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Generalization Of Kannan Results For Metric Interval Space Without Space Completeness And Application To Obtain Near Fixed Interval Ellipse*

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Abstract

In the context of 'Metric Interval Space' (MIS), two new near fixed point theorems are derived. The first theorem makes the assumption that the space is complete, whereas the second theorem does not. Furthermore, we have demonstrated through illustrative instances that Wu's [12] earlier near fixed point theorem, based on Banach contraction [11], is independent from our newly suggested theorems. Towards the end, we introduce the concept of an interval ellipse, the equivalence class of interval ellipses and near fixed interval ellipse. In addition, a novel near fixed interval ellipse theorem for MIS is derived and discussed thoroughly.

1 Introduction

Sense of distance is always very crucial for visualization in mathematics. For this reason, metrizable topological spaces are so useful. Urysohn's metrization theorem [1] separates both types of topological spaces. With usual metric on $\mathbb{R}$, it is always very easy to have a geometrical intuition, but as we move ahead, it is really very difficult to visualize. Researchers tried various types of new metric spaces. Actually there are several variations either in settings of metric spaces or in mappings defined there or in both. So a variety of new metric spaces viz partial metric spaces [2], cone metric spaces [3], b-metric spaces [4], Fuzzy metric spaces [5], etc have been evolved. On the other hand mappings are either single valued or multi valued (see [6, 7, 8, 9, 10], and references therein). Besides that, Banach contraction principle [11] has been a torch bearer for analysists since last century to obtain fixed points. Recently, a new geometric approach has gained the momentum; called metric interval space (MIS). Instead of getting usual fixed points with the help of Banach contraction principle in complete metric space, here we study from 'near fixed points' to 'near fixed interval ellipse theorems'.

On the basis of construction of MIS, concepts of convergent sequences, Cauchy sequences, completeness and continuity are redefined; with a novel concept of 'class limit' of a sequence. If $(\mathcal{M}, d)$ is a MIS and 'd' satisfy the null equality, then the limit of a convergent sequence is unique in the sense that it belongs to a unique equivalence class. Inspired with the work of Wu [12], we discussed and elaborated the crucial requirement of the 'Banach contraction principle' for the existence of a unique equivalence class of near fixed points. Since the Banach contraction satisfies Lipschitz condition, so the self map T is uniformly continuous and consequently continuous also.

In our main result (inspired by Kannan contraction [13, 14]), we obtain a unique equivalence class of near fixed points, if metric 'd' satisfies the 'null equality'. This map does not require continuity in entire space. We also studied 'near fixed points' even without the completion of MIS. Here, we obtained same conclusions but with different sufficient conditions. This paper aims to extend the concept of near fixed points to MIS, regardless of their completeness. Based on authors' limited understanding, it seems that there is a lack of appropriate scholarly articles that can be referenced.

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Ermiş et al. [15] introduced the study of fixed ellipse theorems. Since MIS is different from the conventional metric space, so we can't study these theorems in usual way. That's why we examined the geometrical characteristics of non-unique near fixed points of any self mapping in a MIS, in reference to a near fixed interval ellipse, and established its equivalence classes.

In Section 2 we revised the ideas of interval space, null set, and introduced definitions that helped us to understand the concepts of MIS, null equality, near fixed point, convergent and Cauchy sequences in MIS, continuity in MIS etc. We also discussed some examples that highlight the necessity of the contraction condition (5). We proposed two near fixed point theorems in Section 3 based on the ideas covered in Section 2. We assumed MIS to be complete in the first theorem and used the Kannan contraction condition, whereas we omitted the space's completeness in the second theorem. Lastly, in Section 4, we delved into the near fixed interval ellipse theorems within the context of MIS.

2 Preliminaries

Let $\mathcal{M} = \Big\{[l, m] : l, m \in \mathbb{R}, l \leq m\Big\}$ be the collection of all closed and bounded intervals in $\mathbb{R}$. The internal and external compositions on $\mathcal{M}$ are defined as

$$[l, m] \oplus [l', m'] = [l + l', m + m'],$$

and

$$\alpha \odot [l, m] = \begin{cases} [\alpha l, \alpha m] & \text{if } \alpha \geq 0, \\ [\alpha m, \alpha l] & \text{if } \alpha < 0. \end{cases}$$

It is easy to verify that $[0, 0]$ is the zero element of $\mathcal{M}$ and the inverse of $[l, m]; l \neq m$ does not exist with respect to the above compositions. So, $\mathcal{M}$ does not form a vector space.

Also, for any $[l, m]; l \neq m$, we have

$$[l, m] \ominus [l, m] = [l, m] \oplus [-m, -l] = [l - m, m - l] = [-(m - l), (m - l)]. \tag{1}$$

To make a better understanding of (1), we define the null set as

$$\begin{aligned} \Omega &= \Big\{[l, m] \ominus [l, m] : [l, m] \in \mathcal{M}\Big\} \\ &= \Big\{[-(m - l), (m - l)] : [l, m] \in \mathcal{M}\Big\} \\ &= \Big\{[-\alpha, \alpha] : \alpha \geq 0\Big\} \\ &= \Big\{\alpha \odot [-1, 1] : \alpha \geq 0\Big\} \end{aligned}$$

where $\alpha = (m - l)$.

From above discussion, following results are obvious:

1. Ω is generated by $[-1, 1]$.
2. For any $[l, m] \in \mathcal{M}$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha \cdot \beta > 0$
$$(\alpha + \beta) \odot [l, m] = \Big(\alpha \odot [l, m]\Big) \oplus \Big(\beta \odot [l, m]\Big).$$
3. In general, for any $[l, m] \in \mathcal{M}$ and $\alpha, \beta \in \mathbb{R}$, it may be possible that
$$(\alpha + \beta) \odot [l, m] \neq \Big(\alpha \odot [l, m]\Big) \oplus \Big(\beta \odot [l, m]\Big).$$

Now, we consider a relation $\stackrel{\Omega}{=}$ on $\mathcal{M}$ defined as for any $[l,m],[l',m'] \in \mathcal{M}$,

$$[l,m] \stackrel{\Omega}{=} [l',m'] \;\Leftrightarrow\; \exists \text{ some } \omega_1,\omega_2 \in \Omega \text{ such that } [l,m] \oplus \omega_1 = [l',m'] \oplus \omega_2. \tag{2}$$

Wu [12] showed that the relation defined in (2) is an equivalence relation. So, we define the equivalence class of any $[l,m] \in \mathcal{M}$

$$\langle [l,m] \rangle = \Big\{ [l',m'] \in \mathcal{M} : [l,m] \stackrel{\Omega}{=} [l',m'] \Big\}.$$

Before discussing near fixed point theorems in MIS, first we introduce the concepts of MIS and then near fixed points.

