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Some Properties Involving The Generalized Harmonic Polynomials*

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Abstract

In this paper, we define the generalized harmonic polynomials and derive some properties including them using umbral methods and combinatorial identities. Also, we obtain new identities with the help of special matrices.

1 Introduction

The harmonic numbers have interesting applications in many fields of mathematics, such as number theory, combinatorics, analysis and computer science. The harmonic numbers are defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n = 1, 2, \cdots .$$

In recent years, there are various generalizations of the harmonic numbers in the literature [1, 2, 8–10].

In [6], for every ordered pair $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{N}$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0, \quad H_n(\alpha) = \sum_{k=1}^{n} \frac{1}{k\alpha^k}.$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$.

In recent years, the authors used methods of umbral calculus and algebraic operations to make some progress in the theory of generating functions involving harmonic numbers and series [3, 4, 5, 11].

In [3], using umbral operational methods, Dattoli and Germano defined the harmonic based exponential function (GHBEF) and the harmonic polynomials as follows:

$${}_{\theta}e(x) = e^{\widehat{\theta}x} = 1 + \sum_{n=1}^{\infty} \frac{H_n}{n!} x^n,$$

and

$$h_n(x) = x^n + \sum_{i=1}^{n} \binom{n}{i} H_i x^{n-i}, \tag{1}$$

respectively, where $\widehat{\theta} := e^{\partial_z}$ is the vacuum shift operator. It is clear that $\widehat{\theta}^n \widehat{\theta}^m = \widehat{\theta}^{n+m}$. They derived the number of results based on elementary notions relying on the properties of Gaussian integrals.

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In [5], Dattoli et al. defined higher-order harmonic numbers and derived their relevant properties and generating functions by using an umbral-type method. They showed that the combinations of umbral and other techniques yield a very efficient tool to explore the properties of these numbers.

In [4], using methods of umbral calculus and algebraic nature, Dattoli and Srivastava explained some progress in the theory of generating functions including harmonic numbers. For example,

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} H_{n+m} = -e^z \sum_{k=0}^{m} \binom{m}{k} \Phi^{(k)}(z) - H_m \quad \text{for } m = 0, 1, 2, \cdots ,$$

where $\Phi^{(k)}(z)$ denotes the derivative of $\Phi(z)$ of order k with respect to z.

2 Generalized Harmonic Polynomials and Their Applications

In this section, we will define generalized harmonic polynomials $h_n(x, \alpha)$ and give some new properties including them.

Definition 1 *For any real number z and $L(\alpha) = 1 - \frac{1}{\alpha}$, the function*

$$f(z) = \int_{L(\alpha)}^{1} \frac{1 - (1-x)^z}{1-x} dx$$

is called generalized harmonic number umbral vacuum.

Theorem 1 *Let n be a non-negative integer. We have*

$$\widehat{\theta}^n f(z)\,|_{z=0} = H_n(\alpha),$$

or simply

$$\widehat{\theta}^n = H_n(\alpha).$$

Proof. Indeed,

$$\widehat{\theta}^n f(z)\,|_{z=0} = e^{n\partial_z} f(z)\,|_{z=0} = f(n+z)\,|_{z=0} = \left. \int_{L(\alpha)}^{1} \frac{1 - (1-x)^{n+z}}{1-x} dx \right|_{z=0}$$

$$= \int_{L(\alpha)}^{1} \frac{1 - (1-x)^n}{1-x} dx = H_n(\alpha),$$

as claimed. ■

Definition 2 *The binomial expansion*

$$h_0(x, \alpha) = 1, \ \text{and} \ \ h_n(x, \alpha) = \left(x + \widehat{\theta}\right)^n = x^n + \sum_{i=1}^{n} \binom{n}{i} x^{n-i} H_i(\alpha), \quad n > 0 \tag{2}$$

are called generalized harmonic polynomials.

For $\alpha = 1$, $h_n(x, 1) = h_n(x)$ in (1) are the harmonic polynomials [3].

