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A Generalized System Of Fractional Differential Equations Of Beam Type: Existence Of Solutions And Stability*

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Abstract

We study the existence of solutions and their Ulam-stability for a coupled system of nonlinear fractional differential equations. The fractional calculus is used by assuming the Caputo fractional derivative, which indicates a good record for applications. Our method is based on the theory of fixed points. As a consequence, two main theorems on the existence of solutions are proved. Some examples are provided to illustrate the applications of the obtained results. Another main result of Ulam-Hyers stability of the solutions of the system is also discussed.

1 Introduction

The Ulam stability problem deals with the question of whether an approximate solution of a functional equation that is "close" to the exact solution in a certain sense remains "close" under the action of the equation. The study of Ulam stability of fractional order functional equations is an active area of research, and the results depend on the specific form of the equation and the properties of the functions involved. Several techniques, such as fixed point theory and variation methods, have been used to investigate the Ulam stability of fractional-order functional equations. Both classical differential equations and fractional differential equations are used to study Ulam stability, which is a crucial issue. It has numerous uses in biology, economics, and other fields. In order to consider a related differential inequality in Ulam stability, we often first need to demonstrate the existence and uniqueness of solutions for the differential problem under study (see [10, 11, 12, 13]). As such, in the formulation of Ulam-type stability, we generally assume that there exists a resolution to the specific issue such that both solutions are "ϵ-close in the meaning of the employed norm" for any resolution of the relevant differential inequality. For more information and for some applications, the reader is invited to see the research papers [4, 5, 6, 8, 17, 20, 21, 23, 24, 25, 26].

A set of differential equations that describe a beam's behavior usually originates from the ideas of building engineering and continuum mechanics. Euler-Bernoulli beam theory, which ignores shear deformation and makes certain basic assumptions such as minor deflections and rotations, is frequently used to describe beams. In this paper, we are concerned with the study of the existence of solutions and their Ulam-Hyers stability for the following coupled beam system of differential equations:

$$\begin{cases} D^{\alpha}u\left(t\right)=f_{1}\left(t,u\left(t\right),v\left(t\right)\right)+a_{1}g_{1}\left(t,u\left(t\right)\right)+b_{1}h_{1}\left(t,u^{''}\left(t\right)\right), \\ D^{\beta}v\left(t\right)=f_{2}\left(t,u\left(t\right),v\left(t\right)\right)+a_{2}g_{2}\left(t,v\left(t\right)\right)+b_{2}h_{2}\left(t,v^{''}\left(t\right)\right), \end{cases} \tag{1}$$

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under the conditions:

$$\begin{cases} u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases} \tag{2}$$

where, the beam's deflection is given by $u(t)$, $v(t)$, $t \in [0,1]$, $3 < \alpha, \beta \leq 4$, D^{α}, D^{β} denote the Caputo derivatives, and $f_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, g_i, $h_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $a_i, b_i \in \mathbb{R}$. Note that f_i, g_i, h_i, $i = 1,2$ present the distributed load of the beam. Furthermore, extra natural beginning conditions are better satisfied by the Caputo derivative than by other fractional derivatives, which makes it a good choice for modeling dynamic systems with memory effects in many applications. Thus, understanding equations containing the Caputo derivative is essential to providing precise descriptions and analyses of a variety of real-world occurrences.

In the classical case where $\alpha = \beta = 4$, the authors of the paper [27] studied the existence of positive solutions for the following nonlinear coupled system:

$$\begin{cases} U^{(4)}(t) + \beta_1 U''(t) - \alpha_1 U(t) = F_1(t, U(t), V(t)), \\ V^{(4)}(t) + \beta_2 V''(t) - \alpha_2 V(t) = F_2(t, U(t), V(t)), \\ U(0) = U(1) = U''(0) = U''(1) = 0, \\ V(0) = V(1) = V''(0) = V''(1) = 0, \end{cases} \tag{3}$$

where, $F_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $\alpha_i, \beta_i \in \mathbb{R}$ verify the following conditions:

$$\beta_i < 2\pi^2, \quad -\frac{\beta_i^2}{4} \leq \alpha_i, \quad \frac{\alpha_i}{\pi^4} + \frac{\beta_i}{\pi^2} < 1, \quad i = 1,2.$$

An important case to be noted is when $\alpha = \beta = 4$, $f_1(t, u(t), v(t)) = f(t, u(t))$, $f_2(t, u(t), v(t)) = f(t, v(t))$, $g_1(t, u(t)) = u(t)$, $g_2(t, v(t)) = v(t)$, $h_1(t, u''(t)) = u''(t)$, $h_2(t, v''(t)) = v''(t)$ (see [27]). This special case transforms the above fractional differential system into the following ODE:

$$\begin{cases} Z^{(4)}(t) + \beta_1 Z''(t) - \alpha_1 Z(t) = f(t, Z(t)), \\ Z(0) = Z(1) = Z''(0) = Z''(1) = 0. \end{cases} \tag{4}$$

Note that Eq. (4) describes the deformation of an elastic beam supported at its end-points (see [2, 15]). Some research papers dealing with beam equations can be found in [1, 3, 16, 19, 22]. To the best of our knowledge, the above-introduced coupled system has not been treated in its general form as it is presented in (1)–(2). Therefore, based on some of the above-cited papers and taking into account that when α, β tends to the value 4, one can obtain the above two standard differential problems (3)–(4), we think that the above fractional differential system is important to be investigated.

The paper is organized as follows: Section 2 presents some definitions and properties of fractional calculus. In Section 3, we prove our main results for solutions to problem (1). In Section 4, some examples illustrate the two main results. In the last section, two types of stability for the solution of the system have been investigated: Ulam-Hyers stability and generalized Ulam-Hyers stability.

2 Information

In this section, we recall some definitions and properties that will be used in this paper. An auxiliary result is also proved in this section. For more details, we invite the interested reader to read the books [7, 14, 18].

Definition 1 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C([a,b])$ is defined as*

$$J_a^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-\rho)^{\alpha-1} f(\rho)\, d\rho, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{5}$$

Fractional integrals are a useful tool in physics to explain the behavior of memory-containing systems, in which the response at any one moment relies on both the present and previous inputs. Fractional integrals, for example, can be used to represent the stress relaxation or creep behavior in viscoelastic materials. Fractional integrals in fluid dynamics can be used to characterize non-Newtonian fluids, in which the viscosity is dependent on the stress history or the shear rate. In general, systems with complicated dynamics, memory effects, or non-local behaviors are commonly described physically by fractional integrals, which are not well represented by integer-order integrals. It offers a more sophisticated comprehension of several mathematical and scientific phenomena.

