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Criteria Analysis Of Fractional Derivative For Mathematical Modeling Using CNR Concept*

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Abstract

In this work, a new methodical classification process of fractional derivatives is suggested and introduces CNR(Classification rule) concept. CNR concept is a mathematical tool for classifying the criteria of a fractional derivative. More fractional derivatives are proposed as fractional modeling is now the subject of extensive research. The CNR approach effectively eliminates any worries regarding the theoretical basis of operators that arise as a part of fractional modeling. The CNR concept is conceived through the classification of fractional operators so that the classification proposed here also becomes an integral part of fractional calculus. We apply this classification rule to categorize five specific criteria related to fractional derivatives. Using this classification rule, we have proved that linearity is an essential criterion for every fractional operator.

1 Introduction

Recent studies show that fractional calculus with fractional derivatives and their generalizations are useful mathematical method to model real-life problems. But the plethora of definitions makes this field more complex. At the same time, each definition has its own use in various fields. In this work, a classification of the fractional derivatives were done by using a mathematical model defined as the CNR concept. When considering the history and evolution of fractional derivatives, the Reimann-Liovillie derivative and the Caputo derivative are the first to be analyzed the most. The Reimann-Liovillie and Caputo derivatives can be retrieved from the Hilfer fractional derivative by using particular values of the parameter. But ϕ-Hilfer fractional derivative is a generalization of Hilfer fractional derivative [55]. Another important fact is that so many fractional derivatives can be retrieved using ϕ-Hilfer fractional derivative [55]. The M-operator is a generalization of the Katugampola derivative [56], which is used in quantum mechanics. Fractional modelling is widely used in the medical field as well. For example, the nonlocal behavior of fractional derivatives is used to model heat conduction in biological tissues [36], and the beta operator is used to model infectious diseases [1]. The generalized fractional derivative generalizes the beta, katugampola, and conformable fractional derivatives. All the above-mentioned derivatives have their place in the application of fractional calculus. So we cannot avoid them by considering their use in mathematical modelling. Recently, many fractional derivatives have been introduced for describing various phenomena in real life [63], especially fractional derivatives with non-singular kernals. This is a topic clearly stated in the study conducted in 2018 under the leadership of Hong Guang Sun [47]. The study has been done by classifying the areas where fractional derivatives are used: physics, control theory, signal and image processing, mechanics and dynamic systems, biology, environmental science, materials, economics, and multidisciplinary engineering fields. This kind of work reminds us how important the field of fractional derivatives is to mathematical modelling. Recently, fractional modelling has played a major role in the study of the Corona virus [54, 45, 7]. Through all this modelling, new generalized derivative operators have also emerged [32, 34, 30]. But our present system is not able to provide clear criteria for any of them. That is where this article is going to come in handy.

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This work is organized as follows: Section 2 includes a review study about the evolution of criteria analysis and classification of fractional derivatives, and the newly introduced classification of fractional derivatives is mentioned in Section 3, which has mainly four classes. We classify these derivatives by considering the terms included in the definition. The CNR concept introduced in Section 4 also includes the advantages of classification in criteria analysis and proves why linearity is a strict sense criteria for fractional derivatives. Section 5 includes discussion about the concepts described in this work and how the CNR concept is useful for mathematical modelling. Section 6 concludes this article with concluding remarks and future work. The main goals of this work is as follows:

1. Introduce a new classification of fractional derivative based on the nature of the derivative.
2. Introduce the CNR (classification rule) concept for criteria analysis. This is structured using the newly introduced classification.
3. Analyze and classify the existing criteria using the CNR concept. This explains how we can apply the CNR concept to fractional operators, that is, first identify the classification and then check the properties to be satisfied.
4. We integrate the CNR concept with fractional modeling, providing comprehensive guidelines on its application and elucidating how it is employed in the context of fractional modeling.

2 Literature Review

The growing number of definitions reminds us that the study of the laws that the fractional derivative must obey is essential. Tarasov argues that the Leibniz rule is not a necessary condition for fractional derivatives [51], but the Leibniz rule is included in the criteria proposed by Ortigueira [42]. This uncertainty can be studied further using the classification we describe here. In 1977, Ross made an authoritative study of the rules that fractional derivatives should follow [44]. But since then, the field of fractional calculus has undergone many transformations. The concept of fractional derivatives or integrals later became useful for mathematical modelling of many real-life problems. As a result, different definitions of fractional derivatives have been suggested because it doesn't have a clear geometrical definition. Much of it was beyond Ross's vision. Ortigueira and Machado are also interested in analysing the conditions of fractional derivatives in 2015 [42]. Their study also attracted much attention in this area. By the time of their study, the field of fractional calculus had undergone further analysis in the scientific sense. But after 2015, fractional calculus got a new dimension. Fractional derivatives using non-singular kernel functions have attracted more attention in these areas. The complexity of this area has further increased due to the increasing number of fractional derivatives. But Teodoro Machado and Oliveira exposed this complexity and served to make the field more systematic [53]. Apart from that, he put five conditions for the fractional derivatives, which are included in the set of local operators that he suggested. From there, the semi-group property was completely eliminated. So we are forced to accept the situation that we still don't have a generalized format to prove the criterion that fractional derivatives obey. Such a generalized format is intended by the classification proposed in this work. Through this classification, we arrive at the CNR concept, which is more beneficial in the theory of fractional calculus. We need to examine how the classical rules of differentiation translate into the realm of fractional derivatives, such as Leibniz's rule, chain rule, semigroup property, etc. Consider Leibniz's rule, traditionally employed to find the derivative of a product function. However, when dealing with fractional operators, the application of Leibniz's rule becomes more intricate. Each operator exhibits distinct behavior, leading to variations in the formula for determining the derivative of a product function, denoted as $h = fg$. In this context, the formula for $D^\alpha(fg)$ can be termed the Generalized Leibniz Rule. In the realm of fractional modeling, the differentiation of a product function is crucial for gaining a deeper understanding that will prove beneficial in subsequent analyses. The application of Leibniz's rule emerges as a fundamental tool for dissecting and comprehending such scenarios. Another important property that we have to analyze is the semigroup property, that is $D^\alpha D^\beta f = D^{\alpha+\beta} f$. Examining the semigroup property represents a crucial aspect in the realm of fractional analysis. An intriguing study conducted by Nguyen Dinh Cong delves into solutions for fractional differential equations [13]. Nguyen Dinh Cong adeptly employed semigroup concepts to transform a multiterm fractional differential equation into a single-term equation through a reduction process. However, it's noteworthy that the semigroup property is not a global condition, posing a significant challenge for Nguyen Dinh Cong and their research team. To address these issues, a resolution is proposed using the CNR concept, the intricate details of which are thoroughly expounded in Section 5.

