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On $h(x)$ Lucas p-Polynomials*

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Abstract

In this paper, $h(x)$ Lucas p-polynomials and a new Lucas $Q_{p,h}(x)$ matrix are defined where $p(\geq 0)$ is an integer and $h(x)(> 0)$ is a polynomial with real coefficients. The properties of $Q_{p,h}(x)$ matrix, relations among the code matrix elements and correct ability are mentioned.

1 Introduction

The Lucas p-numbers [3] are given by the recurrence relation

$$L_p(n) = L_p(n-1) + L_p(n-p-1)$$

with $n > p+1$ and initial seeds

$$L_p(1) = L_p(2) = \cdots = L_p(p) = 1, \quad L_p(p+1) = p+2$$

where $p = 0, 1, 2, \cdots$.

For $p = 1$, $L_1(n) = L_n$ are known as classical Lucas numbers. The ratio of two consecutive Lucas numbers converges to the irrational number, $\mu = \lim_{n\longrightarrow\infty} \frac{L_{n+1}}{L_n} = \frac{1+\sqrt{5}}{2}$ which is known as golden mean or golden ratio or golden proportion. The Lucas numbers and golden mean are useful in physical sciences, chemical sciences, mathematical sciences, computer sciences, biological sciences, architectures and arts [7, 8, 9, 10, 11, 15].

Extensions of Lucas p-numbers [4] are given by the recurrence relation

$$L_{p,m}(n) = mL_{p,m}(n-1) + L_{p,m}(n-p-1)$$

with initial seeds

$$L_{p,m}(1) = a_1, L_{p,m}(2) = a_2, L_{p,m}(3) = a_3, \cdots, L_{p,m}(p+1) = a_{p+1}$$

where $p(\geq 0)$ is integer, $m(> 0)$, $n > p+1$ and a_1, a_2, $a_3, ..., a_{p+1}$ are arbitrary real or complex numbers.

Different types of polynomials are given by Lucas-like recurrence relations. Such polynomials are known as Lucas polynomials. Lucas polynomials, $L_n(x)$ given by the recurrence relation

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2$$

with initial seeds

$$L_0(x) = 2, \quad L_1(x) = x.$$

$h(x)$ Lucas polynomials, $L_{h,n}(x)$ (where $h(x)$ is a polynomial with real coefficients) are given by Nalli and Haukkanen [5] as follows:

$$L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), \quad n \geq 1$$

with initial seeds

$$L_{h,0}(x) = 2, \quad L_{h,1}(x) = h(x).$$

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In this paper, we introduce $h(x)(>0)$ Lucas p-polynomials, $L_{p,h}(n,x)$ by the recurrence relation

$$L_{p,h}(n,x) = h(x)L_{p,h}(n-1,x) + L_{p,h}(n-p-1,x)$$

with initial seeds

$$L_{p,h}(1,x) = b_1, \quad L_{p,h}(2,x) = b_2, \quad L_{p,h}(3,x) = b_3, \cdots, \quad L_{p,h}(p+1,x) = b_{p+1}$$

where p (≥ 0) is integer, $h(x)$ (>0) is a polynomial with real coefficients, $n > p+1$ and b_1, b_2, b_3, $\cdots$, b_{p+1} are arbitrary real or complex numbers.

1.1 $h(x)$ Lucas p-Polynomials, $L_{p,h}(n,x)$

We consider the recurrence relation for integer p (≥ 0), $h(x)(>0)$ and $n = 0, \pm1, \pm2, \pm3, \ldots,$

$$L_{p,h}(n,x) = h(x)L_{p,h}(n-1,x) + L_{p,h}(n-p-1,x) \tag{1}$$

with the initial seeds

$$L_{p,h}(n,x) = h^{n-1}(x), \quad n = 1,2,3,\cdots,p+1. \tag{2}$$

With the help of (1) and (2) we get the following table: Thus, we get $h(x)$ Lucas p-polynomials,

Table 1: $h(x)$ **Lucas p-polynomials,** $L_{p,h}(n,x)$

$n \to$	...	0	-1	...	$-p+1$	$-p$	$-p-1$	...	$-2p+1$	$-2p$	$-2p-1$	$-2p-2$	...
$L_{p,h}(n,x)$	...	0	0	...	0	1	0	...	0	$-h(x)$	1	0	...

