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Regular Ideal Convergence Of Double Sequences Of Functions In 2-Normed Spaces*

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Abstract

In this study, we introduce the concepts of regular ideal convergence types and regular ideal Cauchy for double sequences of functions in 2-normed spaces. Also, we study some properties of these new concepts.

1 Introduction

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [19] and Schoenberg [37]. This concept was extended to the double sequences by Mursaleen and Edely [29]. The idea of ideal convergence was introduced by Kostyrko et al. [27] as a generalization of statistical convergence. Nuray and Ruckle [31] independently introduced the same concept with another name generalized statistical convergence. Das et al. [5] introduced the concept of ideal convergence of double sequences in a metric space and studied some properties. Tripathy and Tripathy [40] studied on ideal convergence and regular ideal convergence of double sequences. Dündar and Altay [10] introduced ideal convergence and regular ideal convergence of double sequences. Also, Dündar [6] introduced regular ideal convergence and regular ideal Cauchy double sequences of functions. Recently, Dündar and Akın [15] studied regular ideal convergence of double sequence of sets. Akın [32] investigated regular ideal invariant convergence of double sequences. A lot of development have been made in this area after the works [7, 8, 9, 11, 12, 13, 17, 18, 28, 40, 38].

The concept of 2-normed space was initially introduced by Gähler [20, 21] in the 1960's. Since then, this concept has been studied by many authors [23, 24, 26]. Şahiner et al. [39] and Gürdal [26] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [25] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [34] defined statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Sarabadan and Talebi [35] investigated the concepts $\mathcal{I}_2$-convergence, $\mathcal{I}_2$-limit points and $\mathcal{I}_2$-cluster points of double sequences in 2-normed spaces. Dündar and Sever [14] introduced the notions of $\mathcal{I}_2$ and $\mathcal{I}_2^*$-Cauchy double sequences, and studied their some properties with $(AP2)$ in 2-normed spaces. Yegül and Dündar [45, 46, 47] studied statistical convergence and ideal convergence of double sequences of functions and investigated some properties of these convergence types in 2-normed spaces.

Now, we recall some fundamental definitions and notations (See [1–5, 16, 20, 22, 24, 25, 27, 33–36, 40–47]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\|$: $X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$.

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$.

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(iv) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space. As an example of a 2-normed space, we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose that X is a d-dimensional 2-normed space, where $2 \leq d < \infty$.

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $d(K) = \lim\limits_{n\to\infty} \frac{1}{n}|K_n|$, if it exists.

The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ we have $d(K(\varepsilon)) = 0$, where $K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case, we write $st - \lim x = L$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$. Throughout the paper, we take $\mathcal{I}$ as a nontrivial admissible ideal in $\mathbb{N}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter on X, called the filter associated with $\mathcal{I}$.

Throughout the paper, we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. After this, we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Throughout the paper, we take X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be convergent to f if $f_n(x) \overset{\|.,.\|_Y}{\longrightarrow} f(x)$ for each $x \in X$. We write $f_n \overset{\|.,.\|_Y}{\longrightarrow} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), z\| < \varepsilon.$$

The sequence $\{f_n\}$ is said to be $\mathcal{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$,

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each $x \in X$. In this case, we write $f_n \overset{\|.,.\|_Y}{\longrightarrow}_{\mathcal{I}} f$.

The sequence $\{f_n\}$ is said to be (pointwise) $\mathcal{I}^*$-convergent to f, if there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., $\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each $z \in Y$,

$$\lim_{k\to\infty} \|f_{m_k}(x), z\| = \|f(x), z\|$$

and we write $\mathcal{I}^* - \lim\limits_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$.

A double sequence $x = (x_{mn})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write $\lim\limits_{m,n\to\infty} x_{mn} = L$.

A double sequence $x = (x_{mn})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$ for all $m, n \in \mathbb{N}$, that is, $\|x\|_\infty = \sup\limits_{m,n} |x_{mn}| < \infty$. We let the set of all bounded double sequences by L_∞.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, we have $d_2(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}.$$

A double sequence (x_{mn}) of real numbers is said to be regularly convergent, if it is convergent in Pringsheim's sense and the limits $\lim\limits_{m\to\infty} x_{mn}$ and $\lim\limits_{n\to\infty} x_{mn}$ exist for each fixed $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively. If the double sequence (x_{mn}) is regularly convergent to L in X, then

$$\lim_{n\to\infty} \lim_{m\to\infty} x_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} x_{mn} = L$$

and we write $r - \lim\limits_{m,n\to\infty} x_{mn} = L$.