The derivative of a composite function is also important in the theory of differential calculus. so it is important to analyze in fractional calculus as well. Understanding this concept becomes crucial. The commonly employed formula for computing the derivative of a composite function is recognized as the chain rule, and its applicability is widespread in practical problem-solving. In the context of neural networks, the cost function often involves a composition of an activation function and a linear function. Utilizing the chain rule becomes imperative, especially in the application of backpropagation techniques for weight updates. In fractional neural networks, the development of formulas for determining fractional derivatives of composite functions becomes essential for effectively updating weights. Guy Jumarie delves into the exploration of the chain rule concerning fractional operators and arrives at the conclusion that there isn't a unique chain rule for such operators [27]. Jumarie establishes certain rules for the fractional derivative of composite functions and outlines their applications in system modeling, That chain rules for fractional operators are as follows,

1. $D^\alpha(f(g(x))) = (D^\alpha f)(g(x)) * (D^\alpha g)(x)$.
2. $D^\alpha(f(g(x))) = f^\alpha(u)\,(u')^\alpha$.
3. $D^\alpha(f(g(x))) = \left(\frac{f(g(x))}{g(x)}\right)^{1-\alpha} (D'^\alpha D^\alpha g(x)$.
4. $D^\alpha(f(g(x))) = \Gamma(2-\alpha)g^{\alpha-1}(D^\alpha f)D^\alpha g(x)$.

However, Tarasov disproved the validity of the chain rule proposed by Jumarie [50]. This discrepancy needs to be clarified, emphasizing the importance of attaining a thorough comprehension of the criteria governing fractional operators. The introduced CNR concept serves as a valuable tool for categorizing these criteria. Additional criteria under consideration in this analysis include integer-order unification and linearity. These factors hold significance in the field of fractional calculus, but they can be viewed as overarching conditions.

3 Classification of Fractional Derivative Operators

In [53], G. Sales Teodoro, J.A. Tenreiro Machado, and E. Capelas de Oliveira propose a classification of non-integer derivative operators into four classes, such as classical fractional operators, modified operators, local operators, and operators with non-singular kernals. Here we intend to divide fractional derivatives into four classes. While categorizing the fractional derivatives, there must be a criterion for it. The criterion considered here is the nature of the derivative, or the terms included in the definition. Fractional derivatives have been generalized in this way in order to study the rules to be followed by the fractional derivative operators. Fractional derivatives are categorized into four types, such as Type-1, Type-2, Type-3, and Type-4. It should be noted first that the number of parameters and variables used in the kernal functions used in each generalized form can increase. The entire manuscript considers the left derivatives; the right derivatives can also be included in the same way.

3.1 Type-1 Class

The general form of the Type-1 class is:

$$D^\alpha(f(x)) = f_1(\alpha,\beta;x,t)\frac{d^k}{dx^k}\int_a^x f_2(\alpha,\beta;x,t)f(t)dt \tag{3.1}$$

where f_1 and f_2 are outer and inner kernal functions which are continuous on (a,x). In most of the cases f_1 is constant. One of the main attraction of this *Type-1* class of derivatives is its wide class of domain. Domain of this derivatives includes integrable functions, more precisely f satisfy $\int_a^x f_2(\alpha,\beta;x,t)f(t)dt$ is k differentiable. But we cannot argue that the domain is L^1 or its particular subset, it may varying depends on its kernal function f_2. The existence of a derivative for the product of functions $f_2 * f$ on the interval (a,x) is contingent upon the integrability of $f_2 * f$ within that interval. While kernel functions typically adhere to continuity, the potential for developing new fractional operators introduces the possibility of kernel functions being discontinuous. Thus, there's no limitation imposed on the parent space of the kernel function. However, for this definition to make sense, f_2 must be integrable.

d_1 ϕ-Riemann-Liouville fractional derivative [28]

$$D^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\phi'(t)}\right)^n \frac{d}{dx}^n \int_a^x \phi'^{n-\alpha-1} f(t)dt. \tag{3.2}$$

d_2 Riemann-Liouville derivative [29] $\phi(x) = x$ in (3.2).

d_3 Katugampola fractional derivative [33]

$$\phi(x) = x^\rho, \text{ in (3.2)}.$$

d_4 Hadamard [19]

$$\phi(x) = \ln(x) \text{ in (3.2)}.$$

d_5 Riemann derivative [39]

$$\phi(x) = x, \quad a = 0 \text{ in (3.2)}.$$

d_6 Liovillie derivative [10]

$$\phi(x) = x, \quad a = -\infty \text{ in (3.2)}.$$

d_7 Riesz derivative [10]

$$\phi(x) = x, \quad a = -\infty, \quad b = \infty \text{ in (3.2)}.$$

d_8 Feller derivative [38, 65]

$$\phi(x) = x, \quad a = -\infty, \quad b = \infty, \quad 0 < \eta < 1 \text{ in (3.2)}.$$

d_9 Chen derivative [14]

$$\phi(x) = x, \ \ a = c, \text{ in (3.2)}.$$

d_{10} Jumarie derivative [24, 25, 35, 17]

$$\phi(x) = x, \ \ a = 0, \ \ g(x) = f(x) - f(0) \text{ in (3.2)}.$$

d_{11} Weyl derivative [59]

$$\phi(x) = x, \ \ b = \infty \text{ in (3.2)},$$

$$D_\infty^\alpha[f(x)] = (-1)^m\left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}[f(t)]dt.$$

d_{12} Erdélyi-Kober derivative of $f(x)$ [28]

$$\phi(x) = x^{\sigma}, \;\; g(x) = x^{\sigma(\mu+\alpha)} f(x), \;\; \text{in (3.2)}.$$