Definition 2 *The fractional derivative of $f \in C^p([a,b])$ in the sense of Caputo is defined as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(p-\alpha)} \int_a^t (t-\rho)^{p-\alpha-1} f^{(p)}(\rho)\, d\rho, \quad p-1<\alpha<p,\ p\in\mathbb{N}^*,\ t\in[a,b]. \tag{6}$$

We request the next preliminaries.

Lemma 1 *For $\alpha>0$, the general solution of $D^{\alpha}u(t)=0$ is given by*

$$u(t) = a_0 + a_1 t + \ldots + a_{p-1} t^{p-1}, \tag{7}$$

where $a_i \in \mathbb{R}$, $i=0,1,\ldots,p-1$ and $p=[\alpha]+1$.

Lemma 2 *For $\alpha>0$, the fractional integral and differential operators satisfy*

$$J^{\alpha} D^{\alpha} u(t) = u(t) + a_0 + a_1 t + \ldots + a_{p-1} t^{p-1}, \tag{8}$$

with $a_i \in \mathbb{R}$, $i=0,1,\ldots,p-1$ and $p=[\alpha]+1$.

Lemma 3 (Schaefer Fixed Point Theorem) *Let E be a Banach space and $T: E\to E$ be a completely continuous operator. If the subset*

$$V = \{X \in E : X = \mu T X, \quad 0<\mu<1\},$$

is bounded, then T has at least one fixed point in E.

We prove the following auxiliary result:

Lemma 4 *Let $f \in C([0,1],\mathbb{R})$. The solution of the problem*

$$D^{\alpha} u(t) = f(t), \quad 3<\alpha<4, \tag{9}$$

under the conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} f(\rho)\, d\rho - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} f(\rho)\, d\rho \\ &\quad + \frac{t}{2\Gamma(\alpha-2)} \int_0^1 (1-\rho)^{\alpha-3} f(\rho)\, d\rho - \frac{t^3}{6\Gamma(\alpha-2)} \int_0^1 (1-\rho)^{\alpha-3} f(\rho)\, d\rho. \end{aligned} \tag{10}$$

Proof. We begin by applying J^{α} for both sides of (9), and then we use Lemma 2. So, we have

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\rho)^{\alpha-1} f(\rho)\,d\rho-a_0-a_1t-a_2t^2-a_3t^3. \tag{11}$$

Hence, we state that

$$u^{'}(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\rho)^{\alpha-2} f(\rho)\,d\rho-a_1-2a_2t-3a_3t^2,$$

$$u^{''}(t)=\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-\rho)^{\alpha-3} f(\rho)\,d\rho-2a_2-6a_3t.$$

If $t=0$, we get the two values

$$a_0=a_2=0,$$

and for $t=1$, we get

$$a_1=\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\rho)^{\alpha-1} f(\rho)\,d\rho-\frac{1}{2\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3} f(\rho)\,d\rho,$$

$$a_3=\frac{1}{6\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3} f(\rho)\,d\rho.$$

Substituting the above four constants in (11), we obtain (10). Lemma 4 is thus proved. ■

Now, we introduce the following space:

$$E=\left\{u : u\in C^2([0,1])\right\},$$

and on E, we define the norm

$$\|u\|_E=\|u\|_\infty+\left\|u^{'}\right\|_\infty \quad \text{with } \|u\|=\sup_{x\in[0,1]}|u(x)|.$$

For $E\times E$, we can consider the norm

$$\|(u,v)\|_{E\times E}=\|u\|_E+\|v\|_E.$$

It is clear that $\left(E\times E,\|.\|_{E\times F}\right)$ is a product Banach space.

Let us now introduce the application $T:E\times E\to E\times E$, where,

$$T(u,v)(t)=(T_1(u)(t),T_2(u)(t)), \tag{12}$$

with

$$\begin{aligned} T_1(u)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\rho)^{\alpha-1}\left(f_1(\rho,u(\rho),v(\rho))+a_1g_1(\rho,u(\rho))+b_1h_1\left(\rho,u^{''}(\rho)\right)\right)d\rho \\ &\quad -\frac{t}{\Gamma(\alpha)}\int_0^1 (1-\rho)^{\alpha-1}\left(f_1(\rho,u(\rho),v(\rho))+a_1g_1(\rho,u(\rho))+b_1h_1\left(\rho,u^{''}(\rho)\right)\right)d\rho \end{aligned}$$

$$+\frac{t}{2\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left(f_1(\rho,u(\rho),v(\rho))+a_1g_1(\rho,u(\rho))+b_1h_1\left(\rho,u''(\rho)\right)\right)d\rho$$
$$-\frac{t^3}{6\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left(f_1(\rho,u(\rho),v(\rho))+a_1g_1(\rho,u(\rho))+b_1h_1\left(\rho,u''(\rho)\right)\right)d\rho,$$

and

$$\begin{aligned}
T_2(v)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho \\
&\quad -\frac{t}{\Gamma(\beta)}\int_0^1 (1-\rho)^{\beta-1}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho \\
&\quad +\frac{t}{2\Gamma(\beta-2)}\int_0^1 (1-\rho)^{\beta-3}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho \\
&\quad -\frac{t^3}{6\Gamma(\beta-2)}\int_0^1 (1-\rho)^{\beta-3}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho.
\end{aligned}$$

3 Solvability

Differential equations that include fractional and integer derivatives frequently occur in a variety of domains, including dynamical systems, viscoelastic materials, and fractional calculus. The non-standard character of fractional derivatives makes it difficult to guarantee the existence and uniqueness of solutions for these kinds of equations. On the other hand, existence and uniqueness outcomes can be shown under specific circumstances.

In this section, we establish two main results for system (1)–(2). To do that, we need to consider the following hypotheses:

(H_1) The functions $f_1, f_2 : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$, $g_1, g_2, h_1, h_2 : [0,1]\times\mathbb{R}\to\mathbb{R}$, are continuous.

(H_2) There are some nonnegative constants k_i, l_i, m_i, n_i, $(i=1,2)$ such that

$$|f_i(t,u_2,v_2)-f_i(t,u_1,v_1)| \le k_i|u_2-u_1|+l_i|v_2-v_1|,$$
$$|g_i(t,u_2)-g_i(t,u_1)| \le m_i|u_2-u_1|,$$
$$|h_i(t,u_2)-h_i(t,u_1)| \le n_i|u_2-u_1|,$$

for all $t\in[0,1]$ and all $u_i, v_i\in\mathbb{R}, (i=1,2)$.

(H_3) There are some positive constants L_i, $(i=1,6)$, such that

$$|f_1(t,u,v)|\le L_1,\quad |f_2(t,u,v)|\le L_2,\quad |g_1(t,u)|\le L_3,$$
$$|g_2(t,u)|\le L_4,\quad |h_1(t,u)|\le L_5,\quad |h_2(t,u)|\le L_6,$$

for all $t\in[0,1]$ and all $u,v\in\mathbb{R}$.