$$L_{p,h}(n,x) = h(x)L_{p,h}(n-1,x) + L_{p,h}(n-p-1,x) \tag{3}$$

for $h(x)(>0)$, $n = 0, \pm1, \pm2, \pm3, \ldots$ and $p = 0,1,2,3,\ldots$, where $L_{p,h}(n,x) = h^{n-1}(x)$, $n = 1,2,3,4,\ldots,p+1$. The characteristic equation of $h(x)$ Lucas p-polynomials are given by

$$y^{p+1} - h(x)y^p - 1 = 0. \tag{4}$$

The equation (4) has $(p+1)$ roots. The only one positive root $y = \mu_{p,h}(x)$, called golden $(p, h(x))$-proportion, coincides with $\mu_{p,m}$, golden (p,m)-proportion when $h(x) = m$. Also when $h(x) = 1$, $\mu_{p,h}(x)$ coincides with μ_p, golden p-proportion.

In section 2, $Q_{p,h}(x)$ matrix are defined and the properties of $Q_{p,h}(x)$ matrix are illustrated. In section 3, Lucas coding and decoding method are presented with an example. In section 4, code matrix is defined. In section 5, relations among the code matrix elements on $h(x)$ Lucas p-polynomials are established. We also show that the relations, among the code matrix elements for $h(x) = m(>0)$, coincide with the relations among the code matrix elements for the m-extension of the Lucas p-numbers in information theory [1] and the relations among the code matrix elements for $h(x) = 1$, coincide with the generalized relations among the code matrix elements for the coding theory on Lucas p numbers [2]. In section 6, error detection and correction are described as in [2].

2 Lucas $Q_{p,h}(x)$ Matrix

In this paper, we define a new matrix called Lucas $Q_{p,h}(x)$ matrix of order $(p+1)$ on $h(x)$ Lucas p-polynomial where p (≥ 0) is an integer and $h(x)$ (>0) is a polynomial with real coefficients. The $Q_{p,h}(x)$ matrix is given by

$$Q_{p,h}(x) = \begin{pmatrix} h(x) & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} L_{p,h}(2,x) & L_{p,h}(1,x) & \cdots & L_{p,h}(3-p,x) & L_{p,h}(2-p,x) \\ L_{p,h}(2-p,x) & L_{p,h}(1-p,x) & \cdots & L_{p,h}(3-2p,x) & L_{p,h}(2-2p,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{p,h}(0,x) & L_{p,h}(-1,x) & \cdots & L_{p,h}(1-p,x) & L_{p,h}(-p,x) \\ L_{p,h}(1,x) & L_{p,h}(0,x) & \cdots & L_{p,h}(2-p,x) & L_{p,h}(1-p,x) \end{pmatrix}$$

by using (2).

For examples,

$$Q_{1,h}(x) = \begin{pmatrix} h(x) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_{1,h}(2,x) & L_{1,h}(1,x) \\ L_{1,h}(1,x) & L_{1,h}(0,x) \end{pmatrix},$$

$$Det\ Q_{1,h}(x) = -1 = (-1)^1.$$

$$Q_{2,h}(x) = \begin{pmatrix} h(x) & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{2,h}(2,x) & L_{2,h}(1,x) & L_{2,h}(0,x) \\ L_{2,h}(0,x) & L_{2,h}(-1,x) & L_{2,h}(-2,x) \\ L_{2,h}(1,x) & L_{2,h}(0,x) & L_{2,h}(-1,x) \end{pmatrix},$$

$$Det\ Q_{2,h}(x) = 1 = (-1)^2,$$

$$Q_{3,h}(x) = \begin{pmatrix} h(x) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{3,h}(2,x) & L_{3,h}(1,x) & L_{3,h}(0,x) & L_{3,h}(-1,x) \\ L_{3,h}(-1,x) & L_{3,h}(-2,x) & L_{3,h}(-3,x) & L_{3,h}(-4,x) \\ L_{3,h}(0,x) & L_{3,h}(-1,x) & L_{3,h}(-2,x) & L_{3,h}(-3,x) \\ L_{3,h}(1,x) & L_{3,h}(0,x) & L_{3,h}(-1,x) & L_{3,h}(-2,-x) \end{pmatrix},$$