And take the derivative of $g(x)$

$$D^{\alpha}_{a+;\beta,\gamma} f(x) = x^{-\beta(\alpha+\gamma)} \left(\frac{1}{\beta x^{\beta-1}} \frac{d}{dt} \right)^{n} \frac{\beta x^{\beta * n}}{\Gamma(\alpha)} \int_a^x f(t)(x^{\beta} - t^{\beta})^{\alpha-1} t^{\beta\gamma+\beta-1} dt \qquad \alpha > -n.$$

d_{13} R-L type Hilfer derivative [20]

$$\phi(x) = x, \;\; 0 < \alpha < 1,$$

$$D_a^{\alpha,\beta}[f(x)] = \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_a^x f(t)(x-t)^{\alpha-1} dt, \;\; \text{where } t \geq a.$$

d_{14} R-L type K-Hilfer derivative [55]

$$\phi(x) = x, \;\; 0 < \alpha < 1,$$

$${}^{k}D^{\alpha}[f(x)] = \frac{1}{k\Gamma_k(\alpha)} \frac{d}{dx} \int_0^x f(t)(x-t)^{\frac{\alpha}{k}-1} dt, \;\; \text{where } t \geq 0.$$

d_{15} R-L type Hilfer-Katugampola derivative [43]

$$\phi(x) = \frac{t^{\rho}}{\rho}, \;\; 0 < \alpha < 1,$$

$${}^{\rho}D_a^{\alpha,\beta}[f(x)] = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} t^{1-\rho} \frac{d}{dx} \int_a^x f(t)(x^{\rho} - t^{\rho})^{\alpha-1} dt, \;\; \text{where } x > a.$$

d_{16} Prabhakar derivative [18]

$$\phi(x) = x, f_2 = E^{\gamma}_{\alpha,\beta}[\lambda(t-u)^{\alpha}] \;\; \text{in (3.2)},$$

$${}^{P}D^{\alpha,\beta}[f(x)] = \frac{d}{dx}^{n} \int_{t_0}^{t} (t-u)^{\beta-1} E^{\gamma}_{\alpha,\beta}[\lambda(t-u)^{\alpha}] f(u) du.$$

d_{17} Davidson-Essex [53]

$$f_1 = \frac{\alpha}{\Gamma(1-\alpha)}, \;\; f_2 = (x-t)^{-\alpha}, \;\; k = n,$$

$$D_0^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} \frac{d^n}{dt^n}[f(t)] dt.$$

d_{18} Canavati [53]

$$f_1 = \frac{\alpha}{\Gamma(1-\alpha)}, \;\; f_2 = (x-t)^{\beta}, \;\; k = n,$$

$${}_aD_x^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{\beta} \frac{d^n}{dt^n}[f(t)] dt \qquad n = \lfloor \alpha \rfloor, \; \beta = n - \alpha.$$

d_{19} Atangana-Baleanu $R-L$ type [2]

$$f_1 = \frac{B(\alpha)}{1-\alpha}, \;\; f_2 = E_{\alpha}\left(-\alpha \frac{(x-t)^{\alpha}}{1-\alpha} \right), \;\; k = 0,$$

$${}^{ABR-L}_{a}D_x^{\alpha}[f(x)] = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^x f(t) E_{\alpha}\left(-\alpha \frac{(x-t)^{\alpha}}{1-\alpha} \right) dx, \qquad x > a.$$

d_{20} Yang [60]

$$f_1 = \frac{R(\alpha)}{(1-\alpha)}, \;\; f_2 = e^{-\frac{\alpha(x-t)^\alpha}{1-\alpha}}, \;\; k = 0,$$

$$D_s^\alpha[f(x)] = \frac{R(\alpha)}{(1-\alpha)}\frac{d}{dt}\int_a^x f(t)e^{-\frac{\alpha(x-t)^\alpha}{1-\alpha}}dt.$$

d_{21} Generalized $R-L$ type [49]

$$f_1 = \frac{G(\alpha)}{1-\alpha}, \;\; f_2 = E_\beta\left(-\alpha\frac{(x-t)^\beta}{1-\alpha}\right),$$

$$D_x^{\alpha,\beta}[f(x)] = \frac{G(\alpha)}{1-\alpha}\frac{d}{dt}\int_b^x f(t)E_\beta\left(-\alpha\frac{(x-t)^\beta}{1-\alpha}\right)dx, \quad x > b, \; \beta \in [0,1].$$

d_{22} Yang $R-L$ type [61]

$$f_1 = 1, \;\; f_2 = E_{\phi,\phi}((\alpha_1, v_1), ..., (\alpha_n, v_n); (x-t)^\alpha), \;\; k = 0,$$

$${}^{RL}_{E_{\phi,\phi}}D^\alpha[f(x)] = \frac{d}{dt}\int_a^x E_{\phi,\phi}((\alpha_1, v_1), ..., (\alpha_n, v_n); (x-t)^\alpha)f(t)dt.$$

d_{23} General $R-L$ type [62]

$$f_1 = 1, \;\; f_2 = P(x-t), \;\; k = 0,$$

$$D_P^{R-L}[f(x)] = \frac{d}{dt}\int_a^x P(x-t)f(t)dt.$$

3.2 Type-2 Class

General form of *Type-2* class is,

$$D^\alpha(f(x)) = f_1(\alpha,\beta;x,t)\int_a^x f_2(\alpha,\beta;x,t)f^k(t)dt \tag{3.3}$$

Where f_1 and f_2 are outer and inner kernal functions which are continuous on (a, x). In most of the cases f_1 is constant. Domain of *Type-2* class of derivatives is the k differentiable functions. Similar to *Type-1*, we are not imposing any constraints on the kernel function here. While there are fractional operators with both singular and non-singular kernals, all of the kernal functions that are known to exist are continuous on (a, x). It does not imply that $C(a, b)$ is the home of the kernel function. In the future, we intend to define fractional operators with discontinuous kernal functions. However, for this definition to make sense, f_2 must be integrable.