A double sequence (x_{mn}) in X is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-convergent ($r(\mathcal{I}_2, \mathcal{I})$-convergent), if it is $\mathcal{I}_2$-convergent in Pringsheim's sense and for each $\varepsilon > 0$ and each nonzero $z \in X$, the following statements hold:

$$\{m \in \mathbb{N} : \|x_{mn} - L_n, z\| \geq \varepsilon\} \in \mathcal{I}, \text{ for some } L_n \in X \text{ and for each } n \in \mathbb{N},$$

and

$$\{n \in \mathbb{N} : \|x_{mn} - K_m, z\| \geq \varepsilon\} \in \mathcal{I}, \text{ for some } K_m \in X \text{ and for each } m \in \mathbb{N}.$$

If the double sequence (x_{mn}) is regularly $(\mathcal{I}_2, \mathcal{I})$-convergent ($r(\mathcal{I}_2, \mathcal{I})$-convergent) to $L \in X$, then

$$\mathcal{I} - \lim_{n\to\infty} \lim_{m\to\infty} x_{mn} = \mathcal{I} - \lim_{m\to\infty} \lim_{n\to\infty} x_{mn} = L.$$

A double sequence (x_{mn}) in X is said to be $r(\mathcal{I}_2^*, \mathcal{I}^*)$-convergent, if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$), $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$) such that the limits

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} x_{mn}, \quad \lim_{\substack{m\to\infty \\ m\in M_1}} x_{mn} \quad \text{and} \quad \lim_{\substack{n\to\infty \\ n\in M_2}} x_{mn},$$

exist for each fixed $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively.

A double sequence (x_{mn}) in X is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence ($r(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence), if it is $\mathcal{I}_2$-Cauchy sequence in Pringsheim's sense and for each $\varepsilon > 0$ and each nonzero $z \in X$, there exist $k_n = k_n(\varepsilon, z) \in \mathbb{N}$ and $l_m = l_m(\varepsilon, z) \in \mathbb{N}$ such that the following statements hold:

$$A_1(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}, z\| \geq \varepsilon\} \in \mathcal{I}, \ (n \in \mathbb{N})$$

and

$$A_2(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - x_{ml_m}, z\| \geq \varepsilon\} \in \mathcal{I}, \ (m \in \mathbb{N}).$$

A double sequence (x_{mn}) is said to be regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence ($r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence), if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$, $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$, $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$), for each $\varepsilon > 0$ and each $z \in X$, there exist $N = N(\varepsilon)$, $s = s(\varepsilon)$, $t = t(\varepsilon)$, $(s,t) \in M$, $k_n = k_n(\varepsilon)$, $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that for all $m, n, s, t, k_n, l_m \geq N$,

$$\begin{aligned} &\|x_{mn} - x_{st}, z\| < \varepsilon, \quad \text{for } (m,n), (s,t) \in M, \\ &\|x_{mn} - x_{k_n n}, z\| < \varepsilon, \quad \text{for each } \ m \in M_1 \ \text{ and for each } \ n \in \mathbb{N}, \\ &\|x_{mn} - x_{ml_m}, z\| < \varepsilon, \quad \text{for each } \ n \in M_2 \text{ and for each } \ m \in \mathbb{N}. \end{aligned}$$

After this, we let X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be two double sequences of functions, f, g be two functions from X to Y.

A double sequence $\{f_{mn}\}$ is said to be convergent to f if for each $x \in X$ and for every $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x, \varepsilon)$ such that for all $m, n \geq k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow} f$.

A double sequence $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$,

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in X$. In this case, we write $\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ or $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_{\mathcal{I}_2} f$.

A double sequence $\{f_{mn}\}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}_2^*$-convergent to f, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N} \setminus M \in \mathcal{I}_2$) such that for each $x \in X$, each $z \in Y$ and all $(m,n) \in M$, $\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ and we write $\mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$.

A double sequence $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-Cauchy sequence, if for every $\varepsilon > 0$ and each $x \in X$, there exist $s = s(\varepsilon, x)$, $t = t(\varepsilon, x) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \in \mathcal{I}_2.$$

A double sequence $\{f_{mn}\}$ is said to be $\mathcal{I}_2^*$- Cauchy sequence, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N} \setminus M \in \mathcal{I}_2$) and for every $\varepsilon > 0$ and each $x \in X$, $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for all $(m,n), (s,t) \in M$ and each $z \in Y$ $\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon$, whenever $m, n, s, t > k_0$. In this case, we write $\lim_{m,n,s,t\to\infty} \|f_{mn}(x) - f_{st}(x), z\| = 0$.

If we take $\mathcal{I} = \mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then $\mathcal{I} = \mathcal{I}_d$ is a non-trivial admissible ideal of $\mathbb{N}$ and the ideal convergence coincides with statistical convergence.