Definition 1 ([12]) *Let $\mathcal{M}$ be the collection of all closed and bounded intervals in $\mathbb{R}$. A MIS is a pair $\Big(\mathcal{M}, d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}\Big)$ such that for any $[l,m],[l',m'],[l'',m''] \in \mathcal{M}$, the following conditions are satisfied:*

1. $d\Big([l,m],[l',m']\Big) = 0 \;\Leftrightarrow\; [l,m] \stackrel{\Omega}{=} [l',m']$;
2. $d\Big([l,m],[l',m']\Big) = d\Big([l',m'],[l,m]\Big)$;
3. $d\Big([l,m],[l',m']\Big) \leq d\Big([l,m],[l'',m'']\Big) + d\Big([l'',m''],[l',m']\Big)$.

Definition 2 ([12]) *A map $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ is said to satisfy the null equality iff for any $[l,m],[l',m'] \in \mathcal{M}$ and $\omega_1,\omega_2 \in \Omega$, the following conditions are satisfied:*

1. $d\Big([l,m] \oplus \omega_1,[l',m']\Big) = d\Big([l,m],[l',m']\Big)$;
2. $d\Big([l,m],[l',m'] \oplus \omega_2\Big) = d\Big([l,m],[l',m']\Big)$;
3. $d\Big([l,m] \oplus \omega_1,[l',m'] \oplus \omega_2\Big) = d\Big([l,m],[l',m']\Big)$.

To understand these definitions better, Wu [12] proposed the following example. Here we mention this example but to avoid repetition, we will not give the proof.

Example 1 *Let $\mathcal{M}$ be the collection of all closed and bounded intervals. Consider a non-negative real valued function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ by the rule*

$$d\Big([x,y],[z,w]\Big) = |(x+y)-(z+w)| \text{ for any } [x,y],[z,w] \in \mathcal{M}. \tag{3}$$

Then, $(\mathcal{M},d)$ form a complete MIS such that d satisfies the null equality.

Definition 3 ([12]) *Let $\mathcal{M}$ be the collection of all closed and bounded intervals in $\mathbb{R}$ and T be a self map defined on $\mathcal{M}$. A point $[l,m] \in \mathcal{M}$ is said to be a near fixed point of T iff $T([l,m]) \stackrel{\Omega}{=} [l,m]$.*

By above definition and condition (2), we see that $T([l,m]) \stackrel{\Omega}{=} [l,m]$ iff $\exists$ some $\omega_1,\omega_2 \in \Omega$ such that

$$T([l,m]) \oplus \omega_1 = [l,m] \oplus \omega_2.$$

In this context, we proceed to establish the formal definitions of convergent sequence, Cauchy sequence, completeness and continuity in MIS.

Definition 4 **([12])** *Let $(\mathcal{M}, d)$ be a MIS. A sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$ where $[l_n, m_n] \in \mathcal{M}\ \forall\ n \in \mathbb{N}$, is said to be convergent iff*

$$\lim_{n\to\infty} d\Big([l_n, m_n], [l, m]\Big) = 0 \text{ for some } [l, m] \in \mathcal{M}.$$

The point $[l, m]$ is called the limit of the sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$.

One can notice that if d satisfy the null equality then any $[l', m'] \in \langle [l, m] \rangle$ is also a limit of the sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$. So, here is the requirement to introduce the concept of class limit of sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$.

Definition 5 **([12])** *Let $(\mathcal{M}, d)$ be a MIS such that d satisfy the null equality. If the sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$ where $[l_n, m_n] \in \mathcal{M}\ \forall\ n \in \mathbb{N}$, is such that*

$$\lim_{n\to\infty} d\Big([l_n, m_n], [l, m]\Big) = 0 \text{ for some } [l, m] \in \mathcal{M}.$$

Then the equivalence class $\langle [l, m] \rangle$ is said to be the class limit of sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$.

Definition 6 **([12])** *Let $(\mathcal{M}, d)$ be a MIS. A sequence $\{[l_n, m_n]\}_{n=1}^{\infty}$ where $[l_n, m_n] \in \mathcal{M}\ \forall\ n \in \mathbb{N}$, is said to be Cauchy iff for any $\epsilon > 0$, $\exists\, p \in \mathbb{N}$ such that*

$$d\Big([l_s, m_s], [l_t, m_t]\Big) < \epsilon \text{ for all } t, s > p.$$

It is easy to show that in a *MIS* $(\mathcal{M}, d)$, every convergent sequence is a Cauchy sequence but the converse may not be true. Indeed, let $\{[l_n, m_n]\}_{n=1}^{\infty}$ be a convergent sequence in $\mathcal{M}$ convergent to $[l, m] \in \mathcal{M}$. Then for any $\epsilon > 0$, $\exists\, p \in \mathbb{N}$ such that

$$d\Big([l_n, m_n], [l, m]\Big) < \frac{\epsilon}{2} \text{ for } n > p. \tag{4}$$

Therefore for $s, t > p$ we have

$$\begin{aligned} d\Big([l_s, m_s], [l_t, m_t]\Big) &\le d\Big([l_s, m_s], [l, m]\Big) + d\Big([l, m], [l_t, m_t]\Big) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ (using (4))} \\ &= \epsilon \end{aligned}$$

which shows that $\{[l_n, m_n]\}_{n=1}^{\infty}$ is a Cauchy sequence.

The following example illustrates that converse may not be true.

Example 2 *Let $\mathcal{M} = \Big\{[x, y] : x \le y,\ 0 \le x, y < 1\Big\}$. Consider a non-negative real valued function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ by the rule*

$$d\Big([x, y], [z, w]\Big) = |(x + y) - (z + w)| \text{ for any } [x, y], [z, w] \in \mathcal{M}.$$

Then one can easily verify that $(\mathcal{M}, d)$ is a MIS. Define a sequence $\{[l_n, m_n]\}_{n=2}^{\infty}$ such that $[l_n, m_n] = [\frac{1}{n}, 1 - \frac{1}{n}]; n \ge 2$ which is a Cauchy sequence but not convergent in $\mathcal{M}$.