d_1 Caputo-Fabrizio [11]

$$f_1 = \frac{M(\alpha)}{1-\alpha}, \;\; f_2 = e^{-\frac{\alpha(x-t)}{1-\alpha}}, \;\; k = 0,$$

$$D_t^\alpha[f(x)] = \frac{M(\alpha)}{1-\alpha}\int_a^x f'^{-\frac{\alpha(x-t)}{1-\alpha}}dt.$$

d_2 Atangana-Baleanu Caputo type [2]

$$f_1 = \frac{B(\alpha)}{1-\alpha}, \;\; f_2 = E_\alpha\left(-\alpha\frac{(x-t)^\alpha}{1-\alpha}\right), \;\; k = 0,$$

$${}^{ABC}_{a}D_x^\alpha[f(x)] = \frac{B(\alpha)}{1-\alpha}\int_a^x f'(t)E_\alpha\left(-\alpha\frac{(x-t)^\alpha}{1-\alpha}\right)dx, \quad x > b.$$

d_3 Generalized Caputo type [49]

$$f_1 = \frac{G(\alpha)}{1-\alpha}, \;\; f_2 = E_\beta\left(-\alpha\frac{(x-t)^\beta}{1-\alpha}\right), \;\; k=0,$$

$$^{gC}D_x^{\alpha,\beta}[f(x)] = \frac{G(\alpha)}{1-\alpha}\int_b^x f'(t)E_\beta\left(-\alpha\frac{(x-t)^\beta}{1-\alpha}\right)dx, \quad x>b, \;\beta\in[0,1],$$

d_4 Sun-Hao-Zhang-Baleanu [48]

$$f_1 = \frac{M(\alpha)}{(1-\alpha)^{\frac{1}{\alpha}}}, \;\; f_2 = e^{-\frac{\alpha(x-t)^\alpha}{1-\alpha}}, \;\; k=0,$$

$$^{SE}D_x^\alpha[f(x)] = \frac{M(\alpha)}{(1-\alpha)^{\frac{1}{\alpha}}}\int_a^x f'^{-\frac{\alpha(x-t)^\alpha}{1-\alpha}}dt.$$

d_5 Caputo-Fabrizio with Gaussian kernel [12]

$$f_1 = \frac{1+\alpha^2)}{\sqrt{\pi^\alpha(1-\alpha)}}, \;\; f_2 = e^{-\frac{\alpha(x-t)^2}{1-\alpha}}, \;\; k=0,$$

$$^{CF}D^\alpha[f(x)] = \frac{1+\alpha^2)}{\sqrt{\pi^\alpha(1-\alpha)}}\int_a^x f'^{-\frac{\alpha(x-t)^2}{1-\alpha}}dt \quad f(a)=0.$$

d_6 Yang Liouville-Caputo type [61]

$$f_1 = 1, \;\; f_2 = E_{\phi,\phi}((\alpha_1,v_1),...,(\alpha_n,v_n);(x-t)^\alpha), \;\; k=0,$$

$$^{C}_{E_{\phi,\phi}}D^\alpha[f(x)] = \int_a^x E_{\phi,\phi}((\alpha_1,v_1),...,(\alpha_n,v_n);(x-t)^\alpha)\frac{d}{dt}f(t)dt.$$

d_7 General Liouville-Caputo type [62]

$$f_1 = 1, \;\; f_2 = P(x-t), \;\; k=0,$$

$$D_P^C[f(x)] = \int_a^x P(x-t)f'(t)dt.$$

d_8 R-L type Hilfer derivative [20]

$$\phi(x) = x, \;\; 0<\alpha<1,$$

$$D_a^{\alpha,\beta}[f(x)] = \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_a^x f(t)\,(x-t)^{\alpha-1}\,dt, \;\; \text{where} \;\; t\geq a.$$

d_9 R-L type K-Hilfer derivative [55]

$$\phi(x) = x, \;\; 0<\alpha<1,$$

$$^kD^\alpha[f(x)] = \frac{1}{k\Gamma_k(\alpha)}\frac{d}{dx}\int_0^x f(t)\,(x-t)^{\frac{\alpha}{k}-1}\,dt, \;\; \text{where} \;\; t\geq 0.$$

d_{10} R-L type Hilfer-Katugampola derivative [43]

$$\phi(x) = \frac{t^\rho}{\rho}, \;\; 0<\alpha<1,$$

$$^\rho D_a^{\alpha,\beta}[f(x)] = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}t^{1-\rho}\int_a^x f'^\rho - t^\rho)^{\alpha-1}dt, x>a \;\; \text{where} \;\; \rho>0,$$

$$D^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^n(t)dt.$$

d_{13} ϕ-Capito fractional derivative [4]

$$D^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \phi'^{n-\alpha-1} \left(\frac{1}{\phi'(t)} \right)^n f^n(t)dt \tag{3.4}$$

d_{14} Caputo-Katugampola fractional derivative [6]

$$\phi(x) = x^r \quad \text{in (3.4).}$$

d_{15} Caputo Derivative [9] in (3.4).

d_{16} Caputo-Hadamard [5]

$$\phi(x) = \ln(x) \quad \text{in (3.4).}$$

d_{17} Liovillie- Caputo derivative [40]

$$\phi(x) = x, \;\; a = -\infty \quad \text{in (3.4).}$$

d_{18} Caputo-Riesz derivative [26]

$$\phi(x) = x \quad \text{in (3.4).}$$

d_{19} Prabhakar derivative [18]

$$\phi(x) = x, \;\; f_2 = E^{\gamma}_{\alpha,\beta}[\lambda(t-u)^{\alpha}] \quad \text{in (3.4),}$$

$${}^{P}D^{\alpha,\beta}[f(x)] = \int_{t_0}^{t} (t-u)^{\beta-1} E^{\gamma}_{\alpha,\beta}[\lambda(t-u)^{\alpha}] f^n(u)du.$$

d_{20} ϕ fractional derivative [23]

$$D^{-\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\phi'(t)f(t)}{(\phi(x)-\phi(t))^{1-\alpha}} dt. \tag{3.5}$$

3.3 Type-3 Class

This *Type-3* class mainly includes the local operators. The general form of this class of derivatives is,

$$D^{\alpha} f(x) = \lim_{x \to y} \frac{kf(g(x,t,y)) - f(y)}{(x^{\alpha} - y^{\alpha})^{1-\beta}(x-y)^{\beta}} \qquad h = (x-y).$$

The kernal function g belongs to $C(a,b)$.