Theorem 1 *Suppose that the hypothesis (H_2) holds and $M+N \in]0,1[$, where*

$$M = \max\left\{ \frac{1}{3\Gamma(\alpha+1)} \begin{pmatrix} 6k_1 + 6m_1|a_1| + 3\alpha(\alpha-1)k_1 \\ +3m_1\alpha(\alpha-1)|a_1| + 6n_1|b_1| \\ +\alpha(\alpha-1)n_1|b_1| \\ +2n_1|b_1|\alpha(\alpha-1)(\alpha-2) \end{pmatrix}, \frac{l_1}{\Gamma(\alpha+1)}(2+\alpha(\alpha-1)) \right\},$$

$$N = \max\left\{ \frac{1}{3\Gamma(\beta+1)} \begin{pmatrix} 6k_2 + 6m_2|a_2| + 3\beta(\beta-1)k_2 \\ +3m_2\beta(\beta-1)|a_2| + 6n_2|b_2| \\ +\beta(\beta-1)n_2|b_2| \\ +2n_2|b_2|\beta(\beta-1)(\beta-2) \end{pmatrix}, \frac{k_2}{\Gamma(\beta+1)}(2+\beta(\beta-1)) \right\}.$$

Then, system (1)–(2) has a unique solution.

Proof. To proceed with the proof, we need to show that T admits only one fixed point.

We shall show that T is a contraction mapping: Assume that $(u_1,v_1), (u_2,v_2) \in E \times F$. Then, for all $t \in [0,1]$, we get

$$\begin{aligned}
&|T_1(u_2)(t) - T_1(u_1)(t)| \\
\leq\ & \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} \left(f_1(\rho, u_2(\rho), v_2(\rho)) + a_1 g_1(\rho, u_2(\rho)) + b_1 h_1\left(\rho, u_2''(\rho)\right)\right) d\rho \right. \\
& - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} \left(f_1(\rho, u_2(\rho), v_2(\rho)) + a_1 g_1(\rho, u_2(\rho)) + b_1 h_1\left(\rho, u_2''(\rho)\right)\right) d\rho \\
& + \frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-\rho)^{\alpha-3} \left(f_1(\rho, u_2(\rho), v_2(\rho)) + a_1 g_1(\rho, u_2(\rho)) + b_1 h_1\left(\rho, u_2''(\rho)\right)\right) d\rho \\
& - \frac{1}{6\Gamma(\alpha-2)} \int_0^1 (1-\rho)^{\alpha-3} \left(f_1(\rho, u_2(\rho), v_2(\rho)) + a_1 g_1(\rho, u_2(\rho)) + b_1 h_1\left(\rho, u_2''(\rho)\right)\right) d\rho \\
& - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} \left(f_1(\rho, u_1(\rho), v_2(\rho)) + a_1 g_1(\rho, u_1(\rho)) + b_1 h_1\left(\rho, u_1''(\rho)\right)\right) d\rho \\
& + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} \left(f_1(\rho, u_1(\rho), v_1(\rho)) + a_1 g_1(\rho, u_1(\rho)) + b_1 h_1\left(\rho, u_1''(\rho)\right)\right) d\rho \\
& - \frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-\rho)^{\alpha-3} \left(f_1(\rho, u_1(\rho), v_1(\rho)) + a_1 g_1(\rho, u_1(\rho)) + b_1 h_1\left(\rho, u_1''(\rho)\right)\right) d\rho \\
& \left. + \frac{1}{6\Gamma(\alpha-2)} \int_0^1 (1-\rho)^{\alpha-3} \left(f_1(\rho, u_1(\rho), v_1(\rho)) + a_1 g_1(\rho, u_1(\rho)) + b_1 h_1\left(\rho, u_1''(\rho)\right)\right) ds \right|, \qquad (13)
\end{aligned}$$

which leads to

$$\begin{aligned}
&|T_1(u_2)(t) - T_1(u_1)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-\rho)^{\alpha-1} \left(f_1(\rho, u_2(\rho), v_2(\rho)) - f_1(\rho, u_1(\rho), v_1(\rho))\right) d\rho \right|
\end{aligned}$$

$$+\frac{|a_1|}{\Gamma(\alpha)}\left|\int_0^t (t-\rho)^{\alpha-1}\left(g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right)d\rho\right|$$
$$+\frac{|b_1|}{\Gamma(\alpha)}\left|\int_0^t (t-\rho)^{\alpha-1}\left(h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right)d\rho\right|$$
$$+\frac{1}{\Gamma(\alpha)}\left|\int_0^1 (1-\rho)^{\alpha-1}\left(f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right)d\rho\right|$$
$$+\frac{|a_1|}{\Gamma(\alpha)}\left|\int_0^1 (1-\rho)^{\alpha-1}\left(g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right)d\rho\right|$$
$$+\frac{|b_1|}{\Gamma(\alpha)}\left|\int_0^1 (1-\rho)^{\alpha-1}\left(h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right)d\rho\right|$$
$$+\frac{1}{3\Gamma(\alpha-2)}\left|\int_0^1 (1-\rho)^{\alpha-3}\left(f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right)d\rho\right|$$
$$+\frac{|a_1|}{3\Gamma(\alpha-2)}\left|\int_0^1 (1-\rho)^{\alpha-3}\left(g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right)d\rho\right|$$
$$+\frac{|b_1|}{3\Gamma(\alpha-2)}\left|\int_0^1 (1-\rho)^{\alpha-3}\left(h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right)d\rho\right|.$$

Then, we have

$$\begin{aligned}
&\left\|T_1(u_2)-T_1(u_1)\right\|_\infty\\
=\ &\sup_{[0,1]}\left|T_1(u_2)(t)-T_1(u_1)(t)\right|\\
\leq\ &\frac{2}{\Gamma(\alpha)}\int_0^1 (1-\rho)^{\alpha-1}\left|f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right|d\rho\\
&+\frac{2|a_1|}{\Gamma(\alpha)}\int_0^1 (1-\rho)^{\alpha-1}\left|g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right|d\rho\\
&+\frac{2|b_1|}{\Gamma(\alpha)}\int_0^1 (1-\rho)^{\alpha-1}\left|h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right|d\rho\\
&+\frac{1}{3\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left|f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right|d\rho\\
&+\frac{|a_1|}{3\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left|g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right|d\rho\\
&+\frac{|b_1|}{3\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left|h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right|d\rho.
\end{aligned}$$