Definition 7 **([12])** *A MIS $(\mathcal{M}, d)$ is said to be complete iff every Cauchy sequence in $\mathcal{M}$ converges to some point in $\mathcal{M}$.*

Definition 8 *Let $(\mathcal{M}, d), (\mathcal{M}', d)$ be any two MIS. A function $T : (\mathcal{M}, d) \to (\mathcal{M}', d)$ is said to be continuous at a point $[l, m] \in \mathcal{M}$ if for any $\epsilon > 0$, $\exists$ some $\delta > 0$ such that*

$$d\Big([l', m'], [l, m]\Big) < \delta \;\Rightarrow\; d\Big(T([l', m']), T([l, m])\Big) < \epsilon.$$

If T is continuous at all points of $\mathcal{M}$, we say that T is continuous on $\mathcal{M}$.

The well known Banach contraction principle [11] states that: *Suppose (X, d) be a complete metric space and T be a self map on X such that*

$$d\Big(T(x), T(y)\Big) \le \alpha \cdot d(x, y) \text{ for any } x, y \in X \text{ and } \alpha \in (0, 1).$$

Then the self map T has a unique fixed point.

Since $(\mathcal{M}, d)$ in Example 1 is not a metric space and so the conventional approach of studying the fixed points of the self map T on $\mathcal{M}$ is not applicable. Instead, we focus on analyzing the near fixed points within these spaces. Recently, Wu [12] worked in this direction and proved the following near fixed point theorem:

Theorem 1 *Let $(\mathcal{M}, d)$ be a complete MIS. Suppose a self map $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ satisfies the condition*

$$d\Big(T([x, y]), T([z, w])\Big) \le \alpha \cdot d\Big([x, y], [z, w]\Big) \tag{5}$$

where $\alpha \in (0, 1)$ and $[x, y], [z, w] \in \mathcal{M}$. Then, T has a near fixed point $[l, m] \in \mathcal{M}$ satisfying $T([l, m]) \stackrel{\Omega}{=} [l, m]$. Additionally, assume that d satisfies the null equality. Then the uniqueness lies in the sense that there is a unique equivalence class $\langle [l, m] \rangle$ of $[l, m]$ such that any point $[l', m'] \in \mathcal{M}$ is a near fixed point of T iff $[l', m'] \in \langle [l, m] \rangle$.

Now we will discuss some examples to demonstrate the crucial requirement of the contraction condition (5) for the existence of a unique equivalence class of near fixed points. Without that, the uniqueness of an equivalence class of near fixed points is uncertain.

Example 3 *Continued from example 1, we see that the metric d defined in (3) satisfies the null equality. Here, we consider the following two cases:*

1. *Suppose we define a self map $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ such that $T([x, y]) = [x, y]$ for any $[x, y] \in \mathcal{M}$. Then for any $[x, y], [z, w] \in \mathcal{M}$, we have*

$$\begin{aligned} d\Big(T([x, y]), T([z, w])\Big) &= d\Big([x, y], [z, w]\Big) \\ &= |(x + y) - (z + w)| \\ &= d\Big([x, y], [z, w]\Big). \end{aligned}$$

   Though $\nexists$ any $\alpha \in (0, 1)$ such that the contraction condition (5) is satisfied, yet each point $[l, m] \in \mathcal{M}$ is a near fixed point of T. Consequently, there exist infinitely many equivalence classes of near fixed points corresponding to each near fixed point in $\mathcal{M}$.

2. *Suppose we define a self map $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ such that $T([x, y]) = (2 \odot [x, y]) \oplus [-1, 1]$ for any*

$[x,y]\in\mathcal{M}$. Then for any $[x,y],[z,w]\in\mathcal{M}$, we have

$$\begin{aligned} d\Big(T([x,y]),T([z,w])\Big) &= d\Big((2\odot[x,y])\oplus[-1,1],(2\odot[z,w])\oplus[-1,1]\Big)\\ &= d\Big([2x-1,2y+1],[2z-1,2w+1]\Big)\\ &= |(2x+2y)-(2z+2w)|\\ &= 2|(x+y)-(z+w)|\\ &= 2\cdot d\Big([x,y],[z,w]\Big). \end{aligned}$$

Though $\nexists$ any $\alpha\in(0,1)$ such that the contraction condition (5) is satisfied yet the self mapping has a unique equivalence class of near fixed point $\langle[-1,1]\rangle$ in $\mathcal{M}$.

Corollary 1 *The contraction condition (5) implies the continuity of the self map T in the whole space.*

Proof. Let $T:(\mathcal{M},d)\to(\mathcal{M},d)$ satisfies the condition (5). Let $[x_0,y_0]\in\mathcal{M}$ be arbitrary. Choose $\epsilon>0$ be arbitrary and take $0<\delta<\frac{\epsilon}{\alpha}$. Then

$$d\Big(T([x,y],T([x_0,y_0])\Big)\le\alpha\cdot d\Big([x,y],[x_0,y_0]\Big)<\alpha\cdot\delta \text{ if } d\Big([x,y],[x_0,y_0]\Big)<\delta.$$

So, $d\Big(T([x,y],T([x_0,y_0])\Big)<\epsilon$ whenever $d\Big([x,y],[x_0,y_0]\Big)<\delta$. This shows that T is continuous at $[x_0,y_0]$. Since $[x_0,y_0]$ is arbitrary. Hence the result is obvious. ∎

3 Main Results

Now we prove another near fixed point theorem in complete MIS based on the metric contraction condition which was proposed by Kannan [13].

Theorem 2 *Let $(\mathcal{M},d)$ be a complete MIS. Suppose a self map $T:(\mathcal{M},d)\to(\mathcal{M},d)$ satisfies the condition*

$$d\Big(T([x,y]),T([z,w])\Big)\le\alpha\cdot\Big\{d\Big([x,y],T([x,y])\Big)+d\Big([z,w],T([z,w])\Big)\Big\} \tag{6}$$

where $\alpha\in(0,\frac{1}{2})$ and $[x,y],[z,w]\in\mathcal{M}$. Then, T has a near fixed point $[l,m]\in\mathcal{M}$ satisfying $T([l,m])\stackrel{\Omega}{=}[l,m]$. Additionally, assume that d satisfies the null equality. Then the uniqueness lies in the sense that there is a unique equivalence class $\langle[l,m]\rangle$ of $[l,m]$ such that any point $[l',m']\in\mathcal{M}$ is a near fixed point of T iff $[l',m']\in\langle[l,m]\rangle$.