$$Det\ Q_{3,h}(x) = -1 = (-1)^3,$$

$$Q_{4,h}(x) = \begin{pmatrix} h(x) & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} L_{4,h}(2,x) & L_{4,h}(1,x) & L_{4,h}(0,x) & L_{4,h}(-1,x) & L_{4,h}(-2,x) \\ L_{4,h}(-2,x) & L_{4,h}(-3,x) & L_{4,h}(-4,x) & L_{4,h}(-5,x) & L_{4,h}(-6,x) \\ L_{4,h}(-1,x) & L_{4,h}(-2,x) & L_{4,h}(-3,x) & L_{4,h}(-4,x) & L_{4,h}(-5,x) \\ L_{4,h}(0,x) & L_{4,h}(-1,x) & L_{4,h}(-2,x) & L_{4,h}(-3,x) & L_{4,h}(-4,x) \\ L_{4,h}(1,x) & L_{4,h}(0,x) & L_{4,h}(-1,x) & L_{4,h}(-2,x) & L_{4,h}(-3,x) \end{pmatrix},$$

$$Det\ Q_{4,h}(x) = 1 = (-1)^4,$$

and so on. Thus, $Det\ Q_{p,h}(x) = (-1)^p$, which is independent of $h(x)$.

2.1 Properties of $Q_{p,h}(x)$

Theorem 1 *For a given integer n ($n = 0, \pm1, \pm2, \pm3, \ldots$) the nth power of the $Q_{p,h}(x)$ matrix is given by*

$$Q^n_{p,h}(x) = \begin{pmatrix} L_{p,h}(n+1,x) & L_{p,h}(n,x) & \cdots & L_{p,h}(n-p+2,x) & L_{p,h}(n-p+1,x) \\ L_{p,h}(n-p+1,x) & L_{p,h}(n-p,x) & \cdots & L_{p,h}(n-2p+2,x) & L_{p,h}(n-2p+1,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{p,h}(n-1,x) & L_{p,h}(n-2,x) & \cdots & L_{p,h}(n-p,x) & L_{p,h}(n-p-1,x) \\ L_{p,h}(n,x) & L_{p,h}(n-1,x) & \cdots & L_{p,h}(n-p+1,x) & L_{p,h}(n-p,x) \end{pmatrix}$$

where $L_{p,h}(n,x) = h^{n-1}(x),\ n = 1,2,3,4,\cdots,p+1.$

Proof. When $p = 1$, we have to prove

$$Q^n_{1,h}(x) = \begin{pmatrix} L_{1,h}(n+1,x) & L_{1,h}(n,x) \\ L_{1,h}(n,x) & L_{1,h}(n-1,x) \end{pmatrix}. \tag{5}$$

We will prove it by mathematical induction. For $n = 1$,

$$Q_{1,h}(x) = \begin{pmatrix} h(x) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_{1,h}(2,x) & L_{1,h}(1,x) \\ L_{1,h}(1,x) & L_{1,h}(0,x) \end{pmatrix} \text{ by (2)},$$

which is true for $n = 1$. For $n = 2$,

$$Q^2_{1,h}(x) = \begin{pmatrix} h^2(x)+1 & h(x) \\ h(x) & 1 \end{pmatrix} = \begin{pmatrix} L_{1,h}(3,x) & L_{1,h}(2,x) \\ L_{1,h}(2,x) & L_{1,h}(1,x) \end{pmatrix} \text{ by (2)},$$

which is true for $n = 2$. Suppose (5) is true for integer $n = k$, then

$$Q^k_{1,h}(x) = \begin{pmatrix} L_{1,h}(k+1,x) & L_{1,h}(k,x) \\ L_{1,h}(k,x) & L_{1,h}(k-1,x) \end{pmatrix}.$$

Now, we can write

$$\begin{aligned} Q^{k+1}_{1,h}(x) &= Q^k_{1,h}(x)Q_{1,h}(x) = \begin{pmatrix} L_{1,h}(k+1,x) & L_{1,h}(k,x) \\ L_{1,h}(k,x) & L_{1,h}(k-1,x) \end{pmatrix} \begin{pmatrix} h(x) & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} L_{1,h}(k+2,x) & L_{1,h}(k+1,x) \\ L_{1,h}(k+1,x) & L_{1,h}(k,x) \end{pmatrix} \text{ by (2)}. \end{aligned}$$