Lemma 1 *Let n be a positive integer. Then the following identity holds*

$$\sum_{i=0}^{\infty}\binom{i}{n}x^{i}=\frac{x^{n}}{(1-x)^{n+1}}. \tag{3}$$

Theorem 2 *The generating function of $h_n(x,\alpha)$ is given by*

$$\sum_{n=1}^{\infty}h_n(x,\alpha)z^n=\frac{xz}{1-xz}-\frac{1}{1-z(x+1)}\ln\left(1-\frac{z}{\alpha(1-xz)}\right),$$

where $|xz|<1$.

Proof. From (2) and (3), we have

$$\begin{aligned}\sum_{n=1}^{\infty}h_n(x,\alpha)z^n&=\sum_{n=1}^{\infty}(xz)^n+\sum_{n=1}^{\infty}\sum_{i=1}^{n}\binom{n}{i}x^{n-i}H_i(\alpha)z^n\\&=\frac{1}{1-xz}-1+\sum_{i=1}^{\infty}x^{-i}H_i(\alpha)\sum_{n=i}^{\infty}\binom{n}{i}x^nz^n\\&=\frac{xz}{1-xz}+\sum_{i=1}^{\infty}x^{-i}H_i(\alpha)\frac{(xz)^i}{(1-xz)^{i+1}}\\&=\frac{xz}{1-xz}+\frac{1}{1-xz}\sum_{i=1}^{\infty}H_i(\alpha)\left(\frac{z}{1-xz}\right)^i.\end{aligned}$$

Since $\sum\limits_{i=0}^{\infty}H_i(\alpha)x^i=\frac{1}{1-x}\ln\left(\frac{\alpha}{\alpha-x}\right)$, we write

$$\sum_{n=1}^{\infty}h_n(x,\alpha)z^n=\frac{xz}{1-xz}-\frac{1}{1-z(x+1)}\ln\left(1-\frac{z}{\alpha(1-xz)}\right).$$

Thus, we have the proof. ∎

Theorem 3 *For integer $n\geq 2$, we have*

$$\sum_{i=1}^{n-1}\left(x^{n-i}-h_{n-i}(x,\alpha)\right)\left((n-2i-1)\left(g_i(x,\alpha)-x^i\right)-(x+1)\left(x^i-h_i(x,\alpha)\right)\right)=0,$$

where

$$g_n(x,\alpha)=\frac{\left(x+\frac{1}{\alpha}\right)^{n+1}-x^{n+1}}{n+1}-x^n.$$

Proof. Define a function as

$$A(z):=\sum_{n=1}^{\infty}\left(x^n-h_n(x,\alpha)\right)z^n=\left(\frac{1-z(x+1)}{\ln\left(1-\frac{z}{\alpha(1-xz)}\right)}\right)^{-1}. \tag{4}$$

Take the partial derivative of $A(z)$ respect to z, we write

$$A'(z)=(A(z))^2\frac{1+x}{\ln\left(1-\frac{z}{\alpha(1-xz)}\right)}-\frac{A(z)}{(1-xz)(\alpha(1-xz)-z)\ln\left(1-\frac{z}{\alpha(1-xz)}\right)},$$

and from here,

$$\begin{aligned}(1+x)(A(z))^2 &= \sum_{n=1}^{\infty}\left(\frac{x^n-\left(x+\frac{1}{\alpha}\right)^n}{n}\right)z^n\sum_{n=0}^{\infty}(n+1)\left(x^{n+1}-h_{n+1}(x,\alpha)\right)z^n \\ &+\sum_{n=0}^{\infty}\left(\left(x+\frac{1}{\alpha}\right)^{n+1}-x^{n+1}\right)z^n\sum_{n=1}^{\infty}\left(x^n-h_n(x,\alpha)\right)z^n.\end{aligned}$$

Using product of generating functions, we get

$$\begin{aligned}(1+x)(A(z))^2 &= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(i+1)\left(x^{i+1}-h_{i+1}(x,\alpha)\right)\left(\frac{x^{n-i}-\left(x+\frac{1}{\alpha}\right)^{n-i}}{n-i}\right)z^n \\ &+\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\left(\left(x+\frac{1}{\alpha}\right)^{i+1}-x^{i+1}\right)\left(x^{n-i}-h_{n-i}(x,\alpha)\right)z^n. \qquad (5)\end{aligned}$$