Proof. Let $[l_0,m_0]\in\mathcal{M}$ be arbitrary, and with the help of self mapping T the iterative sequence $\{[l_n,m_n]\}_{n=1}^{\infty}$ is defined as

$$[l_1,m_1]=T([l_0,m_0]),[l_2,m_2]=T^2([l_0,m_0]),...,[l_n,m_n]=T^n([l_0,m_0]).$$

First we have to show that $\{[l_n,m_n]\}_{n=1}^{\infty}$ is a Cauchy sequence. Since, the self map T satisfies the condition (6), so we have

$$\begin{aligned} d\Big([l_1,m_1],[l_2,m_2]\Big) &= d\Big(T([l_0,m_0]),T([l_1,m_1])\Big)\\ &\le \alpha\cdot\Big\{d\Big([l_0,m_0],T([l_0,m_0])\Big)+d\Big([l_1,m_1],T([l_1,m_1])\Big)\Big\}\\ &= \alpha\cdot\Big\{d\Big([l_0,m_0],[l_1,m_1]\Big)+d\Big([l_1,m_1],[l_2,m_2]\Big)\Big\}, \end{aligned}$$

which implies that

$$d\Big([l_1, m_1], [l_2, m_2]\Big) \leq \frac{\alpha}{1-\alpha} \cdot d\Big([l_0, m_0], [l_1, m_1]\Big).$$

Also

$$\begin{aligned} d\Big([l_2, m_2], [l_3, m_3]\Big) &= d\Big(T([l_1, m_1]), T([l_2, m_2])\Big) \\ &\leq \alpha \cdot \Big\{d\Big([l_1, m_1], T([l_1, m_1])\Big) + d\Big([l_2, m_2], T([l_2, m_2])\Big)\Big\} \\ &= \alpha \cdot \Big\{d\Big([l_1, m_1], [l_2, m_2]\Big) + d\Big([l_2, m_2], [l_3, m_3]\Big)\Big\}, \end{aligned}$$

which implies that

$$d\Big([l_2, m_2], [l_3, m_3]\Big) \leq \frac{\alpha}{1-\alpha} \cdot d\Big([l_1, m_1], [l_2, m_2]\Big) \leq \Big(\frac{\alpha}{1-\alpha}\Big)^2 \cdot d\Big([l_0, m_0], [l_1, m_1]\Big).$$

In general, for any positive integer n, we have

$$d\Big([l_n, m_n], [l_{n+1}, m_{n+1}]\Big) \leq \Big(\frac{\alpha}{1-\alpha}\Big)^n \cdot d\Big([l_0, m_0], [l_1, m_1]\Big).$$

If p is any positive integer, then

$$\begin{aligned} d\Big([l_n, m_n], [l_{n+p}, m_{n+p}]\Big) &\leq d\Big([l_n, m_n], [l_{n+1}, m_{n+1}]\Big) + d\Big([l_{n+1}, m_{n+1}], [l_{n+2}, m_{n+2}]\Big) \\ &\quad +... + d\Big([l_{n+p-1}, m_{n+p-1}], [l_{n+p}, m_{n+p}]\Big) \\ &\leq \Big[\Big(\frac{\alpha}{1-\alpha}\Big)^n + \Big(\frac{\alpha}{1-\alpha}\Big)^{n+1} + ... + \Big(\frac{\alpha}{1-\alpha}\Big)^{n+p-1}\Big] \cdot d\Big([l_0, m_0], [l_1, m_1]\Big) \\ &= [\beta^n + \beta^{n+1} + ... + \beta^{n+p-1}] \cdot d\Big([l_0, m_0], [l_1, m_1]\Big) \\ &< \frac{\beta^n}{1-\beta} \cdot d\Big([l_0, m_0], [l_1, m_1]\Big). \end{aligned}$$

where $\beta = \frac{\alpha}{1-\alpha}$.

Since $\alpha \in (0, \frac{1}{2})$ which implies $\beta \in (0, 1)$ and so

$$d\Big([l_n, m_n], [l_{n+p}, m_{n+p}]\Big) \to 0 \quad \text{as } n \to \infty.$$

This shows that $\{[l_n, m_n]\}_{n=1}^{\infty}$ is a Cauchy sequence. Since the MIS $(\mathcal{M}, d)$ is complete, there exists some $[l, m] \in \mathcal{M}$ such that

$$d\Big([l_n, m_n], [l, m]\Big) \to 0 \text{ as } n \to \infty$$

or

$$\lim_{n\to\infty} [l_n, m_n] = \langle [l, m] \rangle.$$

As d satisfies the null equality, we show that any point $[l', m'] \in \langle [l, m] \rangle$ is a near fixed point of T. Since $[l', m'] \in \langle [l, m] \rangle$ we have

$$[l', m'] \oplus \omega_1 = [l, m] \oplus \omega_2 \text{ for some } \omega_1, \omega_2 \in \Omega.$$

Now

$$\begin{aligned}
d\Big([l',m'],T([l',m'])\Big) &= d\Big([l',m']\oplus\omega_1,T([l',m'])\Big)\\
&\le d\Big([l',m']\oplus\omega_1,[l_n,m_n]\Big)+d\Big([l_n,m_n],T([l',m'])\Big)\\
&= d\Big([l',m']\oplus\omega_1,[l_n,m_n]\Big)+d\Big(T([l_{n-1},m_{n-1}]),T([l',m'])\Big)\\
&\le d\Big([l',m']\oplus\omega_1,[l_n,m_n]\Big)+\alpha\cdot\Big\{d\Big([l_{n-1},m_{n-1}],T([l_{n-1},m_{n-1}])\Big)\\
&\quad +d\Big([l',m'],T([l',m'])\Big)\Big\},
\end{aligned}$$

which implies that

$$(1-\alpha)\cdot d\Big([l',m'],T([l',m'])\Big)\le d\Big([l,m]\oplus\omega_2,[l_n,m_n]\Big)+\alpha\cdot d\Big([l_{n-1},m_{n-1}],[l_n,m_n]\Big).$$

So

$$(1-\alpha)\cdot d\Big([l',m'],T([l',m'])\Big)\le d\Big([l,m],[l_n,m_n]\Big)+\alpha\cdot d\Big([l_{n-1},m_{n-1}],[l_n,m_n]\Big).$$

which implies that

$$d\Big([l',m'],T([l',m'])\Big)=0 \ \text{ as } n\to\infty,$$

i.e., $T([l',m'])\stackrel{\Omega}{=}[l',m']$ for any point $[l',m']\in\langle[l,m]\rangle$.