If we take $\mathcal{I}_2 = \mathcal{I}_{d_2} = \{A \subset \mathbb{N}\times\mathbb{N} : d_2(A) = 0\}$, then $\mathcal{I}_2 = \mathcal{I}_{d_2}$ is a nontrivial strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$ and the ideal convergence coincides with statistical convergence.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfies the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. (hence $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$).

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property $(AP2)$, if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Throughout the paper, we let convergent instead of pointwise convergent.

Now, we begin with quoting the following lemmas due to Yegül and Dündar [46, 47] which are needed throughout the paper.

Lemma 1 ([46, Theorem 3.2]) *For each $x \in X$ and each nonzero $z \in Y$,*

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \textit{implies} \quad \mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 2 ([46, Theorem 3.8]) *For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \textit{implies} \quad \mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 3 ([46, Theorem 3.9]) *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property $(AP2)$. For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \textit{implies} \quad \mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 4 ([47, Theorem 3.2]) *The double sequence of functions $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent if and only if it is $\mathcal{I}_2$-Cauchy sequence in 2-normed spaces.*

Lemma 5 ([47, Theorem 3.3]) *If the double sequence of functions $\{f_{mn}\}$ is a $\mathcal{I}_2^*$-Cauchy sequence, then it is $\mathcal{I}_2$-Cauchy sequence in 2-normed spaces.*

Lemma 6 ([47, Theorem 3.5]) *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property $(AP2)$. Then, the concepts $\mathcal{I}_2$-Cauchy double sequence and $\mathcal{I}_2^*$-Cauchy double sequence of functions coincide in 2-normed spaces.*

2 Main Results

In this section, firstly, we define the concepts of regular convergence and regular Cauchy double sequence of functions in 2-normed spaces and investigate the relation between them.

Definition 1 *A double sequence $\{f_{mn}\}$ is said to be regularly convergent, if it is convergent in Pringsheim's sense and for each $z \in Y$ and each $x \in X$ the limits*

$$\lim_m \|f_{mn}(x), z\|,\ (n \in \mathbb{N}) \text{ and } \lim_n \|f_{mn}(x), z\|,\ (m \in \mathbb{N})$$

exist. If $\{f_{mn}\}$ is regularly convergent to f, then for each $z \in Y$ and each $x \in X$

$$\lim_n \lim_m \|f_{mn}(x), z\| = \lim_m \lim_n \|f_{mn}(x), z\| = \|f(x), z\|.$$

In this case, for each $z \in Y$ and each $x \in X$ we write

$$r - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad \text{or} \quad f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_r f.$$

Example 1 *Let a double sequence $\{f_{mn}\}$ in $(X, \|.,.\|)$ as following*

$$f_{mn}(x) = \begin{cases} n, & \text{if } m \leq 3, \\ 1, & \text{if } m \geq 4. \end{cases}$$

It is clear that $\{f_{mn}\}$ is convergent to 1 in Pringsheim's sense but it isn't regular convergent.

Definition 2 *A double sequence $\{f_{mn}\}$ is said to be regularly Cauchy double sequence, if for every $\varepsilon > 0$, each $x \in X$ and each $z \in Y$, there exist $N \in \mathbb{N}$, $m_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that we have*

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

for all $m, n, s, t > N$,

$$\|f_{mn}(x) - f_{k_n n}(x), z\| < \varepsilon,\ (n \in \mathbb{N}),$$

for all $m, k_n > m_0$ and

$$\|f_{mn}(x) - f_{m l_m}(x), z\| < \varepsilon,\ (m \in \mathbb{N}),$$

for all $n, l_m > n_0$.

Theorem 1 *If a double sequence $\{f_{mn}\}$ is regularly convergent, then it is a regularly Cauchy double sequence.*

Proof. Let a double sequence $\{f_{mn}\}$ be a regularly convergent. Then, $\{f_{mn}\}$ is convergent in Pringsheim's sense and clearly that $\{f_{mn}\}$ is a Cauchy sequence. Also, for every $\varepsilon > 0$, each $x \in X$ and each $z \in Y$ there exist $m_0, n_0 \in \mathbb{N}$ such that

$$\|f_{mn}(x) - f_n(x), z\| < \frac{\varepsilon}{2},\ (n \in \mathbb{N}),$$

for some f_n and all $m > m_0$, and

$$\|f_{mn}(x) - g_m(x), z\| < \frac{\varepsilon}{2},\ (m \in \mathbb{N}),$$

for some g_m and all $n > n_0$. Then, for each $z \in Y$ and each $x \in X$ we have

$$\begin{aligned}\|f_{mn}(x) - f_{k_n n}(x), z\| &\leq \|f_{mn}(x) - f_n(x), z\| + \|f_{k_n n}(x) - f_n(x), z\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\ (n \in \mathbb{N}),\end{aligned}$$

for some f_n and all $m, k_n > m_0$. Similarly, for each $z \in Y$ and each $x \in X$ we have

$$\begin{aligned}\|f_{mn}(x) - f_{ml_m}(x), z\| &\leq \|f_{mn}(x) - g_m(x), z\| + \|f_{ml_m}(x) - g_m(x), z\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\ (m \in \mathbb{N}),\end{aligned}$$

for some g_m and all $n, l_m > n_0$. Hence, $\{f_{mn}\}$ is a regularly Cauchy double sequence. ■