Using (H_2), we have:

$$\begin{aligned} & \left\| T_1\left(u_2\right)-T_1\left(u_1\right)\right\|_{\infty} \\ \leq\ & \frac{2}{\Gamma(\alpha+1)}\left(k_1\left\|u_2-u_1\right\|_{\infty}+l_1\left\|v_2-v_1\right\|_{\infty}\right) \\ & +\frac{2\left|a_1\right|}{\Gamma(\alpha+1)} m_1\left\|u_2-u_1\right\|_{\infty}+\frac{2\left|b_1\right|}{\Gamma(\alpha+1)} n_1\left\|\left(u_2-u_1\right)^{''}\right\|_{\infty} \\ & +\frac{1}{3 \Gamma(\alpha-1)}\left(k_1\left\|u_2-u_1\right\|_{\infty}+l_1\left\|v_2-v_1\right\|_{\infty}\right) \\ & +\frac{\left|a_1\right|}{3 \Gamma(\alpha-1)} m_1\left\|u_2-u_1\right\|_{\infty}+\frac{\left|b_1\right|}{3 \Gamma(\alpha-1)} n_1\left\|\left(u_2-u_1\right)^{''}\right\|_{\infty}. \end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned} & \left\| T_1\left(u_2\right)-T_1\left(u_1\right)\right\|_{\infty} \\ \leq\ & \frac{1}{3 \Gamma(\alpha+1)}\left(6 k_1+6 m_1\left|a_1\right|+\alpha(\alpha-1) k_1+m_1 \alpha(\alpha-1)\left|a_1\right|\right)\left\|u_2-u_1\right\|_{\infty} \\ & +\frac{l_1}{3 \Gamma(\alpha+1)}(6+\alpha(\alpha-1))\left\|v_2-v_1\right\|_{\infty}+\frac{n_1\left|b_1\right|}{3 \Gamma(\alpha+1)}(6+\alpha(\alpha-1))\left\|\left(u_2-u_1\right)^{''}\right\|_{\infty}. \end{aligned} \tag{15}$$

Also, we can state that

$$\begin{aligned} & \left\| T_2\left(v_2\right)-T_2\left(v_1\right)\right\|_{\infty} \\ \leq\ & \frac{1}{3 \Gamma(\beta+1)}\left(6 k_2+6 m_2\left|a_2\right|+\beta(\beta-1) k_2+m_2 \beta(\beta-1)\left|a_2\right|\right)\left\|v_2-v_1\right\|_{\infty} \\ & +\frac{k_2}{3 \Gamma(\beta+1)}(6+\beta(\beta-1))\left\|u_2-u_1\right\|_{\infty}+\frac{n_2\left|b_2\right|}{3 \Gamma(\beta+1)}(6+\beta(\beta-1))\left\|\left(v_2-v_1\right)^{''}\right\|_{\infty}. \end{aligned} \tag{16}$$

On the other hand, we have

$$\begin{aligned} & T_1^{''}(u)(t) \\ =\ & \frac{1}{\Gamma(\alpha-2)} \int_0^t(t-\rho)^{\alpha-3}\left(f_1(\rho, u(\rho), v(\rho))+a_1 g_1(\rho, u(\rho))+b_1 h_1\left(\rho, u^{''}(\rho)\right)\right) d \rho \\ & -\frac{t}{\Gamma(\alpha-2)} \int_0^1(1-\rho)^{\alpha-3}\left(f_1(\rho, u(\rho), v(\rho))+a_1 g_1(\rho, u(\rho))+b_1 h_1\left(\rho, u^{''}(\rho)\right)\right) d \rho. \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} & \left|T_1^{''}\left(u_2\right)(t)-T_1^{''}\left(u_1\right)(t)\right| \\ \leq\ & \left\lvert\, \frac{1}{\Gamma(\alpha-2)} \int_0^t(t-\rho)^{\alpha-3}\left(f_1\left(\rho, u_2(\rho), v_2(\rho)\right)+a_1 g_1\left(\rho, u_2(\rho)\right)+b_1 h_1\left(\rho, u_2^{''}(\rho)\right)\right) d \rho\right. \\ & -\frac{1}{\Gamma(\alpha-2)} \int_0^1(1-\rho)^{\alpha-3}\left(f_1\left(\rho, u_2(\rho), v_2(\rho)\right)+a_1 g_1\left(\rho, u_2(\rho)\right)+b_1 h_1\left(\rho, u_2^{''}(\rho)\right)\right) d \rho \\ & -\frac{1}{\Gamma(\alpha-2)} \int_0^t(t-\rho)^{\alpha-3}\left(f_1\left(\rho, u_1(\rho), v_1(\rho)\right)+a_1 g_1\left(\rho, u_1(\rho)\right)+b_1 h_1\left(\rho, u_1^{''}(\rho)\right)\right) d \rho \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left(f_1(\rho,u_1(\rho),v_1(\rho))+a_1g_1(\rho,u_1(\rho))+b_1h_1\left(\rho,u_1''(\rho)\right)\right)d\rho\Bigg| \\
\leq\ &\frac{1}{\Gamma(\alpha-2)}\left|\int_0^t (t-\rho)^{\alpha-3}\left(f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right)d\rho\right| \\
&+\frac{|a_1|}{\Gamma(\alpha-2)}\left|\int_0^t (t-\rho)^{\alpha-3}\left(g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right)d\rho\right| \\
&+\frac{|b_1|}{\Gamma(\alpha-2)}\left|\int_0^t (t-\rho)^{\alpha-3}\left(h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right)d\rho\right| \\
&+\frac{1}{\Gamma(\alpha-2)}\left|\int_0^1 (1-s)^{\alpha-3}\left(f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right)d\rho\right| \\
&+\frac{|a_1|}{\Gamma(\alpha-2)}\left|\int_0^1 (1-\rho)^{\alpha-3}\left(g_1(\rho,u_2(\rho))-g_1(\rho,u_1(\rho))\right)d\rho\right| \\
&+\frac{|b_1|}{\Gamma(\alpha-2)}\left|\int_0^1 (1-\rho)^{\alpha-3}\left(h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right)d\rho\right|.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\left\|T_1''(u_2)-T_1''(u_1)\right\|_\infty &= \sup_{[0,1]}\left|T_1''(u_2)-T_1''(u_1)\right| \\
&\leq \frac{2}{\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left|f_1(\rho,u_2(\rho),v_2(\rho))-f_1(\rho,u_1(\rho),v_1(\rho))\right|d\rho \\
&\quad+\frac{2|a_1|}{\Gamma(\alpha-2)}\int_0^1 (1-\rho)^{\alpha-3}\left|g_1(\rho,u_2(\rho))-g_1(s,u_1(\rho))\right|d\rho \\
&\quad+\frac{2|b_1|}{\Gamma(\alpha-2)}\int_0^x (x-\rho)^{\alpha-3}\left|h_1\left(\rho,u_2''(\rho)\right)-h_1\left(\rho,u_1''(\rho)\right)\right|d\rho.
\end{aligned}
$$