Now, if possible, there is another near fixed point $[l',m']$ of T with $[l',m']\notin\langle[l,m]\rangle$ such that $T([l',m'])\stackrel{\Omega}{=}[l',m']$, then the condition (2) implies

$$[l',m']\oplus\omega_1=T([l',m'])\oplus\omega_2 \text{ for some } \omega_1,\omega_2\in\Omega.$$

Also $T([l,m])\stackrel{\Omega}{=}[l,m]$. From condition (2),

$$[l,m]\oplus\omega_3=T([l,m])\oplus\omega_4 \text{ for some } \omega_3,\omega_4\in\Omega.$$

Since d satisfies the null equality. We have

$$\begin{aligned}
d\Big([l',m'],[l,m]\Big) &= d\Big([l',m']\oplus\omega_1,[l,m]\oplus\omega_3\Big)\\
&= d\Big(T([l',m'])\oplus\omega_2,T([l,m])\oplus\omega_4\Big)\\
&= d\Big(T([l',m']),T([l,m])\Big)\\
&\le \alpha\cdot\Big\{d\Big([l',m'],T([l',m'])\Big)+d\Big([l,m],T([l,m])\Big)\Big\}
\end{aligned}$$

which implies $d\Big([l',m'],[l,m]\Big)=0$ or $[l',m']\stackrel{\Omega}{=}[l,m]$ and hence $[l',m']\in\langle[l,m]\rangle$, which contradicts the given assumption. Therefore any $[l',m']\notin\langle[l,m]\rangle$ can not be a near fixed point of T. This completes the proof. ■

On comparing the near fixed point Theorem 1 to Theorem 2, one might be interested in the relationship between the contraction conditions (5) and (6). According to Corollary 1, contraction condition (5) implies continuity of the map in the entire space, but contraction condition (6) does not.

Now we provide two examples to demonstrate that contraction conditions (5) and (6) are independent of one another.

Example 4 *Let* $\mathcal{M} = \Big\{[x, y] : x \leq y,\ 0 \leq x, y \leq 2\Big\}$. *Consider a non-negative real valued function* $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ *by the rule*

$$d\Big([x, y], [z, w]\Big) = |(x + y) - (z + w)| \textit{ for any } [x, y], [z, w] \in \mathcal{M}.$$

Take $\mathcal{A} = \Big\{[x, y] : x \leq y,\ 0 \leq x, y \leq 1\Big\}$ *and* $\mathcal{B} = \Big\{[x, y] : x \leq y,\ 1 < x, y \leq 2\Big\}$. *It is clear that* $\mathcal{A} \cup \mathcal{B} \neq \mathcal{M}$. *Suppose a self map* $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ *such that*

$$T([x, y]) = \begin{cases} \frac{1}{4} \odot [x, y] & \textit{if} \quad [x, y] \in \mathcal{A}, \\ \frac{1}{5} \odot [x, y] & \textit{if} \quad [x, y] \in \mathcal{B}, \\ \frac{1}{6} \odot [x, y] & \textit{if} \quad [x, y] \in \mathcal{M}/\mathcal{A} \cup \mathcal{B}. \end{cases}$$

The contraction condition (5) is not satisfied because the self map T *is not continuous at* $[1, 1]$. *However, it is obvious that the contraction condition (6) is satisfied for any* $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

Example 5 *Let* $\mathcal{M} = \Big\{[x, y] : x \leq y,\ 0 \leq x, y \leq 1\Big\}$. *Consider a non-negative real valued function* $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ *by the rule*

$$d\Big([x, y], [z, w]\Big) = |(x + y) - (z + w)| \textit{ for any } [x, y], [z, w] \in \mathcal{M}.$$

Suppose a self map $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ *such that*

$$T([x, y]) = \frac{1}{3} \odot [x, y] \textit{ for any } [x, y] \in \mathcal{M}.$$

Here, condition (5) is satisfied but if we take $[x, y] = [0, 1], [z, w] = [0, 0]$ *then condition (6) is not satisfied.*

The MIS has been assumed to be complete in all of the results we have seen so far. The goal of our next result is to observe whether we can discuss the near fixed point even without the completion of the MIS.

In this study, we present an alternative near fixed point theorem (based on the result proposed by Kannan [14]) that does not require the completeness of the space. Surprisingly, we arrive at the same conclusion as in near fixed point Theorems 1 and 2, albeit with different sufficient conditions.

Theorem 3 *Let* $(\mathcal{M}, d)$ *be a* MIS. *Suppose a self map* $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ *satisfies:*

1. *For any* $[x, y], [z, w] \in \mathcal{M}$

$$d\Big(T([x, y]), T([z, w])\Big) \leq \alpha \cdot \Big\{d\Big([x, y], T([x, y])\Big) + d\Big([z, w], T([z, w])\Big)\Big\}. \tag{7}$$

2. T *is continuous at a point* $[l, m] \in \mathcal{M}$.

3. *The sequence of iterates* $\{[l_n, m_n]\}_{n=1}^{\infty}$ *corresponding to a point* $[l_0, m_o] \in \mathcal{M}$, *where* $[l_n, m_n] = T^n([l_0, m_0])$, *has a sub-sequence* $\{[l_{n_k}, m_{n_k}]\}_{k=1}^{\infty}$ *such that*

$$d\Big([l_{n_k}, m_{n_k}], [l, m]\Big) \to 0 \quad \textit{as } n_k \to \infty.$$

Then, T has a near fixed point $[l,m] \in \mathcal{M}$ satisfying $T([l,m]) \stackrel{\Omega}{=} [l,m]$. Additionally, assume that d satisfies the null equality. Then the uniqueness lies in the sense that there is a unique equivalence class $\langle [l,m] \rangle$ of $[l,m]$ such that any point $[l',m'] \in \mathcal{M}$ is a near fixed point of T iff $[l',m'] \in \langle [l,m] \rangle$.

Proof. By the given hypothesis, the self mapping T is continuous at point $[l,m]$ and the sequence $\{[l_{n_k}, m_{n_k}]\}_{k=1}^{\infty}$ satisfies

$$d\Big([l_{n_k}, m_{n_k}], [l,m]\Big) \to 0 \text{ as } n_k \to \infty.$$

So

$$d\Big([l_{n_k+1}, m_{n_k+1}], T([l,m])\Big) \to 0 \text{ as } n_k \to \infty.$$

Suppose, if possible, $d\Big([l,m], T([l,m])\Big) > 0$.