Now, we introduce the concepts of regular $(\mathcal{I}_2, \mathcal{I})$ and regular $(\mathcal{I}_2^*, \mathcal{I}^*)$-convergence of double sequences of functions in 2-normed spaces and study some properties of them.

Definition 3 *A double sequence $\{f_{mn}\}$ is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-convergent ($r(\mathcal{I}_2, \mathcal{I})$-convergent), if it is $\mathcal{I}_2$-convergent in Pringsheim's sense and for every $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$ the followings hold:*

$$\{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| \geq \varepsilon\} \in \mathcal{I},\ (n \in \mathbb{N}),$$

for some functions f_n and

$$\{n \in \mathbb{N} : \|f_{mn}(x) - g_m(x), z\| \geq \varepsilon\} \in \mathcal{I},\ (m \in \mathbb{N}),$$

for some functions g_m. If $\{f_{mn}\}$ is regularly $(\mathcal{I}_2, \mathcal{I})$-convergent to f, then for each $x \in X$ and each nonzero $z \in Y$, we have

$$\mathcal{I} - \lim_n \lim_m \|f_{mn}(x), z\| = \mathcal{I} - \lim_m \lim_n \|f_{mn}(x), z\| = \|f(x), z\|.$$

In this case, we write

$$r(\mathcal{I}_2, \mathcal{I}) - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Example 2 *Let the ideals $\mathcal{I} = \mathcal{I}_d$, $\mathcal{I}_2 = \mathcal{I}_{d_2}$ and a double sequence $\{f_{mn}\}$ in $(X, \|.,.\|)$ as following*

$$f_{mn}(x) = \begin{cases} n, & \text{if } m \leq 2, \\ (m+n)x, & \text{if } m = n = k^2,\ k \in \mathbb{N} \text{ and } m \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

For $m \geq 3$, every $\varepsilon > 0$, each $x \in X$ and each $z \in Y$ we have

$$K(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - 0, z\| \geq \varepsilon\} = \{(4,4), (9,9), (16,16), \cdots\},$$

$$K_1(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - 0, z\| \geq \varepsilon\} = \{4, 9, 16, \cdots\},$$

for each $n \in \mathbb{N}$ and

$$K_2(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - 0\| \geq \varepsilon\} = \{4, 9, 16, \cdots\}$$

for each $m \in \mathbb{N}$ and so, $d_2(K(\varepsilon)) = 0$, $d(K_1(\varepsilon)) = 0$ and $d(K_2(\varepsilon)) = 0$. Hence, it is clear that $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent to 0 but it isn't regular $(\mathcal{I}_2, \mathcal{I})$-convergent.

Example 3 *Let the ideals $\mathcal{I}=\mathcal{I}_d$, $\mathcal{I}_2=\mathcal{I}_{d_2}$ and a double sequence $\{f_{mn}\}$ in $(X,\|.,.\|)$ as following*

$$f_{mn}(x)=\begin{cases} mn, & \text{if } m=n=k^3 \text{ and } k\in\mathbb{N}\\ 2, & \text{otherwise.}\end{cases}$$

For every $\varepsilon>0$, each $x\in X$ and each $z\in Y$ we have

$$K(\varepsilon)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-2,z\|\geq\varepsilon\}=\{(1,1),(8,8),(27,27),(64,64),(125,125),\cdots\},$$

$$K_1(\varepsilon)=\{m\in\mathbb{N}:\|f_{mn}(x)-2,z\|\geq\varepsilon\}=\{1,8,27,64,125,\cdots\},$$

for each $n\in\mathbb{N}$ and

$$K_2(\varepsilon)=\{n\in\mathbb{N}:\|f_{mn}(x)-2\|\geq\varepsilon\}=\{1,8,27,64,125,\cdots\}$$

for each $m\in\mathbb{N}$ and so, $d_2(K(\varepsilon))=0$, $d(K_1(\varepsilon))=0$ and $d(K_2(\varepsilon))=0$. Hence, it is clear that $\{f_{mn}\}$ is regularly $(\mathcal{I}_2,\mathcal{I})$-convergent to 2.