Using (H_2) , we obtain

$$
\begin{aligned}
\left\|T_1''(u_2)-T_1''(u_1)\right\|_\infty \leq\ &\frac{2}{\Gamma(\alpha-1)}\left(k_1\|u_2-u_1\|_\infty+l_1\|v_2-v_1\|_\infty\right) \\
&+\frac{2|a_1|}{\Gamma(\alpha-1)}m_1\|u_2-u_1\|_\infty+\frac{2|b_1|}{\Gamma(\alpha-2)}n_1\left\|(u_2-u_1)''\right\|_\infty. \qquad (18)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\|T_1''(u_2)-T_1''(u_1)\right\|_\infty \\
\leq\ &\frac{2k_1+2m_1|a_1|}{\Gamma(\alpha-1)}\|u_2-u_1\|_\infty+\frac{2l_1}{\Gamma(\alpha-1)}\|v_2-v_1\|_\infty+\frac{2n_1|b_1|}{\Gamma(\alpha-2)}\left\|(u_2-u_1)''\right\|_\infty. \qquad (19)
\end{aligned}
$$

With the same arguments, we have

$$\begin{aligned}
&\left\|T_2^{''}(v_2)-T_2^{''}(v_1)\right\|_{\infty}\\
\leq\ &\frac{2k_2+2m_2\left|a_2\right|}{\Gamma(\beta-1)}\left\|v_2-v_1\right\|_{\infty}+\frac{2k_2}{\Gamma(\alpha-1)}\left\|u_2-u_1\right\|_{\infty}+\frac{2n_2\left|b_2\right|}{\Gamma(\beta-2)}\left\|(v_2-v_1)^{''}\right\|_{\infty}.
\end{aligned}\tag{20}$$

Thus,

$$\begin{aligned}
&\left\|T_1(u_2)-T_1(u_1)\right\|_E\\
=\ &\left\|T_1(u_2)-T_1(u_1)\right\|_{\infty}+\left\|T_1^{''}(u_2)-T_1^{''}(u_1)\right\|_{\infty}\\
\leq\ &\frac{1}{\Gamma(\alpha+1)}\left(2k_1+2m_1\left|a_1\right|+\alpha(\alpha-1)k_1+m_1\alpha(\alpha-1)\left|a_1\right|\right)\left\|u_2-u_1\right\|_{\infty}\\
&+\frac{l_1}{\Gamma(\alpha+1)}\left(2+\alpha(\alpha-1)\right)\left\|v_2-v_1\right\|_{\infty}+\frac{n_1\left|b_1\right|}{3\Gamma(\alpha+1)}\left(6+\alpha(\alpha-1)+2\alpha(\alpha-1)(\alpha-2)\right)\left\|(u_2-u_1)^{''}\right\|_{\infty}\\
\leq\ &\frac{1}{3\Gamma(\alpha+1)}\Big(6k_1+6m_1\left|a_1\right|+3\alpha(\alpha-1)k_1+3m_1\alpha(\alpha-1)\left|a_1\right|+6n_1\left|b_1\right|\\
&+\alpha(\alpha-1)n_1\left|b_1\right|+2n_1\left|b_1\right|\alpha(\alpha-1)(\alpha-2)\Big)\left\|u_2-u_1\right\|_E+\frac{l_1}{\Gamma(\alpha+1)}\left(2+\alpha(\alpha-1)\right)\left\|v_2-v_1\right\|_E\\
\leq\ &M\left\|(u_1,u_2),(v_1,v_2)\right\|_{E\times E}.
\end{aligned}\tag{21}$$

On the other hand,

$$\begin{aligned}
&\left\|T_2(v_2)-T_2(v_1)\right\|_E\\
=\ &\left\|T_2(v_2)-T_2(v_1)\right\|_{\infty}+\left\|T_2^{''}(v_2)-T_2^{''}(v_1)\right\|_{\infty}\\
\leq\ &\frac{1}{3\Gamma(\beta+1)}\begin{pmatrix}6k_2+6m_2\left|a_2\right|+3\beta(\beta-1)k_2+3m_2\beta(\beta-1)\left|a_2\right|+6n_2\left|b_2\right|\\+\beta(\beta-1)n_2\left|b_2\right|+2n_2\left|b_2\right|\beta(\beta-1)(\beta-2)\end{pmatrix}\left\|v_2-v_1\right\|_E\\
&+\frac{k_2}{\Gamma(\beta+1)}\left(2+\beta(\beta-1)\right)\left\|u_2-u_1\right\|_E\\
\leq\ &N\left\|(u_1,u_2),(v_1,v_2)\right\|_{E\times E}.
\end{aligned}\tag{22}$$

By (21) and (22), we attain

$$\left\|T(u_2,v_2)-T(u_1,v_1)\right\|_{E\times E}\leq(M+N)\left\|(u_1,u_2),(v_1,v_2)\right\|_{E\times E},\tag{23}$$

where $M+N\in\,]0,1[$. Our conclusion is that T is a contraction mapping. As a result, according to the Banach fixed point theorem, there exists a single fixed point that is the result of the equation (1)–(2). ■

Theorem 2 *Suppose that the hypotheses (H_1) and (H_3) are satisfied. Then, system (1)–(2) has at least one solution.*

Proof. We show that the operator T is completely continuous.

Step 1: Let us take $\rho>0$ and $B_\rho=\left\{(u,v)\in E^2,\left\|(u,v)\right\|_{E\times E}\leq\rho\right\}$. For $(u,v)\in B_\rho$, and for all $x\in[0,1]$, we have

$$\begin{aligned}
\left|T_1(u,v)\right|\ \leq\ &\frac{1}{6\Gamma(\alpha+1)}\left(6t^{\alpha}+6t+3t\alpha(\alpha-1)+t^3\alpha(\alpha-1)\right)\max_{s\in[0,1]}\left|f_1(s,u(s),v(s))\right|\\
&+\frac{\left|a_1\right|}{6\Gamma(\alpha+1)}\left(6t^{\alpha}+6t+3t\alpha(\alpha-1)+t^3\alpha(\alpha-1)\right)\max_{s\in[0,1]}\left|g_1(s,u(s))\right|\\
&+\frac{\left|b_1\right|}{6\Gamma(\alpha+1)}\left(6t^{\alpha}+6t+3t\alpha(\alpha-1)+t^3\alpha(\alpha-1)\right)\max_{s\in[0,1]}\left|h_1\left(s,u^{''}(s)\right)\right|.
\end{aligned}\tag{24}$$