After that, by (4), we also have

$$(A(z))^2=\sum_{n=1}^{\infty}\sum_{i=0}^{n}\left(x^i-h_i(x,\alpha)\right)\left(x^{n-i}-h_{n-i}(x,\alpha)\right)z^n. \qquad (6)$$

By comparing the coefficients on right sides of (5) and (6), we have the proof. ■

Theorem 4 *For positive integer n, we have*

$$\begin{aligned}\frac{d}{dx}h_n(x,\alpha)&=nh_{n-1}(x,\alpha), \\ h_{n+1}(x,\alpha)&=(x+1)h_n(x,\alpha)+g_n(x,\alpha), \qquad (7)\end{aligned}$$

where $g_n(x,\alpha)$ is as before.

Proof. From (2), we have

$$\begin{aligned}h_{n+1}(x,\alpha)&=\left(x+\widehat{\theta}\right)\left(x+\widehat{\theta}\right)^n=x\left(x+\widehat{\theta}\right)^n+\widehat{\theta}\left(x+\widehat{\theta}\right)^n \\ &=xh_n(x,\alpha)+\widehat{\theta}\left(x^n+\sum_{i=1}^{n}\binom{n}{i}x^{n-i}H_i(\alpha)\right) \\ &=xh_n(x,\alpha)+\frac{x^n}{\alpha}+\sum_{i=1}^{n}\binom{n}{i}x^{n-i}H_{i+1}(\alpha),\end{aligned}$$

and taking $H_{n+1}(\alpha)=H_n(\alpha)+\frac{1}{(n+1)\alpha^{n+1}}$, equals to

$$\begin{aligned}& xh_n(x,\alpha)+\frac{x^n}{\alpha}+\sum_{i=1}^{n}\binom{n}{i}x^{n-i}H_i(\alpha)+\sum_{i=1}^{n}\binom{n}{i}\frac{x^{n-i}}{(i+1)\alpha^{i+1}} \\ =\ & (x+1)h_n(x,\alpha)+\frac{x^n}{\alpha}-x^n+\sum_{i=1}^{n}\frac{1}{\alpha^{i+1}}\binom{n}{i}x^{n-i}\int_0^1 y^i dy \\ =\ & (x+1)h_n(x,\alpha)-x^n+\frac{1}{\alpha}\int_0^1\sum_{i=0}^{n}\binom{n}{i}x^{n-i}\left(\frac{y}{\alpha}\right)^i dy.\end{aligned}$$

By Binomial Theorem, we write

$$\begin{aligned}
h_{n+1}(x,\alpha) &= (x+1)h_n(x,\alpha) - x^n + \frac{1}{\alpha}\int_0^1 \left(x+\frac{y}{\alpha}\right)^n dy \\
&= (x+1)h_n(x,\alpha) - x^n + \frac{\left(x+\frac{1}{\alpha}\right)^{n+1} - x^{n+1}}{n+1} \\
&= (x+1)h_n(x,\alpha) + g_n(x,\alpha).
\end{aligned}$$

Thus, the proof is complete. ■

Corollary 1 *For positive integers n and m, we have*

$$h_{n+m+1}(x,\alpha) = (x+1)^{m+1} h_n(x,\alpha) + \sum_{i=0}^{m} (x+1)^{m-i} g_{n+i}(x,\alpha),$$

where $g_n(x,\alpha)$ is as before.