We define two open balls $\mathfrak{B}_1 = \mathfrak{B}_1\Big([l,m], r\Big)$ and $\mathfrak{B}_2 = \mathfrak{B}_2\Big(T([l,m]), r\Big)$ centered at $[l,m]$ and $T([l,m])$ respectively and the radius $r > 0$ is chosen in such a way that

$$r < \frac{1}{4} \cdot d\Big([l,m], T([l,m])\Big).$$

Since the sequences $\{[l_{n_k}, m_{n_k}]\}_{k=1}^{\infty}$ and $\{[l_{n_k+1}, m_{n_k+1}]\}_{k=1}^{\infty}$ converge to $[l,m]$ and $T([l,m])$ respectively, there exists a positive integer p such that

$$[l_{n_k}, m_{n_k}] \in \mathfrak{B}_1 \text{ and } [l_{n_k+1}, m_{n_k+1}] \in \mathfrak{B}_2 \text{ for any } n_k > p,$$

and hence

$$d\Big([l_{n_k}, m_{n_k}], [l_{n_k+1}, m_{n_k+1}]\Big) > r \text{ for any } n_k > p. \tag{8}$$

Also, by using condition (7), we have

$$\begin{aligned}
& d\Big([l_{n_k+1}, m_{n_k+1}], [l_{n_k+2}, m_{n_k+2}]\Big) \\
\leq \;& \alpha \cdot \Big\{ d\Big([l_{n_k}, m_{n_k}], T([l_{n_k}, m_{n_k}])\Big) + d\Big([l_{n_k+1}, m_{n_k+1}], T([l_{n_k+1}, m_{n_k+1}])\Big) \Big\} \\
\leq \;& \alpha \cdot \Big\{ d\Big([l_{n_k}, m_{n_k}], [l_{n_k+1}, m_{n_k+1}]\Big) + d\Big([l_{n_k+1}, m_{n_k+1}], [l_{n_k+2}, m_{n_k+2}]\Big) \Big\},
\end{aligned}$$

which implies that

$$d\Big([l_{n_k+1}, m_{n_k+1}], [l_{n_k+2}, m_{n_k+2}]\Big) \leq \frac{\alpha}{1-\alpha} \cdot d\Big([l_{n_k}, m_{n_k}], [l_{n_k+1}, m_{n_k+1}]\Big).$$

For $n_s > n_t > p$, we have

$$\begin{aligned}
d\Big([l_{n_s}, m_{n_s}], [l_{n_s+1}, m_{n_s+1}]\Big) &\leq \frac{\alpha}{1-\alpha} \cdot d\Big([l_{n_s-1}, m_{n_s-1}], [l_{n_s}, m_{n_s}]\Big) \\
&\leq \left(\frac{\alpha}{1-\alpha}\right)^2 \cdot d\Big([l_{n_s-2}, m_{n_s-2}], [l_{n_s-1}, m_{n_s-1}]\Big) \\
&\;\;\vdots \\
&\leq \left(\frac{\alpha}{1-\alpha}\right)^{n_s-n_t} \cdot d\Big([l_{n_t}, m_{n_t}], [l_{n_t+1}, m_{n_t+1}]\Big).
\end{aligned}$$

The last expression implies that

$$d\Big([l_{n_s}, m_{n_s}], [l_{n_s+1}, m_{n_s+1}]\Big) \to 0 \text{ as } n_s \to \infty$$

which contradicts (8). So our assumption is wrong and hence $d\Big([l,m], T([l,m])\Big) = 0$ or consequently $T([l,m]) \overset{\Omega}{=} [l,m]$ which shows that $[l,m]$ is a near fixed point of T.

The remaining portion of the theorem will follow from Theorem 2. ■

On comparing Theorem 3 with Theorem 2, it is evident that the completeness of the MIS has been excluded. Instead, conditions 2 and 3 have been assumed. The subsequent illustration demonstrates that the conjunction of conditions 2 and 3 in theorem 3 does not ensure the completeness of the space.

Example 6 *Let* $\mathcal{M} = \Big\{[x,y] : x \leq y,\ 0 \leq x, y < 1\Big\}$. *Let us define a non-negative real valued function* $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ *by the rule*

$$d\Big([x,y], [z,w]\Big) = |(x+y) - (z+w)| \textit{ for any } [x,y], [z,w] \in \mathcal{M}.$$

Suppose a self map $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ *satisfies*

$$T([x,y]) = \frac{1}{2} \odot [x,y] \textit{ for any } [x,y] \in \mathcal{M}.$$

Now it is easy to observe that the MIS *is not complete though conditions 2 and 3 in Theorem 3 are satisfied.*

4 Application

We know that the MIS $(\mathcal{M}, d)$ is different from the conventional metric space. So it is not possible to study the fixed ellipse theorems introduced by Ermiş et al. [15], on the MIS $(\mathcal{M}, d)$ in a usual way. Therefore, in this section we will examine the geometrical characteristics of non-unique near fixed points of any self mapping in a MIS. Specifically, we will focus on a near fixed interval ellipse and establish its equivalence class. Firstly, we introduce the concept of an interval ellipse and its equivalence class in a MIS.

An interval ellipse $\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ having foci at $[l_0, m_0], [\bar{l_0}, \bar{m_0}]$ and length of semi-major axis a on the MIS $(\mathcal{M}, d)$ is defined as

$$\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) = \Big\{[l,m] \in \mathcal{M} : d\Big([l,m], [l_0, m_0]\Big) + d\Big([l,m], [\bar{l_0}, \bar{m_0}]\Big) = 2a\Big\}.$$

Now, we consider a relation $\overset{\mathcal{R}}{\approx}$ on set of all interval ellipses defined as

$$\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \overset{\mathcal{R}}{\approx} \mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big) \Leftrightarrow [l_0, m_0] \overset{\Omega}{=} [l_0', m_0'],\ [\bar{l_0}, \bar{m_0}] \overset{\Omega}{=} [\bar{l_0}', \bar{m_0}'] \text{ and } d \text{ satisfy the null equality.}$$

It is easy to verify that the relation $\overset{\mathcal{R}}{\approx}$ is an equivalence relation. So, we define an equivalence class of interval ellipses having length of semi-major axis a, using a relation $\overset{\mathcal{R}}{\approx}$ as

$$\begin{aligned} &\Big\langle \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \Big\rangle \\ = &\Big\{\mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big) : \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \overset{\mathcal{R}}{\approx} \mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big)\Big\} \end{aligned}$$

where $[l_0, m_0], [\bar{l_0}, \bar{m_0}], [l_0', m_0'], [\bar{l_0}', \bar{m_0}'] \in \mathcal{M}$.