Thanks to (H_3), we obtain

$$\left\| T_1(u,v) \right\|_\infty \leq \frac{2\alpha^2 - 2\alpha + 6}{3\Gamma(\alpha+1)} \left(L_1 + |a_1| L_3 + |b_1| L_5 \right). \tag{25}$$

Also,

$$\left\| T_2(u,v) \right\|_\infty \leq \frac{2\beta^2 - 2\beta + 6}{3\Gamma(\beta+1)} \left(L_2 + |a_2| L_4 + |b_2| L_6 \right). \tag{26}$$

On the other hand, we observe that

$$\begin{aligned} \left| T_1^{''}(u,v) \right| \quad \leq \quad & \left(\frac{t^\alpha + t}{\Gamma(\alpha-1)} \right) \max_{s\in[0,1]} \left| f_1(s,u(s),v(s)) \right| + |a_1| \left(\frac{t^\alpha + t}{\Gamma(\alpha-1)} \right) \max_{s\in[0,1]} \left| g_1(s,u(s)) \right| \\ & + |b_1| \left(\frac{t^\alpha + t}{\Gamma(\alpha-1)} \right) \max_{s\in[0,1]} \left| h_1\left(s,u^{''}(s)\right) \right|. \end{aligned}$$

Hence,

$$\left\| T_1^{''}(u,v) \right\|_\infty \leq \frac{2}{\Gamma(\alpha-1)} \left(L_1 + |a_1| L_3 + |b_1| L_5 \right). \tag{27}$$

We can also prove that

$$\left\| T_2^{''}(u,v) \right\|_\infty \leq \frac{2}{\Gamma(\beta-1)} \left(L_2 + |a_2| L_4 + |b_2| L_6 \right). \tag{28}$$

From (25), (26), (27) and (28), we conclude that

$$\begin{aligned} & \left\| T(u,v) \right\|_E \\ \leq \quad & \max\left(\frac{8\alpha^2 - 8\alpha + 6}{3\Gamma(\alpha+1)} \left(L_1 + |a_1| L_3 + |b_1| L_5 \right), \frac{8\beta^2 - 8\beta + 6}{3\Gamma(\beta+1)} \left(L_2 + |a_2| L_4 + |b_2| L_6 \right) \right) < \infty. \end{aligned} \tag{29}$$

Step 2: Let $(u,v) \in B_\rho$ and $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have

$$\begin{aligned} & \left| T_1(u,v)(x_2) - T_1(u,v)(x_1) \right| \\ \leq \quad & \frac{L_1 + |a_1| L_3 + |b_1| L_5}{\Gamma(\alpha+1)} \left(2(x_2 - x_1)^\alpha + x_2^\alpha - x_1^\alpha \right) \\ & + \frac{L_1 + |a_1| L_3 + |b_1| L_5}{2\Gamma(\alpha+1)} \left(\alpha^2 - \alpha + 2 \right)(x_2 - x_1) + \frac{L_1 + |a_1| L_3 + |b_1| L_5}{6\Gamma(\alpha-2)} \left(x_2^3 - x_1^3 \right). \end{aligned} \tag{30}$$

In a similar way, we can find that

$$\begin{aligned} & \left| T_2(u,v)(x_2) - T_2(u,v)(x_1) \right| \\ \leq \quad & \frac{L_2 + |a_2| L_4 + |b_2| L_6}{\Gamma(\beta+1)} \left(2(x_2 - x_1)^\beta + x_2^\beta - x_1^\beta \right) \\ & + \frac{L_2 + |a_2| L_4 + |b_2| L_6}{2\Gamma(\beta+1)} \left(\beta^2 - \beta + 2 \right)(x_2 - x_1) + \frac{L_2 + |a_2| L_4 + |b_2| L_6}{6\Gamma(\beta-2)} \left(x_2^3 - x_1^3 \right) \end{aligned} \tag{31}$$

On the other hand, we obtain

$$\begin{aligned} \left| T_1^{''}(u,v)(x_2) - T_1^{''}(u,v)(x_1) \right| \quad \leq \quad & \frac{L_1 + |a_1| L_3 + |b_1| L_5}{\Gamma(\alpha-1)} \left(2(x_2 - x_1)^{\alpha-2} + x_2^{\alpha-2} - x_1^{\alpha-2} \right) \\ & + \frac{L_1 + |a_1| L_3 + |b_1| L_5}{\Gamma(\alpha-1)} (x_2 - x_1) \end{aligned} \tag{32}$$

and

$$\left| T_2^{''}(u,v)(x_2) - T_2^{''}(u,v)(x_1) \right| \quad \leq \quad \frac{L_2 + |a_2| L_4 + |b_2| L_6}{\Gamma(\beta-1)} \left(2(x_2 - x_1)^{\beta-2} + x_2^{\beta-2} - x_1^{\beta-2} \right)$$

$$+\frac{L_2+|a_2|\,L_4+|b_2|\,L_6}{\Gamma(\beta-1)}(x_2-x_1). \tag{33}$$

The right-hand sides of the inequalities (30), (31), (32) and (33) are independent of (u,v) and tend to zero as $x_2-x_1\to 0$. Consequently, T is an equicontinuous operator. Therefore, T completely continuous.

Next, we consider

$$F=\{(u,v)\in E,(u,v)=\mu T(u,v),0<\mu<1\}.$$

We request to show that F is bounded. For $(u,v)\in F$ and $t\in[0,1]$, we realize that

$$(u,v)(t)=\mu T(u,v)(t).$$

The inequality (29) allows us to write

$$\begin{aligned}\|(u,v)\|_E &\leq \mu\max\left(\frac{8\alpha^2-8\alpha+6}{3\Gamma(\alpha+1)}(L_1+|a_1|\,L_3+|b_1|\,L_5),\frac{8\beta^2-8\beta+6}{3\Gamma(\beta+1)}(L_2+|a_2|\,L_4+|b_2|\,L_6)\right)\\ &< \infty.\end{aligned} \tag{34}$$

Hence, F is bounded.

Using Lemma 3, we state that system (1)–(2) has at least one solution on $[0,1]$. The proof is completed. ■

4 Applications

In this part, we give a few implementations of the key findings as illustrations. Take the next scenario to provide context for the first primary outcome.

