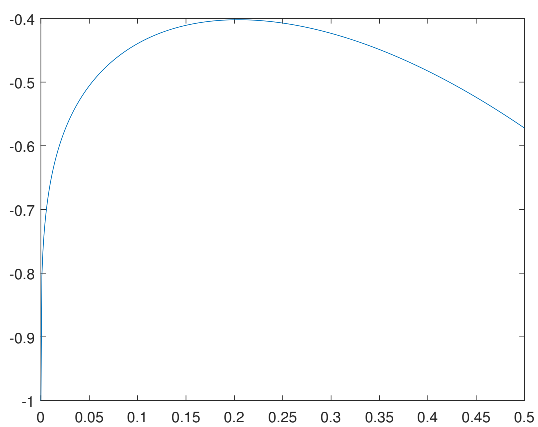

Figure 2: Graph of the function ϕ_2.

Now, $\overline{u}$ is an upper solution for the problem (17) if we have

$$\begin{cases} \left({}^{C}\mathfrak{D}^{\frac{3}{4}}_{0^+}\overline{u}\right)(t) \geq f(t, \overline{u}\,(t)\,, \max\limits_{s\in[t-1,t]} \overline{u}\,(s)), \; t \in \left[0, \frac{1}{2}\right], \\ \overline{u}(t) \geq t, \; t \in [-1, 0]\,. \end{cases}$$

That is

$$\begin{cases} \dfrac{2}{\Gamma\left(\dfrac{3}{4} + 1\right)} \geq 2t^{t^{\frac{7}{4}}} + \dfrac{t}{2} + 1, & t \in \left[0, \dfrac{1}{2}\right], \\ t \geq t, & t \in [-1, 0]\,. \end{cases}$$

Since

$$\phi_3(t) = \frac{2}{\Gamma\left(\frac{3}{4}+1\right)} - 2t^{\frac{7}{4}} - \frac{t}{2} - 1 \geq 0, \text{ for all } t \in \left[0, \frac{1}{2}\right],$$

we obtain $\overline{u}$ is an upper solution for the problem (17).

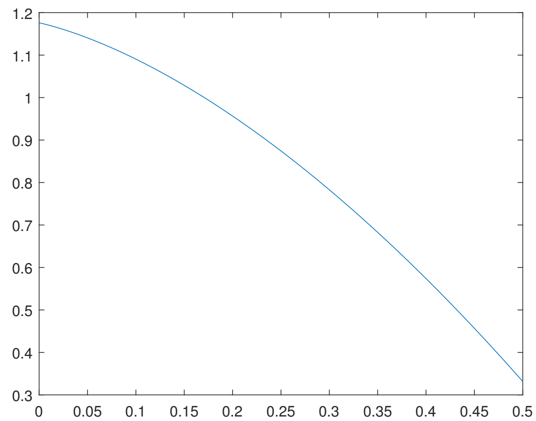

Figure 3: Graph of the function ϕ_3.

Now if we choose $M_1 = -1$ and $M_2 = 0$, we have

$$-\left(M_1 + M_2\right) \frac{T^{\alpha}}{\Gamma\left(1+\alpha\right)} = \frac{1}{2^{\frac{3}{4}}\Gamma\left(1+\frac{3}{4}\right)} = 0.64697 < 1.$$

On the other hand, it is clear that the function f satisfies the other assumptions of Theorem 3, and consequently it follows that the problem (17) admits a unique solution u such that $\underline{u} \leq u \leq \overline{u}$.

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