d_1 Chen [52]

$$g = x, \;\; k = 1,$$

$$D^{\alpha} f(x) = \lim_{y \to x} \frac{f(x) - f(y)}{x^{\alpha} - y^{\alpha}}.$$

d_2 Conformable [31]

$$g = x + hx^{1-\alpha}, \;\; k = 1, \;\; \beta = 1,$$

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}.$$

d_3 Katugampola [33]

$$g = xe^{hx^{-\alpha}}, \;\; k = 1,$$

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{f(xe^{hx^{-\alpha}}) - f(x)}{h}.$$

d_4 M [56]

$$g = xE_\beta(hx^{-\alpha}), \ \ k=1,$$
$$D^\alpha f(x) = \lim_{h\to 0}\frac{f(xE_\beta(hx^{-\alpha})) - f(x)}{h} \qquad \beta > 0.$$

d_5 Beta [1]

$$g = \left(x + h\left(x + \frac{1}{\Gamma(\beta)}\right)\right), \ \ k=1,$$
$${}_0D_x^\beta f(x) = \lim_{h\to 0}\frac{f\left(x + h\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(x)}{h}.$$

d_6 Deformable [64]

$$g = xe^{hx^{-\alpha}}, \ \ k = 1 + hp,$$
$$D^\alpha f(x) = \lim_{h\to 0}\frac{(1+hp)f(x+h\alpha) - f(x)}{h} \qquad p + \alpha = 1.$$

d_7 Generalized conformable [21]

$$g = x + h(K(x))^{1-\alpha}, \ \ k=1,$$
$$D^\alpha f(x) = \lim_{h\to 0}\frac{f(x + h(K(x))^{1-\alpha}) - f(x)}{h}.$$

d_8 Generalized [3]

$$g = \left(x - k(x) + k(x)e^{\frac{h(k(x))^{-\alpha}}{k'(x)}}\right), \ \ k=1,$$
$$D^\alpha f(x) = \lim_{h\to 0}\frac{f\left(x - k(x) + k(x)e^{\frac{h(k(x))^{-\alpha}}{k'(x)}}\right) - f(x)}{h}.$$

d_9 Truncated-ϑ [57]

$$g = x_i H^{\alpha,p,q}_{\gamma,\beta,\delta}(hx^{-\alpha}), \ \ k=1,$$
$$\vartheta^{\alpha,p,q}_{\gamma,\beta,\delta} f(x) = \lim_{h\to 0}\frac{f(x_i H^{\alpha,p,q}_{\gamma,\beta,\delta}(hx^{-\alpha})) - f(x)}{h}.$$

d_{10} General conformable [66]

$$g = x + h\phi(x,p)), \ \ k=1$$
$$D^\alpha f(x) = \lim_{h\to 0}\frac{f(x + h\phi(x,p)) - f(x)}{h}.$$

d_{11} N-conformable [16]

$$g = x + he^{t^{-\alpha}}, \ \ k=1,$$
$$N_1^\alpha f(x) = \lim_{h\to 0}\frac{f(x + he^{t^{-\alpha}}) - f(x)}{h}.$$

d_{12} Multi Index Generalized Derivative [58]

$$g = x + hv(x,\alpha), \ \ k=1,$$
$$\mathbb{N}_v^\alpha f(x) = \lim_{h\to 0}\frac{f(x + hv(t,\alpha) - f(x)}{h}.$$

d_{13} Juan derivative [22]

$$D^\alpha f(x) = \lim_{y\to x}\frac{d^q(f(y) - f(x))}{d(x-y)^\alpha}.$$

3.4 Type-4 Class

This section includes the fractional derivative having series expression in its definition. The general form of this class of derivative is

$$D_a^{\alpha}[f(x)] = \lim_{h\to 0} \frac{p(\alpha,t)}{h^{\alpha}} \sum_{k=-\infty}^{\infty} (-1)^k f_1(\alpha;x,t) f(f_2(\alpha;x,t)).$$

The kernal functions f_1 and f_2 belongs to $C(a,b)$.

d_1 Grünwald-Letnikov derivative

$$f_1 = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad f_2 = x - kh, \quad p = 1,$$

$$^{GL}D_a^{\alpha}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1) f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \qquad nh = b - x.$$

d_2 Unified fractional derivative [41]

$$f_1 = \frac{\Gamma(\alpha+1) f(\frac{x-kh+\gamma h}{2})}{\Gamma\frac{(\alpha+t)}{(2-k+1)}\Gamma\frac{(\alpha-t)}{(2-k+1)}}, \quad f_2 = x - kh, \quad p = 1,$$

$$^{UD}D_a^{\alpha}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\alpha}} e^{\frac{-j\pi(\alpha-t)}{2}} \sum_{k=-\infty}^{\infty} (-1)^k \frac{\Gamma(\alpha+1) f(\frac{x-kh+\gamma h}{2})}{\Gamma\frac{(\alpha+t)}{(2-k+1)}\Gamma\frac{(\alpha-t)}{(2-k+1)}}.$$

d_3 Generalized Conformable derivative [15]

$$f_1 = (-1)^k \binom{\lfloor \alpha+1 \rfloor}{k}, \quad f_2 = x - khT(x,\alpha), \quad p = \frac{1}{h^{\lfloor \alpha+1 \rfloor}},$$

$$G_T^{\alpha}[f(x)] = \lim_{h\to 0} \frac{1}{h^{\lfloor \alpha+1 \rfloor}} \sum_{k=0}^{\lfloor \alpha+1 \rfloor} (-1)^k \binom{\lfloor \alpha+1 \rfloor}{k} f(x - khT(x,\alpha)).$$

Here, the characteristics of kernel functions are not described, due to the fact that it is the kernal function's general form. Depending on what we require, we can explain the kernel function's characteristics. We can describe the properties of kernal functions for each proof that someone wants to use to establish certain properties of the fractional derivative. This allows us to choose which fractional operator to employ and which kernal functions will work best for our purposes.

4 CNR Concept

4.1 Advantage of Classification in Criteria of Fractional Derivative

Ross's criteria

1. fractional derivative preserve analyticity.
2. fractional derivative should satisfy ordinary derivative also.
3. $D^0 f(x) = f(x)$.
4. Fractional derivative should be a linear operator.

5. Fractional derivative should satisfy semigroup property.

Ortigueira and Machado modify this criteria as follows.

1. Fractional derivative should be a linear operator.
2. fractional derivative should satisfy ordinary derivative also.
3. $D^0 f(x) = f(x)$.
4. $D^\alpha D^\beta = D^{\alpha+\beta}$ for every real α, β.
5. Fractional derivative satisfies Generalized Leibniz rule.

Criteria for local operators proposed by Teodoro, Tenreiro, and Oliveira are as follows.

1. Fractional derivative should be a linear operator.
2. fractional derivative should satisfy ordinary derivative also.
3. Derivative of a constant function is 0.
4. Fractional derivative satisfies Chain rule.
5. Fractional derivative satisfies Generalized Leibniz rule.