Theorem 2 *If a double sequence $\{f_{mn}\}$ is regularly convergent, then it is regularly $(\mathcal{I}_2,\mathcal{I})$-convergent to same point.*

Proof. Let a double sequence $\{f_{mn}\}$ is regularly convergent. Then, $\{f_{mn}\}$ is convergent in Pringsheim's sense and by Lemma 1, $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent to same point. Also, for every $\varepsilon>0$, each $x\in X$ and each $z\in Y$ there exist $m_0\in\mathbb{N}$ and $n_0\in\mathbb{N}$ such that

$$\|f_{mn}(x)-f_n(x),z\|<\varepsilon,\ (n\in\mathbb{N}),$$

for some f_n and all $m>m_0$, and

$$\|f_{mn}(x)-g_m(x),z\|<\varepsilon,\ (m\in\mathbb{N}),$$

for some g_m and all $n>n_0$. Then, since $\mathcal{I}$ is admissible ideal so for every $\varepsilon>0$, each $x\in X$ and each nonzero $z\in Y$, we have

$$\{m\in\mathbb{N}:\|f_{mn}(x)-f_n(x),z\|\geq\varepsilon\}\subseteq\{1,2,...,m_0\}\in\mathcal{I},\ (n\in\mathbb{N})$$

and

$$\{n\in\mathbb{N}:\|f_{mn}(x)-g_m(x),z\|\geq\varepsilon\}\subseteq\{1,2,...,n_0\}\in\mathcal{I},\ (m\in\mathbb{N}).$$

Hence, $\{f_{mn}\}$ is regularly $(\mathcal{I}_2,\mathcal{I})$-convergent. ■

Definition 4 *A double sequence $\{f_{mn}\}$ is said to be regularly $(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent ($r(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent), if there exist the sets $M\in\mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus M\in\mathcal{I}_2$), $M_1\in\mathcal{F}(\mathcal{I})$ and $M_2\in\mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\setminus M_1\in\mathcal{I}$ and $\mathbb{N}\setminus M_2\in\mathcal{I}$) such that for each $z\in Y$ and each $x\in X$, the limits*

$$\lim_{\substack{m,n\to\infty\\(m,n)\in M}}\|f_{mn}(x),z\|,\ \lim_{\substack{m\to\infty\\m\in M_1}}\|f_{mn}(x),z\|,\ (n\in\mathbb{N})\text{ and }\lim_{\substack{n\to\infty\\n\in M_2}}\|f_{mn}(x),z\|,\ (m\in\mathbb{N})$$

exist. If $\{f_{mn}\}$ is regularly $(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent to f, then for each $x\in X$ and each $z\in Y$, we have

$$\mathcal{I}^*-\lim_n\lim_m\|f_{mn}(x),z\|=\mathcal{I}^*-\lim_m\lim_n\|f_{mn}(x),z\|=\|f(x),z\|.$$

In this case, we write

$$r(\mathcal{I}_2^*,\mathcal{I}^*)-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|.$$

Theorem 3 *If a double sequence $\{f_{mn}\}$ is regularly $(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent, then it is regularly $(\mathcal{I}_2,\mathcal{I})$-convergent to same point.*

Proof. Suppose that $\{f_{mn}\}$ is regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-convergent. Then, it is $\mathcal{I}_2^*$-convergent and so, by Lemma 2, it is $\mathcal{I}_2$-convergent. Also, there exist the sets $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M_1 = H_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 = H_2 \in \mathcal{I}$) such that for each $z \in Y$ and each $x \in X$,

$$(\forall m \in M_1)\,(\forall \varepsilon > 0)\,(\exists m_0 \in \mathbb{N})\,(\forall m > m_0)\,\|f_{mn}(x) - f_n(x), z\| < \varepsilon,\ (n \in \mathbb{N})$$

for some functions f_n and

$$(\forall n \in M_2)\,(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n > n_0)\,\|f_{mn}(x) - g_m(x), z\| < \varepsilon,\ (m \in \mathbb{N})$$

for some functions g_m. Hence, for each $x \in X$ and each nonzero $z \in Y$ we have

$$A(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| \geq \varepsilon\} \subset H_1 \cup \{1, 2, ..., m_0\},\ (n \in \mathbb{N})$$

and

$$B(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - g_m(x), z\| \geq \varepsilon\} \subset H_2 \cup \{1, 2, ..., n_0\},\ (m \in \mathbb{N}).$$