Hence by mathematical induction, we proved that

$$Q^n_{1,h}(x) = \begin{pmatrix} L_{1,h}(n+1,x) & L_{1,h}(n,x) \\ L_{1,h}(n,x) & L_{1,h}(n-1,x) \end{pmatrix}.$$

When $p = 2$, we have to prove

$$Q^n_{2,h}(x) = \begin{pmatrix} L_{2,h}(n+1,x) & L_{2,h}(n,x) & L_{2,h}(n-1,x) \\ L_{2,h}(n-1,x) & L_{2,h}(n-2,x) & L_{2,h}(n-3,x) \\ L_{2,h}(n,x) & L_{2,h}(n-1,x) & L_{2,h}(n-2,x) \end{pmatrix}. \tag{6}$$

We will prove it by mathematical induction. For $n = 1$,

$$Q_{2,h}(x) = \begin{pmatrix} h(x) & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{2,h}(2,x) & L_{2,h}(1,x) & L_{2,h}(0,x) \\ L_{2,h}(0,x) & L_{2,h}(-1,x) & L_{2,h}(-2,x) \\ L_{2,h}(1,x) & L_{2,h}(0,x) & L_{2,h}(-1,x) \end{pmatrix} \text{ by (2)},$$

which is true for $n = 1$.

For $n = 2$,

$$Q_{2,h}^2(x) = \begin{pmatrix} h^2(x) & h(x) & 1 \\ 1 & 0 & 0 \\ h(x) & 1 & 0 \end{pmatrix} = \begin{pmatrix} L_{2,h}(3,x) & L_{2,h}(2,x) & L_{2,h}(1,x) \\ L_{2,h}(1,x) & L_{2,h}(0,x) & L_{2,h}(-1,x) \\ L_{2,h}(2,x) & L_{2,h}(1,x) & L_{2,h}(0,x) \end{pmatrix} \text{ by (2)},$$

which is true for $n = 2$.

Suppose (6) is true for integer $n = k$, then

$$Q_{2,h}^k(x) = \begin{pmatrix} L_{2,h}(k+1,x) & L_{2,h}(k,x) & L_{2,h}(k-1,x) \\ L_{2,h}(k-1,x) & L_{2,h}(k-2,x) & L_{2,h}(k-3,x) \\ L_{2,h}(k,x) & L_{2,h}(k-1,x) & L_{2,h}(k-2,x) \end{pmatrix}.$$

Now, we can write

$$\begin{aligned} Q_{2,h}^{k+1}(x) &= Q_{2,h}^k(x) Q_{2,h}(x) \\ &= \begin{pmatrix} L_{2,h}(k+1,x) & L_{2,h}(k,x) & L_{2,h}(k-1,x) \\ L_{2,h}(k-1,x) & L_{2,h}(k-2,x) & L_{2,h}(k-3,x) \\ L_{2,h}(k,x) & L_{2,h}(k-1,x) & L_{2,h}(k-2,x) \end{pmatrix} \begin{pmatrix} h(x) & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} L_{2,h}(k+2,x) & L_{2,h}(k+1,x) & L_{2,h}(k,x) \\ L_{2,h}(k,x) & L_{2,h}(k-1,x) & L_{2,h}(k-2,x) \\ L_{2,h}(k+1,x) & L_{2,h}(k,x) & L_{2,h}(k-1,x) \end{pmatrix} \text{ by (2)}. \end{aligned}$$

Hence by mathematical induction, we proved that

$$Q_{2,h}^n(x) = \begin{pmatrix} L_{2,h}(n+1,x) & L_{2,h}(n,x) & L_{2,h}(n-1,x) \\ L_{2,h}(n-1,x) & L_{2,h}(n-2,x) & L_{2,h}(n-3,x) \\ L_{2,h}(n,x) & L_{2,h}(n-1,x) & L_{2,h}(n-2,x) \end{pmatrix}.$$

Therefore, by mathematical induction it can be proved for all values of p. Hence the theorem. ∎

Theorem 2 $Q_{p,h}^n(x) = h(x) Q_{p,h}^{n-1}(x) + Q_{p,h}^{n-(p+1)}(x)$.