Using the criteria explained and the classification, the criteria itself can be classified into two. strict sense criteria and wide sense criteria.
Strict sense criteria(SSC):- The criteria can be proved in every class of derivatives.
Wide sense criteria(WSC):- The criteria cannot be proved in every class of derivatives.
After classifying the criteria in this way, the next question is which criteria will be included in SSC and which criteria will be included in WSC. At the end of this work, we can come to an agreement about that. We need to re-specify WSC itself because there are criteria that some operators obey only in special cases. For example, the Caputo-Fabrizio operator, ABC operator, Generalized Caputo operator, and General liovillie Caputo operator obey the zero-order unification rule only in special cases. The semigroup property can also be included in the same category. But in this work, everything is contained in a single class called WSC, because we are concerned with the rules that the fractional derivative must obey. The criteria can be clearly categorized from the table 1. Let us first consider the linearity property, which all studies undoubtedly accept. section 4.2 will be a guide of how a property should be proved using the classification suggested here. This method of proving criterion in the generalized format is called CNR(classification rule) concept. In many scientific and engineering applications where fractional calculus is used, the criterion analysis of fractional operators is essential. Non-local behaviours, long-range dependencies, and memory are all modelled using fractional operators, such as fractional derivatives and integrals. The following are some major points that emphasise how crucial criterion analysis is for fractional operators:

1. The correctness of mathematical models used to explain occurrences in the actual world is strongly influenced by the choice of fractional operators. Criteria analysis aids in the selection of suitable fractional operators so that the behaviour of the system is correctly described in the model.
2. It is crucial to comprehend the physical meaning of fractional operators. By applying criteria analysis, researchers may determine if a given fractional operator is consistent with the physical properties of the system they are modelling, leading to a more meaningful representation.
3. Different system dynamics can result from different fractional operators. Criteria analysis is a useful tool for assessing how the selection of fractional operators affects the system's general behaviour, stability, and convergence over time.

4. The selection of fractional operators determines the stability and efficiency of algorithms in numerical simulations and computational investigations. The process of choosing operators that preserve computational efficiency and numerical stability is aided by criteria analysis.
5. The choice of fractional operators should be in line with observed behaviours when using fractional calculus to analyse experimental data. By helping to match the model with actual data, criteria analysis increases the prediction capacity of the model.
6. Researchers can better appreciate the importance of fractional operators in capturing system complexity by comparing fractional and classical models with the aid of criteria analysis.
7. Fractional orders are frequently used as parameters in fractional operators. By assisting in the selection of suitable fractional orders, criteria analysis guarantees that the model accurately captures the fundamental physical characteristics of the system.

To sum up, criterion analysis for fractional operators is crucial since it helps to guarantee the precision, usefulness, and effectiveness of fractional calculus models in a variety of engineering and scientific fields. To obtain relevant and trustworthy findings, researchers and practitioners must carefully evaluate and choose fractional operators according to predetermined criteria.

4.2 Linearity Property of Derivatives Using Generalized Classification

4.2.1 Linearity of Type-1 Generalization

Theorem 4.1 *If*

$$D^\alpha[f(x)] = f_1(\alpha,\beta;x,t)\frac{d}{dx}^k \int_a^x f_2(\alpha,\beta;x,t)f(t),$$

then $D^\alpha(\lambda f + g) = \lambda D^\alpha f + D^\alpha g$.

Proof.

$$\begin{aligned}
D^\alpha[(\lambda f + g)] =& f_1(\alpha,\beta;x,t)\frac{d}{dx}^k \int_a^x f_2(\alpha,\beta;x,t)(\lambda f(t) + g(t))\\
=& f_1(\alpha,\beta;x,t)\frac{d}{dx}^k \int_a^x \lambda f_2(\alpha,\beta;x,t)f(t) + f_2(\alpha,\beta;x,t)g(t)\\
=& f_1(\alpha,\beta;x,t)\frac{d}{dx}^k \left[\int_a^x \lambda f_2(\alpha,\beta;x,t)f(t) + \int_a^x f_2(\alpha,\beta;x,t)g(t)\right]\\
=& f_1(\alpha,\beta;x,t)\frac{d}{dx}^k \int_a^x \lambda f(t)f_2(\alpha,\beta;x,t) + f_1(\alpha,\beta;x,t)\frac{d}{dx}^k \int_a^x g(t)f_2(\alpha,\beta;x,t)\\
=& \lambda D^\alpha f + D^\alpha g.
\end{aligned}$$

■

Theorem 4.1 proves the linearity of the generalized version of Type-1 class of derivatives. Now we cannot verify the linearity of derivatives in this class separately. Linearity of Type-2 class of derivatives as follows.

4.2.2 Linearity of Type-2 Generalization

Theorem 4.2 *If*

$$D^\alpha[f(x)] = f_1(\alpha,\beta;x,t)\int_a^x f_2(\alpha,\beta;x,t)f^k(t), \tag{4.1}$$

then $D^\alpha(\lambda f + g) = \lambda D^\alpha f + D^\alpha g$.

Proof.

$$
\begin{aligned}
D^{\alpha}[(\lambda f+g)] =& f_1(\alpha,\beta;x,t)\int_a^x f_2(\alpha,\beta;x,t)\frac{d^k}{dt}(\lambda f(t)+g(t))\\
=& f_1(\alpha,\beta;x,t)\int_a^x \lambda f_2(\alpha,\beta;x,t)f^k(t)+f_2(\alpha,\beta;x,t)g^k(t)\\
=& f_1(\alpha,\beta;x,t)\left[\int_a^x \lambda f_2(\alpha,\beta;x,t)f^k(t)+\int_a^x f_2(\alpha,\beta;x,t)g^k(t)\right]\\
=& f_1(\alpha,\beta;x,t)\int_a^x \lambda f_2(\alpha,\beta;x,t)f^k(t)+f_1(\alpha,\beta;x,t)\int_a^x f_2(\alpha,\beta;x,t)g^k(t)\\
=& \lambda D^{\alpha}f+D^{\alpha}g.
\end{aligned}
$$

■

4.2.3 Linearity of Type-3 generalization

Theorem 4.3 *If*

$$D^{\alpha}f(x)=\lim_{x\to y}\frac{kf(g(x,t))-f(y)}{(x^{\alpha}-y^{\alpha})^{1-\beta}(x-y)^{\beta}} \qquad h=(x-y),$$

then $D^{\alpha}(\lambda f+h)=\lambda D^{\alpha}f+D^{\alpha}g.$

Proof.

$$
\begin{aligned}
D^{\alpha}(\lambda f+h) &= \lim_{x\to y}\frac{k(\lambda f+h)(g(x,t))-(\lambda f+h)(y)}{(x^{\alpha}-y^{\alpha})^{1-\beta}(x-y)^{\beta}}\\
&= \lim_{x\to y}\frac{k(\lambda f)(g(x,t))-\lambda(f)(y)+\lambda kh(g(x,t))-\lambda h(y)}{(x^{\alpha}-y^{\alpha})^{1-\beta}(x-y)^{\beta}}\\
&= \lambda D^{\alpha}f+D^{\alpha}g.
\end{aligned}
$$

■

4.3 Linearity of Type-4 Generalizations:

Type-4 includes only two derivatives, its linearity is proved in [41, 29], but we have to prove the linearity in the generalized format.