Example 1 *Consider the following system:*

$$\begin{cases} D^{\frac{7}{2}}u(t)=\frac{e^{-t}}{5+(t+1)e^t}u(t)+\frac{v(t)}{10+t}+\frac{1}{4}u(t)-\frac{1}{3}u''(t),\\ D^{\frac{13}{4}}v(t)=\frac{u(t)e^{-t}+v(t)}{5+t}+\frac{1}{2}v(t)-\frac{1}{30}v''(t),\\ u(0)=u(1)=u''(0)=u''(1)=0,\\ v(0)=v(1)=v''(0)=v''(1)=0,\end{cases} \tag{35}$$

For this example, we get

$$\begin{aligned}&\alpha=\tfrac{7}{2},\quad \beta=\tfrac{13}{4},\quad a_1=a_2=1,\quad b_1=b_2=-1,\\ &f_1(t,u(t),v(t))=\tfrac{e^{-t}}{5+(t+1)e^t}u(t)+\tfrac{v(t)}{10+t},\\ &f_2(t,u(t),v(t))=\tfrac{u(t)e^{-t}+v(t)}{5+t},\\ &g_1(t,u(t))=\tfrac{1}{4}u(t),\quad g_2(t,u(t))=\tfrac{1}{2}u(t),\\ &h_1(t,u(t))=\tfrac{1}{3}u''(t),\quad h_2(t,u(t))=\tfrac{1}{30}u''(t).\end{aligned}$$

It is clear that, for all $u_1,u_2,v_1,v_2\in\mathbb{R}$ and $t\in[0,1]$, we have

$$\begin{aligned}&|f_1(t,u_2,v_2)-f_1(t,u_1,v_1)|\leq\tfrac{1}{6}|u_2-u_1|+\tfrac{1}{10}|v_2-v_1|,\\ &|f_2(t,u_2,v_2)-f_2(t,u_1,v_1)|\leq\tfrac{1}{5}|u_2-u_1|+\tfrac{1}{5}|v_2-v_1|,\\ &|g_1(t,u_2)-g_1(t,u_1)|\leq\tfrac{1}{4}|u_2-u_1|,\\ &|g_2(t,u_2)-g_2(t,u_1)|\leq\tfrac{1}{2}|v_2-v_1|,\\ &|h_1(t,u_2)-h_1(t,u_1)|\leq\tfrac{1}{3}|u_2-u_1|,\\ &|h_2(t,u_2)-h_2(t,u_1)|\leq\tfrac{1}{30}|v_2-v_1|.\end{aligned}$$

We also have

$$M = \max\left\{\frac{1061}{945\sqrt{\pi}}, \frac{86}{1575\sqrt{\pi}}\right\} = \frac{1061}{945\sqrt{\pi}} = 0,6334446,$$

$$N = \max\left\{\frac{2113}{960\Gamma\left(\frac{17}{4}\right)}, \frac{197}{80\Gamma\left(\frac{17}{4}\right)}\right\} = \frac{197}{80\Gamma\left(\frac{17}{4}\right)} = 0,1484667.$$

The stipulations of Theorem 1 are true. Because of this, there is only one solution to problem (35) on $[0,1]$. *We take the following case into consideration to highlight the second main consequence:*

Example 2 *Consider the following system:*

$$\begin{cases} D^{\frac{11}{3}}u(t) = \frac{e^{-2t}}{5+|\cos(u(t))|+|\sin(u(t))|} + \frac{1}{4}\sin(u(t)) - \frac{1}{3}\cos(u''(t)), \\ D^{\frac{7}{2}}v(t) = \frac{\cos(u(t))+\sin(v(t))}{10+t+e^{t}} + \frac{1}{2}\cos(v(t)) - \frac{1}{30}\sin\left(v''(t)\right), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0. \end{cases} \tag{36}$$

For this example, we have

$$\begin{aligned} &\alpha = \tfrac{11}{3}, \quad \beta = \tfrac{7}{2}, \quad a_1 = a_2 = 1, \quad b_1 = b_2 = -1, \\ &f_1(t, u(t), v(t)) = \tfrac{e^{-2t}}{5+|\cos(u(t))|+|\sin(u(t))|}, \\ &f_2(t, u(t), v(t)) = \tfrac{\cos(u(t))+\sin(v(t))}{10+t+e^{t}}, \\ &g_1(t, u(t)) = \tfrac{1}{4}\sin(u(t)), \quad g_2(t, u(t)) = \tfrac{1}{2}\cos(u(t)), \\ &h_1(t, u(t)) = \tfrac{1}{3}\cos(u''(t)), \quad h_2(t, u(t)) = \tfrac{1}{30}\sin(u''(t)). \end{aligned}$$

For $t \in [0,1]$, *it is clear that* $f_1, f_2, g_1, g_2, h_1, h_2$ *are continuous and bounded functions. Using Theorem 2, system (36) has at least one solution on* $[0,1]$.

Example 3 *Consider the following system:*

$$\begin{cases} D^{\frac{39}{10}}u(t) = \frac{e^{-2t}}{5+|\cos(u(t))|+|\sin(u(t))|} + \frac{1}{4}\sin(u(t)) - \frac{1}{3}\cos(u''(t)), \\ D^{\frac{19}{5}}v(t) = \frac{\cos(u(t))+\sin(v(t))}{10+t+e^{t}} + \frac{1}{2}\cos(v(t)) - \frac{1}{30}\sin(v''(t)), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0. \end{cases} \tag{37}$$

For this example, we have

$$\begin{aligned} &\alpha = \tfrac{11}{3}, \quad \beta = \tfrac{7}{2}, \quad a_1 = a_2 = 1, \quad b_1 = b_2 = -1, \\ &f_1(t, u(t), v(t)) = \tfrac{e^{-2t}}{5+|\cos(u(t))|+|\sin(u(t))|}, \\ &f_2(t, u(t), v(t)) = \tfrac{\cos(u(t))+\sin(v(t))}{10+t+e^{t}}, \\ &g_1(t, u(t)) = \tfrac{1}{4}\sin(u(t)), \quad g_2(t, u(t)) = \tfrac{1}{2}\cos(u(t)), \\ &h_1(t, u(t)) = \tfrac{1}{3}\cos(u''(t)), \quad h_2(t, u(t)) = \tfrac{1}{30}\sin(u''(t)). \end{aligned}$$

For $t \in [0,1]$, *it is clear that* $f_1, f_2, g_1, g_2, h_1, h_2$ *are continuous and bounded functions. Hence, using Theorem 2, System (37) has at least one solution on* $[0,1]$.

5 Ulam Stability

In this section, we first introduce some Ulam stability definitions. Then, we prove some results regarding the introduced concepts. For more details, we refer the reader to [9, 24, 25, 26].