Definition 9 *Let $(\mathcal{M}, d)$ be a MIS and $T : \mathcal{M} \to \mathcal{M}$. An interval ellipse $\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ is known as a near fixed interval ellipse of self map T iff $T([l,m]) \overset{\Omega}{=} [l,m] : [l,m] \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$.*

Theorem 4 *Let $\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ be any interval ellipse in a MIS $(\mathcal{M}, d)$ such that d satisfies the null equality. Consider the map $\psi : \mathcal{M} \to \mathbb{R}^+ \cup \{0\}$ defined as*

$$\psi([l,m]) = d\Big([l,m], [l_0, m_0]\Big) + d\Big([l,m], [\bar{l_0}, \bar{m_0}]\Big) \text{ for any } [l,m] \in \mathcal{M}. \tag{9}$$

Suppose a self map $T : (\mathcal{M}, d) \to (\mathcal{M}, d)$ satisfies

1.
$$d\Big([l,m], T([l,m])\Big) \leq \psi([l,m]) + \psi(T([l,m])) - 4a. \tag{10}$$

2.
$$\psi(T([l,m])) \leq 2a. \tag{11}$$

Then, $\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ is a near fixed interval ellipse of the self map T satisfying

$$T([l,m]) \overset{\Omega}{=} [l,m] \text{ for all } [l,m] \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big).$$

Again, if for any $[l,m] \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ and $[l', m'] \in \mathcal{M} \setminus \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$, the contraction condition (5) is satisfied. Then, the self map T has a unique equivalence class of near fixed interval ellipses $\Big\langle \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \Big\rangle$ i.e., if $\mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big)$ is a near fixed interval ellipse of T, then

$$\mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big) \in \Big\langle \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \Big\rangle$$

or

$$\Big\langle \mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big) \Big\rangle = \Big\langle \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \Big\rangle.$$

Equivalently, if $\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ and $\mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big)$ are the near fixed interval ellipses of T, then

$$\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \overset{\mathcal{R}}{\approx} \mathfrak{E}\Big([l_0', m_0'], [\bar{l_0}', \bar{m_0}'], 2a\Big).$$

Proof. Let $[l,m] \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ be any arbitrary point. Using conditions (9) and (10), we have

$$\begin{aligned}
d\Big([l,m], T([l,m])\Big) &\leq \psi([l,m]) + \psi(T([l,m])) - 4a \\
&= d\Big([l,m], [l_0, m_0]\Big) + d\Big([l,m], [\bar{l_0}, \bar{m_0}]\Big) + d\Big(T([l,m]), [l_0, m_0]\Big) \\
&\quad + d\Big(T([l,m]), [\bar{l_0}, \bar{m_0}]\Big) - 4a \\
&= d\Big(T([l,m]), [l_0, m_0]\Big) + d\Big(T([l,m]), [\bar{l_0}, \bar{m_0}]\Big) - 2a,
\end{aligned}$$

which implies that

$$d\Big([l,m],T([l,m])\Big) \leq d\Big(T([l,m]),[l_0,m_0]\Big) + d\Big(T([l,m]),[\bar{l_0},\bar{m_0}]\Big) - 2a. \tag{12}$$

From condition (11), $\psi(T([l,m])) \leq 2a$, which implies

$$d\Big(T([l,m]),[l_0,m_0]\Big) + d\Big(T([l,m]),[\bar{l_0},\bar{m_0}]\Big) \leq 2a.$$

This demonstrates that, the point $T([l,m])$ not lying on the exterior of the interval ellipse $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$. Consequently, there are two cases according to the position of the point $T([l,m])$.

1. If the point $T([l,m])$ lies in the interior of the interval ellipse $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$. In this case

   $$d\Big(T([l,m]),[l_0,m_0]\Big) + d\Big(T([l,m]),[\bar{l_0},\bar{m_0}]\Big) < 2a.$$

   So, from the inequality (12), we have

   $$d\Big([l,m],T([l,m])\Big) < 0$$

   which is a contradiction.

2. If the point $T([l,m])$ lies on the interval ellipse $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$. In this case

   $$d\Big(T([l,m]),[l_0,m_0]\Big) + d\Big(T([l,m]),[\bar{l_0},\bar{m_0}]\Big) = 2a.$$

   So, from the inequality (12), we have

   $$d\Big([l,m],T([l,m])\Big) \leq 0$$

   which implies $T([l,m]) \overset{\Omega}{=} [l,m]$ i.e., $[l,m]$ is a near fixed point of self map T and consequently, $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$ is a near fixed interval ellipse of self map T.

It can be easily shown that each member of this equivalence class $\Big\langle \mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)\Big\rangle$ is also a near fixed interval ellipse of self map T. To show the uniqueness, suppose there are two equivalence classes of near fixed interval ellipses $\Big\langle \mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)\Big\rangle$ and $\Big\langle \mathfrak{E}\Big([l_0',m_0'],[\bar{l_0}',\bar{m_0}'],2a\Big)\Big\rangle$ of the self map T. This indicates that, T satisfies conditions (10) and (11) for each of the near fixed interval ellipses

$$\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big) \in \Big\langle \mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)\Big\rangle,$$

and

$$\mathfrak{E}\Big([l_0',m_0'],[\bar{l_0}',\bar{m_0}'],2a\Big) \in \Big\langle \mathfrak{E}\Big([l_0',m_0'],[\bar{l_0}',\bar{m_0}'],2a\Big)\Big\rangle.$$

Let $[l, m] \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$ and $[l', m'] \in \mathfrak{E}\Big([l'_0, m'_0], [\bar{l_0}', \bar{m_0}'], 2a\Big)$ be any two points. Therefore $T([l, m]) \stackrel{\Omega}{=} [l, m], T([l', m']) \stackrel{\Omega}{=} [l', m']$ and then condition (2) implies

$$T([l, m]) \oplus \omega_1 = [l, m] \oplus \omega_2 \text{ and } T([l', m']) \oplus \omega_3 = [l', m'] \oplus \omega_4$$

for some $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$.