Theorem 4.4 *If*

$$D_a^{\alpha}[f(x)]=\lim_{h\to 0}\frac{p(\alpha,t)}{h^{\alpha}}\sum_{k=-\infty}^{\infty}(-1)^k f_1(\alpha;x,t)f(f_2(\alpha;x,t)),$$

then $D^{\alpha}(\lambda f+f_1)=\lambda D^{\alpha}f+D^{\alpha}f_1.$

Proof.

$$
\begin{aligned}
D_a^\alpha[\lambda f(x)+g(x)] &= \lim_{h\to 0}\frac{p(\alpha,t)}{h^\alpha}\sum_{k=-\infty}^{\infty}(-1)^k f_1(\alpha;x,t)\big((\lambda f+g)(f_2(\alpha;x,t)\big)\\
&= \lim_{h\to 0}\frac{p(\alpha,t)}{h^\alpha}\sum_{k=-\infty}^{\infty}(-1)^k f_1(\alpha;x,t)\big(\lambda f(f_2(\alpha;x,t)+g(f_2(\alpha;x,t)\big)\\
&= \lim_{h\to 0}\frac{p(\alpha,t)}{h^\alpha}\sum_{k=-\infty}^{\infty}\Big(\lambda(-1)^k f_1(\alpha;x,t)f(f_2(\alpha;x,t)+(-1)^k f_1(\alpha;x,t)g(f_2(\alpha;x,t)\Big)\\
&= \lim_{h\to 0}\frac{p(\alpha,t)}{h^\alpha}\Bigg(\sum_{k=-\infty}^{\infty}\lambda(-1)^k f_1(\alpha;x,t)f(f_2(\alpha;x,t)+\sum_{k=-\infty}^{\infty}(-1)^k f_1(\alpha;x,t)g(f_2(\alpha;x,t)\Bigg)\\
&=\lambda\sum_{k=-\infty}^{\infty}\lim_{h\to 0}\frac{p(\alpha,t)}{h^\alpha}(-1)^k f_1(\alpha;x,t)f(f_2(\alpha;x,t)\\
&\quad+\lim_{h\to 0}\frac{p(\alpha,t)}{h^\alpha}\sum_{k=-\infty}^{\infty}(-1)^k f_1(\alpha;x,t)g(f_2(\alpha;x,t)\\
&=\lambda D_a^\alpha f+D_a^\alpha g.
\end{aligned}
$$

■

Here we have only proved the linearity. No other properties proposed by Ross [44], Ortigueira [42], and Teodoro [53] are proved here. So many derivatives have real-life applications that do not satisfy the above-mentioned conditions, for example Chen [52], conformable [31], Katugampola [33], M-operatior [56], beta [1], truncated-ϑ [57] violating the semi-group structure. Similarly generalized Leibniz rule is not a global condition for fractional derivative [51]. If these conditions are considered as necessary for a fractional derivative, then so many derivatives are ruled out from fractional derivatives. Therefore, we prioritize each set of rules for each derivative class, such as the study of Teodoro [53]. If someone introduces a new condition for a fractional derivative, we can verify it using the CNR concept. Using CNR concept, generalized Leibniz rule and chain rule is a WSC because generalized Leibniz rule satisfies only Type-1 and Type-2, similarly chain rule satisfies only Type-3 and Type-4 [53]. Table 1 is the foundation of CNR concept, we can add more rows to describe the properties of fractional derivatives in depth. We analyse only five criterion in this table for describing CNR concept and make Table 3.3. Before concluding this work, we considered the integer order unification as also a strict sense criterion for fractional derivative. But so many fractional derivatives defined only in non integer order, for example the derivatives with non singular kernal. The criterion "Integer order unification" is included in SSC without considering this problem or without using CNR concept. Because we need the backing of integer order calculus with a clear theory and geometry foundation. Otherwise fractional calculus cannot be consider as the generalization of integer order calculus. We simply demonstrated the linearity of fractional operators due to the generalised kernal function. However, the characteristics of the kernel function can be explained in order to meet the above criteria. For instance, the generalised form of the kernal function makes it impossible to establish the Leibnitz rule and the chain rule using the CNR notion. For this reason, we may state that the generalised version of kernel functions will be important to the future work on this CNR concept.

5 Discussion and How CNR Concept is Useful in Mathematical Modeling

This work establishes a possible way to classify the fractional derivatives into four classes and shows a way to apply the rules to the derivatives of each class. By dividing the fractional derivatives into four classes, this is a good resource for the efforts to make this field more approachable and systematic. The classification

Criteria	$Type-1$	$Type-2$	$Type-3$	$Type-4$
Linearity	✓	✓	✓	✓
Semigroup	⊗	⊗	×	×
Integer order unification	✓	✓	✓	✓
Generalized Leibniz rule	✓	✓	×	×
Chain rule	⊗	⊗	✓	✓

Table 1: Criteria analysis of fractional operators

SSC	WSC
Linearity	Semigroup
Integer order unification	Generalized Leibniz rule
	Chain rule

Table 2: Classification of criterion of fractional operators

introduced here is more systematic than others because this is not just a mathematical fantasy. Also, the CNR concept is worthy of the development of fractional calculus through the derivatives introduced as a part of new fractional modelling. This work is useful to prove the results in the generalized format, so it is very easy to verify all the individual derivatives under that generalized format. Table-1 gives the clarity about it. The symbol ✓,×,⊗ resapectively represents the property is satisfied, not satisfied and satisfied partially. The partially satisfied property, which satisfies for certain unique values of the parameter used in the operator, is nothing but it is not a global property. Table-2 represents the classification of fractional operator criterion. In summary, the characteristics of fractional operators, including their order and type, influence how fractional modeling describes real-world phenomena with memory, non-local interactions, and complex dynamics. These operators provide a powerful mathematical framework for capturing the behavior of systems that cannot be fully represented by traditional calculus. The attributes must also be switched when the fractional and integer operators are switched. However, the characteristics of an integer calculus do not necessarily have to be a fractional operator's criterion, therefore we must have a thorough comprehension of these characteristics. In the subject of fractional modelling, optimising the order of the fractional operator is crucial. In a similar vein, selecting the fractional operator in the CNR idea also heavily depends on optimising the kernal function.