Proof. By Theorem 1,

$$Q_{p,h}^n(x) = \begin{pmatrix} L_{p,h}(n+1,x) & L_{p,h}(n,x) & \cdots & L_{p,h}(n-p+2,x) & L_{p,h}(n-p+1,x) \\ L_{p,h}(n-p+1,x) & L_{p,h}(n-p,x) & \cdots & L_{p,h}(n-2p+2,x) & L_{p,h}(n-2p+1,x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{p,h}(n-1,x) & L_{p,h}(n-2,x) & \cdots & L_{p,h}(n-p,x) & L_{p,h}(n-p-1,x) \\ L_{p,h}(n,x) & L_{p,h}(n-1,x) & \cdots & L_{p,h}(n-p+1,x) & L_{p,h}(n-p,x) \end{pmatrix}.$$

When $p = 1$,

$$\begin{aligned} Q_{1,h}^n(x) &= \begin{pmatrix} L_{1,h}(n+1,x) & L_{1,h}(n,x) \\ L_{1,h}(n,x) & L_{1,h}(n-1,x) \end{pmatrix} \\ &= \begin{pmatrix} h(x)L_{1,h}(n,x) + L_{1,h}(n-1,x) & h(x)L_{1,h}(n-1,x) + L_{1,h}(n-2,x) \\ h(x)L_{1,h}(n-1,x) + L_{1,h}(n-2,x) & h(x)L_{1,h}(n-2,x) + L_{1,h}(n-3,x) \end{pmatrix} \\ &= h(x) Q_{1,h}^{n-1}(x) + Q_{1,h}^{n-2}(x). \end{aligned}$$

When $p = 2$,

$$\begin{aligned}
& Q_{2,h}^{n}(x) \\
= & \begin{pmatrix} L_{2,h}(n+1,x) & L_{2,h}(n,x) & L_{2,h}(n-1,x) \\ L_{2,h}(n-1,x) & L_{2,h}(n-2,x) & L_{2,h}(n-3,x) \\ L_{2,h}(n,x) & L_{2,h}(n-1,x) & L_{2,h}(n-2,x) \end{pmatrix} \\
= & \begin{pmatrix} h(x)L_{2,h}(n,x)+L_{2,h}(n-2,x) & h(x)L_{2,h}(n-1,x)+L_{2,h}(n-3,x) & h(x)L_{2,h}(n-2,x)+L_{2,h}(n-4,x) \\ h(x)L_{2,h}(n-2,x)+L_{2,h}(n-4,x) & h(x)L_{2,h}(n-3,x)+L_{2,h}(n-5,x) & h(x)L_{2,h}(n-4,x)+L_{2,h}(n-6,x) \\ h(x)L_{2,h}(n-1,x)+L_{2,h}(n-3,x) & h(x)L_{2,h}(n-2,x)+L_{2,h}(n-4,x) & h(x)L_{2,h}(n-3,x)+L_{2,h}(n-5,x) \end{pmatrix} \\
= & \; h(x)Q_{2,h}^{n-1}(x)+Q_{2,h}^{n-3}(x).
\end{aligned}$$

Similarly, we can show that,

$$Q_{p,h}^{n}(x) = h(x)Q_{p,h}^{n-1}(x) + Q_{p,h}^{n-(p+1)}(x).$$

Hence, the properties of $Q_{p,h}^{n}(x)$ can be summarized as

(i) $Q_{p,h}^{n}(x) = h(x)Q_{p,h}^{n-1}(x) + Q_{p,h}^{n-(p+1)}(x)$.

(ii) $Det\ (Q_{p,h}^{n}(x)) = [Det\ (Q_{p,h}(x))]^{n} = (-1)^{pn}$, which is independent of $h(x)$.

(iii) By inverse matrix method, we get $Q_{p,h}^{-n}(x)$ from $Q_{p,h}^{n}(x)$.

For a particular case $p = 1$, $Q_{1,h}^{-n}(x)$ is given by

$$Q_{1,h}^{-n}(x) = \begin{pmatrix} L_{2,h}(2k-1,x) & -L_{2,h}(2k,x) \\ -L_{2,h}(2k,x) & L_{2,h}(2k+1,x) \end{pmatrix}, \quad n = 2k, \text{ even},$$

$$Q_{1,h}^{-n}(x) = \begin{pmatrix} -L_{2,h}(2k,x) & L_{2,h}(2k+1,x) \\ L_{2,h}(2k+1,x) & -L_{2,h}(2k+2,x) \end{pmatrix}, \; n = 2k+1, \text{ odd},$$

where $p = 1, 2, 3, \cdots$ and $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. ■