Proof. From Theorem 4 and the induction method on n, the proof can easily be obtained. ■

Theorem 5 *For positive integer n, we have the following identities:*

$$\begin{aligned}
\sum_{i=1}^{n} \binom{n}{i} (-1)^i H_i(\alpha) &= \frac{\left(1-\frac{1}{\alpha}\right)^n - 1}{n}, \\
\sum_{i=1}^{n} \binom{n}{i} h_i(-1,\alpha) &= H_n(\alpha) - 1. \qquad (8)
\end{aligned}$$

Proof. Setting $x=-1$ in (2) and (7), we have

$$(-1)^n + \sum_{i=1}^{n} \binom{n}{i} (-1)^{n-i} H_i(\alpha) = h_n(-1,\alpha) = g_{n-1}(-1,\alpha) = \frac{\left(\frac{1}{\alpha}-1\right)^n - (-1)^n}{n} - (-1)^{n-1}.$$

Thus, we have the first sum. Secondly, applying Binomial Theorem, we get

$$\begin{aligned}
H_n(\alpha) = \widehat{\theta}^n &= \left(\left(\widehat{\theta}-1\right)+1\right)^n = 1 + \sum_{i=1}^{n} \binom{n}{i} \left(\widehat{\theta}-1\right)^i \\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} H_r(\alpha) \\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} h_i(-1,\alpha),
\end{aligned}$$

as claimed. ■

With the help of (8), we can easily deduce the sum in the following corollary.

Corollary 2 *For positive integer n, we have*

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{1}{i}\left(\frac{1}{\alpha}-1\right)^i - \frac{(-1)^i}{i}\right) = H_n(\alpha).$$

Lemma 2 ([7]) *Let n and m be non-negative integers. Then*

$$\sum_{k=0}^{n} \binom{n}{k} k^{\underline{m}} x^k = x^m n^{\underline{m}} (1+x)^{n-m} \quad [x \neq -1 \text{ and } m \neq n], \tag{9}$$

where $x^{\underline{m}}$ stands for following factorial defined by

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1).$$

Let s and r be any real numbers such that $s \geq r \geq 1$, we define an $n \times n$ matrix $A_n = [a_{i,j}]$, where $a_{i,j} = h_i(r,j) - r^i$ and an $n \times n$ matrix $B_n = [b_{i,j}]$, where $b_{i,j} = h_i(s,j) - s^i$. Now, we can give the following theorem.

Theorem 6 *For positive integer n, we have*

$$A_n = P_n B_n, \tag{10}$$

where the $n \times n$ matrix $P_n = [p_{i,j}]$ with $p_{i,j} = \binom{i}{j}(r-s)^{i-j}$.

Proof. It is clear $a_{1,1} = 1$ for $i = j = 1$. For $i = 1$, $j > 1$, we write

$$a_{1,j} = p_{1,1}b_{1,j} + p_{1,2}b_{2,j} + \cdots + p_{1,n}b_{n,j} = p_{1,1}b_{1,j} = h_1(r,j) - r.$$

For $i > 1$ and $j > 1$, we obtain

$$\begin{aligned} a_{i,j} &= \sum_{k=1}^{n} p_{i,k}b_{k,j} = \sum_{k=1}^{i} \binom{i}{k}(r-s)^{i-k} \sum_{t=1}^{k} \binom{t}{k} s^{k-t} H_t(j) \\ &= \sum_{t=1}^{i} s^{-t} H_t(j) \sum_{k=t}^{i} \binom{i}{k}\binom{t}{k}(r-s)^{i-k} s^k. \end{aligned}$$

Then (9) yields that

$$a_{i,j} = \sum_{t=1}^{i} s^{-t} H_t(j)(r-s)^i \binom{i}{t}\left(\frac{r-s}{s}\right)^{-t}\left(1+\frac{s}{r-s}\right)^{i-t} = \sum_{t=1}^{i}\binom{i}{t} H_t(j)\, r^{i-t} = h_i(r,j) - r^i,$$

as claimed. ■

Corollary 3 *Let s and r be real numbers such that $s \geq r \geq 1$. We have that for positive integers n and m,*

$$\sum_{i=0}^{n} \binom{n}{i}(r-s)^{n-i} h_i(s,m) = h_n(r,m).$$

Proof. Equating (n,m)-entries of (10) gives the claimed result. ■

We define the $n \times n$ matrix $C_n = [c_{i,j}]$ with entries

$$c_{i,j} = h_i(j,\alpha) - j^i.$$

Now, we will give the matrices $L_n = [l_{i,j}]$ and $U_n = [u_{i,j}]$ yielding from the LU-decomposition of $C_n = L_n U_n$ in the following theorem:

Theorem 7 *We have*

$$l_{i,j}=\begin{cases}\frac{\alpha(-1)^i}{j!}\sum\limits_{t=1}^{j}\binom{j-1}{t-1}(-1)^t\left(h_i(t,\alpha)-t^i\right), & \text{for } i>j,\\ 1, & \text{for } i=j,\\ 0, & \text{for } i<j,\end{cases}$$

and

$$u_{i,j}=\begin{cases}\frac{i!}{\alpha}\binom{j-1}{i-1}, & \text{for } i\le j,\\ 0, & \text{for } i>j.\end{cases}$$

Proof. For LU-decomposition of the matrix C_n, we have to prove that

$$c_{i,j}=\sum_{1\le k\le \min\{i,j\}} l_{i,k}u_{k,j}.$$

For $i\neq j$,

$$c_{i,j}=\sum_{1\le k\le \min\{i,j\}} l_{i,k}u_{k,j}=\sum_{k=1}^{i}(-1)^k\binom{j-1}{k-1}\sum_{t=1}^{k}\binom{k-1}{t-1}(-1)^t\left(h_i(t,\alpha)-t^i\right).$$

Using

$$\sum_{i=1}^{n}\sum_{j=1}^{i}f(i,j)=\sum_{j=1}^{n}\sum_{i=j}^{n}f(i,j)\ \text{ and }\ \binom{n}{k}\binom{k}{j}=\binom{n}{j}\binom{n-j}{k-j},$$

we write

$$\begin{aligned}c_{i,j}&=\sum_{t=1}^{i}\sum_{k=t}^{i}(-1)^{k+t}\binom{j-1}{k-1}\binom{k-1}{t-1}\left(h_i(t,\alpha)-t^i\right)\\&=\sum_{t=1}^{i}(-1)^t\left(h_i(t,\alpha)-t^i\right)\sum_{k=t}^{i}(-1)^k\binom{j-1}{k-1}\binom{k-1}{t-1}\\&=\sum_{t=1}^{i}\binom{j-1}{t-1}(-1)^t\left(h_i(t,\alpha)-t^i\right)\sum_{k=t}^{i}(-1)^k\binom{j-t}{k-t}.\end{aligned}$$

And then by

$$\sum_{i=1}^{n}(-1)^i\binom{x}{i}=(-1)^n\binom{x-1}{n}\ \text{ and }\ \binom{n}{k}=(-1)^k\binom{-n+k-1}{k},$$

we obtain

$$c_{i,j}=\sum_{t=1}^{i}\binom{j-1}{t-1}\binom{i-j}{i-t}\left(h_i(t,\alpha)-t^i\right).$$

From $\binom{n}{k}=\binom{n}{n-k}$,

$$\begin{aligned}c_{i,j}&=\sum_{k=1}^{i}\binom{i}{k}H_k(\alpha)\sum_{t=1}^{i}\binom{j-1}{j-t}\binom{i-j}{t-j}t^{i-k}\\&=\sum_{k=1}^{i}\binom{i}{k}H_k(\alpha)j^{i-k}=h_i(j,\alpha)-j^i,\end{aligned}$$

as claimed. Similarly, for $i=j$, it seen that the sum is equal to $c_{i,i}=h_i(i,\alpha)-i^i$. So, we have the proof. ■

Corollary 4 *Let n and m be positive integers such that $1 \leq m \leq n$. We have*

$$\sum_{i=1}^{n}\sum_{t=1}^{i}(-1)^{i}\binom{m-1}{i-1}\binom{i-1}{t-1}(-1)^{t}\left(h_{n}\left(t,\alpha\right)-t^{n}\right)=h_{n}\left(m,\alpha\right)-m^{n}.$$

Proof. Equating (n, m)-entries of matrix $C_n = L_n U_n$ gives the claimed result. ∎

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