Since $\mathcal{I}$ is admissible ideal, it is clear that

$$H_1 \cup \{1, 2, ..., m_0\} \in \mathcal{I} \text{ and } H_2 \cup \{1, 2, ..., n_0\} \in \mathcal{I}$$

and so we have $A(\varepsilon) \in \mathcal{I}$ and $B(\varepsilon) \in \mathcal{I}$. This shows that the $\{f_{mn}\}$ is regularly $(\mathcal{I}_2, \mathcal{I})$-convergent. ∎

Theorem 4 *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP), $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property $(AP2)$. If a double sequence $\{f_{mn}\}$ is regularly $(\mathcal{I}_2, \mathcal{I})$-convergent, then it is regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-convergent to same point.*

Proof. Let a double sequence $\{f_{mn}\}$ be $r(\mathcal{I}_2, \mathcal{I})$-convergent. Then, $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent and so by Lemma 3, $\{f_{mn}\}$ is $\mathcal{I}_2^*$-convergent to same point. Also, for every $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$ we have

$$A(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| \geq \varepsilon\} \in \mathcal{I},\ (n \in \mathbb{N}),$$

for some functions f_n and

$$C(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - g_m(x), z\| \geq \varepsilon\} \in \mathcal{I},\ (m \in \mathbb{N}),$$

for some functions g_m.

Now, for each $x \in X$ and each nonzero $z \in Y$ put

$$A_1 = \{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| \geq 1\},\ (n \in \mathbb{N})$$

and

$$A_p = \left\{m \in \mathbb{N} : \frac{1}{p} \leq \|f_{mn}(x) - f_n(x), z\| < \frac{1}{p-1}\right\},\ (n \in \mathbb{N}),$$

for $p \geq 2$, some functions f_n. It is clear that $A_p \cap A_i = \emptyset$ for $p \neq i$ and $A_p \in \mathcal{I}$ for each $p \in \mathbb{N}$. By the property (AP) there is a countable family of sets $\{B_1, B_2, \ldots\}$ in $\mathcal{I}$ such that $A_i \bigtriangleup B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

For each $x \in X$ and each $z \in Y$ we prove that

$$\lim_{\substack{m\to\infty \\ m\in M_1}} \|f_{mn}(x) - f_n(x), z\| = 0, (n \in \mathbb{N}),$$

for some function f_n and $M_1 = \mathbb{N}\backslash B \in \mathcal{F}(\mathcal{I})$. Let $\delta > 0$ be given. Choose $p \in \mathbb{N}$ such that $1/p < \delta$. Then, for each $x \in X$ and each nonzero $z \in Y$ we have

$$\{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| \geq \delta\} \subset \bigcup_{i=1}^{p} A_i,\ (n \in \mathbb{N}),$$

for some function f_n. Since $A_i \triangle B_i$ is a finite set for $i = 1, \dots, p$, there exists $m_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{i=1}^{p} B_i\Big) \cap \{m : m \geq m_0\} = \Big(\bigcup_{i=1}^{p} A_i\Big) \cap \{m : m \geq m_0\}.$$

If $m \notin B$ and $m \geq m_0$, then

$$m \notin \bigcup_{i=1}^{p} B_i \text{ and } \text{ so } m \notin \bigcup_{i=1}^{p} A_i.$$

Thus, for each $x \in X$ and each $z \in Y$ we have

$$\|f_{mn}(x) - f_n(x), z\| < \frac{1}{p} < \delta, \ (n \in \mathbb{N}),$$

and so this implies that

$$\lim_{\substack{m\to\infty \\ m \in M_1}} \|f_{mn}(x) - f_n(x), z\| = 0, \ (n \in \mathbb{N}),$$

for some function f_n. Hence, for each $x \in X$ and each $z \in Y$ we have

$$\mathcal{I}^* - \lim_{m\to\infty} \|f_{mn}(x) - f_n(x), z\| = 0, \ (n \in \mathbb{N}),$$

for some function f_n.

Similarly, for the set

$$C(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - g_m(x), z\| \geq \varepsilon\} \in \mathcal{I}, \ (m \in \mathbb{N}),$$

for each $x \in X$ and each $z \in Y$ we have

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_{mn}(x) - g_m(x), z\| = 0, \ (m \in \mathbb{N}),$$

for some function g_m. Consequently, $\{f_{mn}\}$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-convergent. ∎

Now, for a double sequence $\{f_{mn}\}$, we give the definitions of $r(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence and $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence in 2-normed spaces and study some properties of them.