3 Lucas $Q_{p,h}(x)$ Coding and Decoding Method

Lucas $Q_{p,h}(x)$ matrix is used to develop applications of the coding theory. We represent the initial message in the form of the square matrix, M of order $(p+1)$ where $p = 1, 2, 3, \cdots$. We take $Q_{p,h}^{n}(x)$ matrix of order $(p+1)$ as a coding matrix and its inverse matrix $Q_{p,h}^{-n}(x)$ as a decoding matrix. We make a transformation $M \times Q_{p,h}^{n}(x) = E$ as coding and a transformation $E \times Q_{p,h}^{-n}(x) = M$ as decoding. We represent E as code matrix.

3.1 Example of Lucas $Q_{p,h}(x)$ Coding and Decoding Method

We represent the initial message in form of the square matrix of order 2

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.$$

Let us assume that all elements of the matrix are positive integers i.e., m_1,m_2,m_3,$m_4 > 0$. For $p = 1$, let us select $Q_{1,h}^{n}(x)$ matrix as the coding matrix for any value of n. For $n = 4$, we have

$$Q_{1,h}^{4}(x) = \begin{pmatrix} h^4(x) + 3h^2(x) + 1 & h^3(x) + 2h(x) \\ h^3(x) + 2h(x) & h^2(x) + 1 \end{pmatrix}.$$

So

$$Q_{1,h}^{-4}(x) = \begin{pmatrix} h^2(x)+1 & -h^3(x)-2h(x) \\ -h^3(x)-2h(x) & h^4(x)+3h^2(x)+1 \end{pmatrix}.$$

Then the $Q_{1,h}^4(x)$ coding of the message, M consists of the multiplication of the message, M by the matrix $Q_{1,h}^4(x)$ that is

$$\begin{aligned} M \times Q_{1,h}^4(x) &= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} h^4(x)+3h^2(x)+1 & h^3(x)+2h(x) \\ h^3(x)+2h(x) & h^2(x)+1 \end{pmatrix} \\ &= \begin{pmatrix} m_1(h^4(x)+3h^2(x)+1)+m_2(h^3(x)+2h(x)) & m_1(h^3(x)+2h(x))+m_2(h^2(x)+1) \\ m_3(h^4(x)+3h^2(x)+1)+m_4(h^3(x)+2h(x)) & m_3(h^3(x)+2h(x))+m_4(h^2(x)+1) \end{pmatrix} \\ &= \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E \qquad (7) \end{aligned}$$

where

$$e_1 = m_1(h^4(x)+3h^2(x)+1)+m_2(h^3(x)+2h(x))$$

$$e_2 = m_1(h^3(x)+2h(x))+m_2(h^2(x)+1),$$

$$e_3 = m_3(h^4(x)+3h^2(x)+1)+m_4(h^3(x)+2h(x)),$$

$$e_4 = m_3(h^3(x)+2h(x))+m_4(h^2(x)+1).$$

Then the code message, E is sent to the channel. The decoding of the code message, E is given by

$$\begin{aligned} & \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \begin{pmatrix} h^2(x)+1 & -h^3(x)-2h(x) \\ -h^3(x)-2h(x) & h^4(x)+3h^2(x)+1 \end{pmatrix} \\ = & \begin{pmatrix} e_1(h^2(x)+1)+e_2(-h^3(x)-2h(x)) & e_1(-h^3(x)-2h(x))+e_2(h^4(x)+3h^2(x)+1) \\ e_3(h^2(x)+1)+e_4(-h^3(x)-2h(x)) & e_3(-h^3(x)-2h(x))+e_4(h^4(x)+3h^2(x)+1) \end{pmatrix} \\ = & \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = M. \end{aligned}$$

4 Determinant of the Code Matrix, E

The code matrix, E is given by the following formula $E = M \times Q_{p,h}^n(x)$. From the matrix theory [6] we have

$$Det\ E = Det\ (M \times Q_{p,h}^n(x)) = Det\ M \times Det\ Q_{p,h}^n(x) = Det\ M \times (-1)^{pn} = (-1)^{pn} \times Det\ M. \qquad (8)$$

5 Relations Among the Code Matrix Elements on $h(x)$ Lucas p-Polynomials

5.1 Case 1: $p = 1$

Similar to [2], we obtain

$$\frac{e_1}{e_2} \approx \mu_{1,h}(x) \ \text{ and } \ \frac{e_3}{e_4} \approx \mu_{1,h}(x)$$

where $\mu_{1,h}(x) = \frac{h(x)+\sqrt{h^2(x)+4}}{2}$, e_1, e_2, e_3, e_4 are given in (7).