Now

$$\begin{aligned} d\Big([l, m], [l', m']\Big) &= d\Big([l, m] \oplus \omega_2, [l', m'] \oplus \omega_4\Big) \\ &= d\Big(T([l, m]) \oplus \omega_1, T([l', m']) \oplus \omega_3\Big) \\ &= d\Big(T([l, m]), T([l', m'])\Big) \\ &\leq \alpha \cdot d\Big([l, m], [l', m']\Big). \end{aligned}$$

This shows that $(1-\alpha) \cdot d\Big([l, m], [l', m']\Big) \leq 0$. Since $\alpha \in (0, 1)$ which implies $[l, m] \stackrel{\Omega}{=} [l', m']$, a contradiction. Hence, $\Big\langle \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)\Big\rangle$ is a unique equivalence class of a near fixed interval ellipses of the self map T. This completes the proof. ■

From the geometric perspective, it can be observed that for any point $[l, m]$ belonging to the set $\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)$, condition (10) serves to verify that the point $T([l, m])$ does not lie within the interior of an interval ellipse. Similarly, condition (11) confirms that the point $T([l, m])$ is not situated outside of an interval ellipse. So from conditions (10) and (11), we have

$$T([l, m]) \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big) \text{ for all } [l, m] \in \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big),$$

and hence

$$T\Big(\mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big)\Big) = \mathfrak{E}\Big([l_0, m_0], [\bar{l_0}, \bar{m_0}], 2a\Big).$$

The subsequent illustration demonstrates Theorem 4.

Example 7 *Continued from Example 1. Consider an interval ellipse*

$$\begin{aligned} \mathfrak{E}\Big([0, 1], [2, 3], 10\Big) &= \Big\{[l, m] \in \mathcal{M} : d([l, m], [0, 1]) + d([l, m], [2, 3]) = 10\Big\} \\ &= \Big\{[l, m] \in \mathcal{M} : |(l + m) - 1| + |(l + m) - 5| = 10\Big\}. \end{aligned}$$

Choose $[l', m'] \in \mathcal{M}$ such that $d([l', m'], [0, 1]) + d([l', m'], [2, 3]) < 10$. Suppose a self map T defined on $\mathcal{M}$ as

$$T([l, m]) = \begin{cases} [l, m] & \text{if } [l, m] \in \mathfrak{E}\Big([0, 1], [2, 3], 10\Big), \\ [l', m'] & \text{if } [l, m] \in \mathcal{M} \backslash \mathfrak{E}\Big([0, 1], [2, 3], 10\Big), \end{cases}$$

and

$$d\Big([l, m], [l', m']\Big) \leq \alpha \cdot d\Big([l, m], [\bar{l}, \bar{m}]\Big),$$

where $\alpha \in (0,1)$ *and* $[l,m] \in \mathfrak{E}\Big([0,1],[2,3],10\Big), [\bar{l},\bar{m}] \in \mathcal{M} \setminus \mathfrak{E}\Big([0,1],[2,3],10\Big)$. *Then, the self map* T *satisfies all the conditions of Theorem 4 and hence there exists a unique equivalence class of near fixed interval ellipses* $\Big\langle \mathfrak{E}\Big([0,1],[2,3],10\Big)\Big\rangle$.

Now we provide two examples which illustrate how conditions (10) and (11) affect the persistence of a unique equivalence class of near fixed interval ellipses.

Example 8 *Continued from Example 1, consider an interval ellipse* $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$ *defined on* $\mathcal{M}$. *Choose* $[l',m'] \in \mathcal{M}$ *such that* $d([l',m'],[l_0,m_0]) + d([l',m'],[\bar{l_0},\bar{m_0}]) > \beta$ *where* $\beta > 2a$. *Suppose a self map* $T : (\mathcal{M},d) \to (\mathcal{M},d)$ *such that*

$$T([l,m]) = [l',m'] \text{ for any } [l,m] \in \mathcal{M}.$$

Then, the self map T *satisfies condition (10) but not (11) of Theorem 4. One can easily prove that* T *does not nearly fix the interval ellipse* $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$.

Example 9 *Continued from Example 1, consider an interval ellipse* $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$ *defined on* $\mathcal{M}$. *Suppose a self map* $T : (\mathcal{M},d) \to (\mathcal{M},d)$ *such that*

$$T([l,m]) = [l_0,m_0] \text{ for any } [l,m] \in \mathcal{M}.$$

Then, the self map T *satisfies condition (11) but not (10) of Theorem 4. One can easily prove that* T *does not nearly fix the interval ellipse* $\mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$.

Corollary 2 *Theorem 4 still holds if we replace condition (11) by the following condition:*

$$d\Big([l,m],T([l,m])\Big) + \psi(T([l,m])) \leq 2a. \tag{13}$$

Proof. Let $[l,m] \in \mathfrak{E}\Big([l_0,m_0],[\bar{l_0},\bar{m_0}],2a\Big)$ be any arbitrary point. Using conditions (9) and (10), we have

$$\begin{aligned} d\Big([l,m],T([l,m])\Big) &\leq \psi([l,m]) + \psi(T([l,m])) - 4a \\ &= \psi(T([l,m])) - 2a \\ &\leq \psi(T([l,m])) - \Big(d([l,m],T([l,m])) + \psi(T([l,m]))\Big) \text{ (using condition (13))} \\ &= -d\Big([l,m],T([l,m])\Big) \end{aligned}$$

which implies $2 \cdot d\Big([l,m],T([l,m])\Big) \leq 0$ and hence $T([l,m]) \stackrel{\Omega}{=} [l,m]$. The remaining portion of the theorem will follow from Theorem 4. ∎

Corollary 3 *Theorem 4 still holds if we replace the contraction condition (5) by the contraction condition (6).*

5 Conclusion

The consideration of space completeness has been a fundamental aspect in all previous investigations of near fixed point theorems within the context of MIS. This study utilizes the Kannan contraction condition [13, 14] as a basis to provide two original findings in MIS. The first finding assumes the completeness of the space, whereas the second finding does not make this assumption. We have demonstrated the independence of our new finding from Wu's [12] earlier Banach contraction based near fixed point theorem. Additionally, another result is provided that shows how to approach near fixed points in the space without it being complete with additional sufficient conditions. In the framework of metric spaces, Ermiş et al. [15] introduced the fixed ellipse theorem. The key point to remember is that MIS is not the same as standard metric space on $\mathbb{R}$. We looked at the near fixed interval ellipse theorems in the context of MIS by providing definitions for the words interval ellipse, its equivalence class, and near fixed interval ellipse of a self map.

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