Definition 5 *A double sequence $\{f_{mn}\}$ is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence ($r(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence), if it is $\mathcal{I}_2$-Cauchy sequence in Pringsheim's sense and for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, there exist $k_n \in \mathbb{N}$ and $l_m \in \mathbb{N}$ such that the following statements hold:*

$$A_1(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - f_{k_n n}(x), z\| \geq \varepsilon\} \in \mathcal{I}, \ (n \in \mathbb{N}),$$

and

$$A_2(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - f_{m l_m}(x), z\| \geq \varepsilon\} \in \mathcal{I}, \ (m \in \mathbb{N}).$$

Theorem 5 *If a double sequence $\{f_{mn}\}$ is regularly Cauchy double sequence, then $\{f_{mn}\}$ is regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.*

Proof. Let the double sequence $\{f_{mn}\}$ be a regularly Cauchy double sequence. Then, for every $\varepsilon > 0$, each $x \in X$ and each $z \in Y$, there exist $N \in \mathbb{N}$, $m_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that we have

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

for all $m, n, s, t > N$,

$$\|f_{mn}(x) - f_{k_n n}(x), z\| < \varepsilon, \ (n \in \mathbb{N}),$$

for all $m, k_n > m_0$ and

$$\|f_{mn}(x) - f_{ml_m}(x), z\| < \varepsilon, \ (m \in \mathbb{N}),$$

for all $n, l_m > n_0$. Then, for every $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$ there exist $s, t, k_n, l_m \in \mathbb{N}$ such that we have

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \\ &\subseteq (\mathbb{N}\times\{1,2,\ldots,m_0\}) \cup (\{1,2,\ldots,n_0\}\times\mathbb{N}), \end{aligned}$$

$$B(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - f_{k_n n}(x), z\| \geq \varepsilon\} \subseteq \{1,2,\ldots,m_0\}, \ (n \in \mathbb{N})$$

and

$$C(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - f_{ml_m}(x), z\| \geq \varepsilon\} \subseteq \{1,2,\ldots,n_0\}, \ (m \in \mathbb{N}).$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, we have

$$(\mathbb{N}\times\{1,2,\ldots,m_0\}) \cup (\{1,2,\ldots,n_0\}\times\mathbb{N}) \in \mathcal{I}_2,$$

$$\{1,2,\ldots,m_0\} \in \mathcal{I} \text{ and } \{1,2,\ldots,n_0\} \in \mathcal{I}.$$

Then, we have

$$A(\varepsilon) \in \mathcal{I}_2, \ B(\varepsilon) \in \mathcal{I} \text{ and } C(\varepsilon) \in \mathcal{I}$$

and so, $\{f_{mn}\}$ is regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence of functions. ∎

Theorem 6 *If a double sequence $\{f_{mn}\}$ is $r(\mathcal{I}_2, \mathcal{I})$-convergent, then $\{f_{mn}\}$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.*

Proof. Let $\{f_{mn}\}$ be a $r(\mathcal{I}_2, \mathcal{I})$-convergent. Then, $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent and by Lemma 4, it is $\mathcal{I}_2$-Cauchy double sequence. Also, for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, we have

$$A_1\Big(\frac{\varepsilon}{2}\Big) = \Big\{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}, \ (n \in \mathbb{N}),$$

for some functions f_n and

$$A_2\Big(\frac{\varepsilon}{2}\Big) = \Big\{n \in \mathbb{N} : \|f_{mn}(x) - g_m(x), z\| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}, \ (m \in \mathbb{N}),$$

for some functions g_m.

Since $\mathcal{I}$ is an admissible ideal, for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, the sets

$$A_1^c\Big(\frac{\varepsilon}{2}\Big) = \Big\{m \in \mathbb{N} : \|f_{mn}(x) - f_n(x), z\| < \frac{\varepsilon}{2}\Big\}, \ (n \in \mathbb{N})$$

for some functions f_n and

$$A_2^c\Big(\frac{\varepsilon}{2}\Big) = \Big\{n \in \mathbb{N} : \|f_{mn}(x) - g_m(x), z\| < \frac{\varepsilon}{2}\Big\}, \ (m \in \mathbb{N})$$

for some functions g_m, are nonempty and belong to $\mathcal{F}(\mathcal{I})$. For $k_n \in A_1^c(\frac{\varepsilon}{2})$ ($n \in \mathbb{N}$ and $k_n > 0$), we have

$$\|f_{k_n n}(x) - f_n(x), z\| < \frac{\varepsilon}{2}$$

for some functions f_n. Now, for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, we define the set

$$B_1(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - f_{k_n n}(x), z\| \geq \varepsilon\}, \ (n \in \mathbb{N}),$$

where $k_n \in \mathbb{N}$. Let $m \in B_1(\varepsilon)$ and $k_n \in A_1^c(\frac{\varepsilon}{2})$ ($n \in \mathbb{N}$ and $k_n > 0$). For each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, we have

$$\begin{aligned} \varepsilon \leq \|f_{mn}(x) - f_{k_n n}(x), z\| &\leq \|f_{mn}(x) - f_n(x), z\| + \|f_{k_n n}(x) - f_n(x), z\| \\ &< \|f_{mn}(x) - f_n(x), z\| + \frac{\varepsilon}{2} \end{aligned}$$

for some functions f_n. This shows that

$$\frac{\varepsilon}{2} < \|f_{mn}(x) - f_n(x), z\| \text{ and so } m \in A_1(\frac{\varepsilon}{2}).$$