5.2 Case 2: $p = 2$

In this case, let the message,

$$M = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}.$$

Then, the $Q^n_{2,h}(x)$ coding of the message, M is

$$M \times Q^n_{2,h}(x) = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} = E.$$

Similar to [2], we obtain

$$\frac{e_1}{e_2} \approx \mu_{2,h}(x), \quad \frac{e_2}{e_3} \approx \mu_{2,h}(x), \quad \frac{e_1}{e_3} \approx \mu^2_{2,h}(x),$$
$$\frac{e_4}{e_5} \approx \mu_{2,h}(x), \quad \frac{e_5}{e_6} \approx \mu_{2,h}(x), \quad \frac{e_4}{e_6} \approx \mu^2_{2,h}(x),$$
$$\frac{e_7}{e_8} \approx \mu_{2,h}(x), \quad \frac{e_8}{e_9} \approx \mu_{2,h}(x), \quad \frac{e_7}{e_9} \approx \mu^2_{2,h}(x),$$

where

$$\mu_{2,h}(x) = \frac{s^2 + 2sh(x) + 4h^2(x)}{6s} \quad \text{and} \quad s = \sqrt{108 + 8h^3(x) + 12\sqrt{81 + 12h^3(x)}}.$$

5.3 Case 3: Generalized Relations Among the Code Elements

In general, similar to [2], when $p = t$ and $n > p+1 = t+1$, The generalized relations among the code matrix elements are

$\frac{e_1}{e_2} \approx \mu_{t,h}(x)$; $\frac{e_2}{e_3} \approx \mu_{t,h}(x)$; $\cdots$; $\frac{e_t}{e_{t+1}} \approx \mu_{t,h}(x)$,

$\frac{e_1}{e_3} \approx \mu^2_{t,h}(x)$; $\frac{e_2}{e_4} \approx \mu^2_{t,h}(x)$; $\cdots$; $\frac{e_{t-1}}{e_{t+1}} \approx \mu^2_{t,h}(x)$,

. . .

. . .

$\frac{e_1}{e_{t+1}} \approx \mu^t_{t,h}(x)$,

where e_1, e_2, e_3, $\cdots$, e_t, e_{t+1} are the first row elements of the code matrix, E. We also obtain similar type of relations among the elements of the second row, third row, $\cdots$, $(t+1)$th row of the code matrix, E where $\mu_{t,h}(x)$ is golden $(t, h(x))$-proportion.

6 Error Detection and Correction

The error-correcting codes [12, 13]) are used widely in information and communication system for the protection of the data from noise. The main purpose of the coding theory are the detection and correction of errors arising in the code message, E under influence of noise in the communication channel. For the Lucas coding and decoding method the correct ability is 93.33% like to [2] and it does not depend on $h(x)$ for a case $p = 1$. Also, the correct ability of this method is 99.80% for $p = 2$ [2]. In general, for $p = t$ and $n > p+1 = t+1$ the correct ability of this method is $\frac{2^{(t+1)^2}-2}{2^{(t+1)^2}-1}$ which depends on p but not on $h(x)$. Hence, for large value of p the correct ability of this method is

$$\frac{2^{(p+1)^2} - 2}{2^{(p+1)^2} - 1} \approx 1 = 100\%.$$

7 Conclusion

The Lucas coding and decoding method is the main application of the Lucas $Q_{p,h}(x)$ matrix. The correct ability of this method for the simple case $p = 1$ is equal 93.33% that exceeds essentially all well-known correcting codes. The correct ability of this method for the case $p = 2$ is 99.80%. The correct ability of this method increases as p increases and it is independent of $h(x)$ but $h(x)$ being a polynomial, improves the security in coding theory.

In future, we hope that the Lucas $Q_{p,h}(x)$ matrices have wide applications in mathematics, physics, genetics, chemical sciences, computer sciences, information and coding theory etc. Future works can be extended for some new codes on the k-Lucas sequence and balancing sequence [14].

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