5.1 Influence of CNR Concept in Mathematical Modelling

Fractional calculus has been widely used to model real-life problems in recent years. Primarily, we consider modelling the COVID pandemic using fractional operators [8]. Here, the ABC and CF operators are used to model the spread of COVID-19. This ABC derivative is also used to explain the nature of HIV virus spreading [37]. So the ABC fractional operator is so important to the mathematical modelling of pandemic spread. But this ABC derivative doesn't follow the semigroup property, and the integer order unification is partially satisfied. This operator actually has a non-singular kernal but doesn't follow the properties of an integer-order derivative. At the same time, this has several implications for mathematical modelling. At this stage, the classification mentioned here is useful to categorize this operator in $Type-2$ class and identify the properties it satisfies. This clarity will help researchers who are interested in fractional modelling in the future. If Atangana-Baleanu had rejected this operator as not meeting the criteria, it would have been a big loss for us. This same situation occurs for all fractional operators with non-singular kernal. So this CNR concept may become the key tool for mathematical modelling.

5.2 Influence of the CNR Concept on Analyzing Properties such as the Semigroup Property, Chain Rule, and Leibniz Rule in Mathematical Modeling.

In Nguyen Dinh Cong's research, semigroups were notably employed [13]. However, a significant challenge arose due to the non-global nature of the semigroup property, prompting questions about its acceptability as a condition. The CNR concept emerges as a viable solution to this dilemma, offering an explicit resolution. It is crucial to note that these investigations specifically focused on Caputo-type operators. The CNR concept categorizes all Caputo-type operators under Type-1, where the semigroup property is not an absolute criterion but rather a weak one. Consequently, adherence to the semigroup condition is not obligatory for every order; it suffices to comply with a specific set of real numbers. This flexibility mirrors the situation with classical derivatives. The validation of Nguyen Dinh Cong's work through the CNR concept not only addresses these concerns but also serves as a significant catalyst for advancing further studies on fractional differential equations. Exploring solutions to fractional differential equations is a key aspect of advancing fractional modeling. Fractional modeling inherently involves the formulation of fractional differential equations. A thorough examination of the semigroup property holds the potential to introduce novel methods for effectively solving these fractional differential equations.

Fractional neural networks extend traditional neural network frameworks by integrating principles from fractional calculus. Their incorporation contributes to the wider investigation of sophisticated mathematical tools within the realms of artificial intelligence and machine learning. The progress of fractional neural networks encompasses tasks such as establishing the definitions of fractional derivatives within neural network architectures, formulating training algorithms that incorporate fractional calculus, and examining how fractional operators impact the network's performance and capabilities. Exploring the Leibniz rule and the chain rule for fractional operators is instrumental in advancing the field of fractional neural networks. In the mathematical theory of neural networks, the primary objective is to determine the optimal weight vector. Currently, backpropagation heavily relies on conventional gradient descent methods. This domain calls for the development of novel techniques to minimize the cost function in the context of backpropagation. Insufficient understanding of the chain rule formula has a detrimental impact on the advancement of fractional neural networks. Specifically, when dealing with integer-order derivatives, adherence to the rules for a chosen set of points is adequate. CNR concept allows us to assert that the chain rule may be viewed as a relatively lenient criterion. Consequently, we can confidently employ fractional derivatives of composite functions even when they deviate from the entire set of real numbers. As an illustration, Jumarie proposed specific chain rule formulas for certain system modeling applications [27]. However, Tarasov contradicted these formulas by employing the function x^n [50]. By leveraging the CNR concept, Jumarie can sidestep Tarasov's argument since he did not employ functions resembling x^n. To eliminate any ambiguity, the CNR concept can be universally applied across diverse systems. However, clarity is essential– must explicitly outline the set of fractional orders required for modeling, identify the specific functions to be employed, and formulate rules within this designated framework for predetermined points. It's important to note that these rules may not necessarily be applicable to other functions or orders beyond the specified set. Within a fractional neural network, the application of chain rule formulas for backpropagation is facilitated by the CNR concept. However, it is imperative that our cost function aligns with the order of the fractional derivative. Additionally, the choice of the fractional operator plays a crucial role. It is essential to explicitly state the fractional operator, categorize its type, and delineate the specific conditions it adheres to. Sousa and Oliveira have introduced a Leibniz-type rule specifically tailored for the ψ-Hilfer fractional operator [46]. By employing the CNR concept, we can acknowledge the significance of their contribution, emphasizing that the Leibniz rule is applicable exclusively to ψ-Hilfer fractional operators. The robustness of this rule is confirmed within the framework of the CNR concept, as any attempt to disprove it would require the selection of an alternative fractional operator. However, the CNR concept affirms the validity of this rule not only for the ψ-Hilfer fractional operator but also for all operators derived from it, given that the ψ-Hilfer fractional operator serves as a generalized fractional operator. This combination of ideas enhances the understanding of generalized fractional operators, representing a comprehensive set of fractional operators. The Leibniz rule tailored for ψ-Hilfer fractional operators can be applied universally in ψ-Hilfer fractional modeling.

6 Conclusion and Future Works

This work exposes the lack of clarity about the rules that fractional derivatives should obey and finds a remedy for it. This clarity about the rules will also help new fractional derivative operators. That is, if someone proposes a new fractional derivative, we can see what class it belongs to and what rules that operator should obey. Future research pertaining to the CNR idea is entirely dependent on the fractional operator's criteria. This has a positive effect on Stability Analysis, Convergence Analysis, Regularization Techniques, Spectral Analysis, and Generalized Fractional Operators. The CNR concept proposed in this work provides great freedom for researchers in the field of fractional modelling. Future research possibilities for these investigations lie in clarifying which classes of derivatives obey which rules. It is certainly not a small task, the increasing number of definitions of fractional derivatives must be scrutinized to propose new rules and to study and categorize existing rules in greater depth.

7 Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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