Hence, we have

$$B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2}).$$

Similarly, for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$ we have

$$\|f_{ml_m}(x) - g_m(x), z\| < \frac{\varepsilon}{2}, \ (m \in \mathbb{N})$$

for some functions g_m and $l_m \in A_2^c(\frac{\varepsilon}{2})$ ($m \in \mathbb{N}$ and $l_m > 0$). Therefore, for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, it can be seen that

$$B_2(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - g_{ml_m}(x), z\| \geq \varepsilon\} \subset A_2(\frac{\varepsilon}{2}), \quad (m \in \mathbb{N}).$$

Hence, we have

$$B_1(\varepsilon) \in \mathcal{I} \text{ and } B_2(\varepsilon) \in \mathcal{I}.$$

This shows that $\{f_{mn}\}$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence. ■

Definition 6 *A double sequence $\{f_{mn}\}$ is said to be regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence ($r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence), if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$, $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$, $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$), for each $\varepsilon > 0$, each $x \in X$ and each $z \in Y$ there exist $N \in \mathbb{N}$, $m_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that we have*

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon, \ ((m, n), (s, t) \in M),$$

for all $m, n, s, t > N$,

$$\|f_{mn}(x) - f_{k_n n}(x), z\| < \varepsilon, \ (m \in M_1, n \in \mathbb{N}),$$

for all $m, k_n > m_0$ and

$$\|f_{mn}(x) - f_{ml_m}(x), z\| < \varepsilon, \ (n \in M_2, m \in \mathbb{N}),$$

for all $n, l_m > n_0$.

Theorem 7 *If a double sequence $\{f_{mn}\}$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence, then it is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.*

Proof. Since a double sequence $\{f_{mn}\}$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence, it is $\mathcal{I}_2^*$-Cauchy sequence. We know that $\mathcal{I}_2^*$-Cauchy sequence implies $\mathcal{I}_2$-Cauchy sequence by Lemma 5. Also, since the double sequence $\{f_{mn}\}$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence so there exist the sets $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$), for each $\varepsilon > 0$, each $x \in X$ and each $z \in Y$ there exist $m_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that we have

$$\|f_{mn}(x) - f_{k_n n}(x), z\| < \varepsilon, \ (m \in M_1, n \in \mathbb{N}),$$

for all $m, k_n > m_0$ and

$$\|f_{mn}(x) - f_{ml_m}(x), z\| < \varepsilon, \ (n \in M_2, m \in \mathbb{N}),$$

for all $n, l_m > n_0$. Therefore, for $H_1 = \mathbb{N} \backslash M_1 \in \mathcal{I}, H_2 = \mathbb{N} \backslash M_2 \in \mathcal{I}$ and for each $\varepsilon > 0$, each $x \in X$ and each nonzero $z \in Y$, there exist $k_n \in \mathbb{N}$ and $l_m \in \mathbb{N}$ such that we have

$$A_1(\varepsilon) = \{m \in \mathbb{N} : \|f_{mn}(x) - f_{k_n n}(x), z\| \geq \varepsilon\} \subset H_1 \cup \{1, 2, \dots, m_0\}, \ (n \in \mathbb{N}),$$

for $m \in M_1$ and

$$A_2(\varepsilon) = \{n \in \mathbb{N} : \|f_{mn}(x) - f_{ml_m}(x), z\| \geq \varepsilon\} \subset H_2 \cup \{1, 2, \dots, n_0\}, \ (m \in \mathbb{N}),$$

for $n \in M_2$. Since $\mathcal{I}$ is an admissible ideal,

$$H_1 \cup \{1, 2, \dots, m_0\} \in \mathcal{I} \text{ and } H_2 \cup \{1, 2, \dots, n_0\} \in \mathcal{I}.$$

Hence, we have

$$A_1(\varepsilon) \in \mathcal{I} \text{ and } A_2(\varepsilon) \in \mathcal{I}$$

and so $\{f_{mn}\}$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence. ■

The following theorem can be proved as in [7, 30]. Therefore, we give this theorem without proof.

Theorem 8 *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP), $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property $(AP2)$. If a double sequence $\{f_{mn}\}$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence, then it is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence.*

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