Definition 3 *The System (1)–(2) has the Ulam Hyers stability if there exists a real number $\varpi_{f,g,h} > 0$, such that for all $\varsigma_1, \varsigma_2 > 0, t \in [0,1]$ and for each $(u,v) \in E \times E$ solution of the inequality*

$$\begin{cases} |D^{\alpha}u\left(t\right) - f_1\left(t, u\left(t\right), v\left(x\right)\right) - a_1 g_1\left(t, u\left(t\right)\right) - b_1 h_1\left(t, u^{''}\left(t\right)\right)| \leq \varsigma_1, \\ |D^{\beta}v\left(t\right) - f_2\left(t, u\left(t\right), v\left(t\right)\right) - a_2 g_2\left(t, v\left(t\right)\right) - b_2 h_2\left(t, v^{''}\left(t\right)\right)| \leq \varsigma_2, \end{cases} \tag{38}$$

under the conditions:

$$\begin{cases} u\left(0\right) = u\left(1\right) = u^{''}\left(0\right) = u^{''}\left(1\right) = 0, \\ v\left(0\right) = v\left(1\right) = v^{''}\left(0\right) = v^{''}\left(1\right) = 0, \end{cases} \tag{39}$$

there exists $(u^*, v^*) \in E \times E$ a solution of system (1)–(2), such that

$$\|(u - u^*, v - v^*)\|_{E\times E} \leq \varsigma \varpi_{f,g,h}, \varsigma > 0.$$

Definition 4 *The System (1)–(2) has the Ulam Hyers stability in the generalized sense if there is $\Delta_{f,g,h} \in C(R^+, R^+)$; $\Delta_{f,g,h}(0) = 0$, such that for each $\varsigma > 0$, and for any $(u,v) \in E \times E$ solution of (38)–(39), there exists a solution $(u^*, v^*) \in E \times E$ of system (1)–(2) such that*

$$\|(u - u^*, v - v^*)\|_{E\times E} < \Delta_{f,g,h}(\varsigma).$$

Now, we present to the reader the following stability result.

Theorem 3 *If the conditions of Theorem 1 are satisfied, then problem (1)–(2) is Ulam Hyers stable.*

Proof. Let $(u,v) \in E \times E$ be a solution of (38)–(39), and let, by Theorem 1, $(u^*, v^*) \in E \times E$ be the unique solution of (1)–(2). Integrate (38) implies that

$$\begin{aligned} &\Bigg| u(t) - \Bigg[\frac{1}{\Gamma\left(\alpha\right)}\int_0^t \left(t-\rho\right)^{\alpha-1}\left(f_1\left(\rho, u\left(\rho\right), v\left(\rho\right)\right) + a_1 g_1\left(\rho, u\left(\rho\right)\right) + b_1 h_1\left(\rho, u^{''}\left(\rho\right)\right)\right) d\rho \\ &-\frac{t}{\Gamma\left(\alpha\right)}\int_0^1 \left(1-\rho\right)^{\alpha-1}\left(f_1\left(\rho, u\left(\rho\right), v\left(\rho\right)\right) + a_1 g_1\left(\rho, u\left(\rho\right)\right) + b_1 h_1\left(\rho, u^{''}\left(\rho\right)\right)\right) d\rho \\ &+\frac{t}{2\Gamma\left(\alpha-2\right)}\int_0^1 \left(1-\rho\right)^{\alpha-3}\left(f_1\left(\rho, u\left(\rho\right), v\left(\rho\right)\right) + a_1 g_1\left(\rho, u\left(\rho\right)\right) + b_1 h_1\left(\rho, u^{''}\left(\rho\right)\right)\right) d\rho \\ &-\frac{t^3}{6\Gamma\left(\alpha-2\right)}\int_0^1 \left(1-\rho\right)^{\alpha-3}\left(f_1\left(\rho, u\left(\rho\right), v\left(\rho\right)\right) + a_1 g_1\left(\rho, u\left(\rho\right)\right) + b_1 h_1\left(\rho, u''\left(\rho\right)\right)\right) d\rho\Bigg]\Bigg| \\ \leq\ & \frac{\varsigma_1}{\Gamma(\alpha+1)}, \end{aligned} \tag{40}$$

and

$$\Bigg| v(t) - \Bigg[\frac{1}{\Gamma\left(\beta\right)}\int_0^t \left(t-\rho\right)^{\beta-1}\left(f_2\left(\rho, u\left(\rho\right), v\left(\rho\right)\right) + a_2 g_2\left(\rho, u\left(\rho\right)\right) + b_2 h_2\left(\rho, u^{''}\left(\rho\right)\right)\right) d\rho$$

$$
\begin{aligned}
&-\frac{t}{\Gamma(\beta)}\int_0^1 (1-\rho)^{\beta-1}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho\\
&+\frac{t}{2\Gamma(\beta-2)}\int_0^1 (1-\rho)^{\beta-3}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho\\
&-\frac{t^3}{6\Gamma(\beta-2)}\int_0^1 (1-\rho)^{\beta-3}\left(f_2(\rho,u(\rho),v(\rho))+a_2g_2(\rho,u(\rho))+b_2h_2\left(\rho,u''(\rho)\right)\right)d\rho\Bigg]\Bigg|\\
\leq\ &\frac{\varsigma_2}{\Gamma(\beta+1)}. \qquad (41)
\end{aligned}
$$

Using (38), (40) and (41), we have

$$
\begin{aligned}
\|u-u^*\|_\infty \leq\ & \frac{\varsigma_1}{\Gamma(\alpha+1)}+\frac{1}{3\Gamma(\alpha+1)}\left(6k_1+6m_1|a_1|+\alpha(\alpha-1)k_1+m_1\alpha(\alpha-1)|a_1|\right)\\
&\times\|u-u^*\|_\infty+\frac{l_1}{3\Gamma(\alpha+1)}(6+\alpha(\alpha-1))\|v-v^*\|_\infty\\
&+\frac{n_1|b_1|}{3\Gamma(\alpha+1)}(6+\alpha(\alpha-1))\left\|(u-u^*)''\right\|_\infty \qquad (42)
\end{aligned}
$$

and

$$
\begin{aligned}
\|v-v^*\|_\infty \leq\ & \frac{\varsigma_2}{\Gamma(\beta+1)}+\frac{1}{3\Gamma(\beta+1)}\left(6k_2+6m_2|a_2|+\beta(\beta-1)k_2+m_2\beta(\beta-1)|a_2|\right)\\
&\times\|v-v^*\|_\infty+\frac{k_2}{3\Gamma(\beta+1)}(6+\beta(\beta-1))\|u-u^*\|_\infty\\
&+\frac{n_2|b_2|}{3\Gamma(\beta+1)}(6+\beta(\beta-1))\left\|(v-v^*)''\right\|_\infty. \qquad (43)
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
\|(u-u^*)''\|_\infty \leq\ & \frac{\varsigma_1}{\Gamma(\alpha-1)}+\frac{2k_1+2m_1|a_1|}{\Gamma(\alpha-1)}\|u-u^*\|_\infty+\frac{2l_1}{\Gamma(\alpha-1)}\|v-v^*\|_\infty\\
&+\frac{2n_1|b_1|}{\Gamma(\alpha-2)}\left\|(u-u^*)''\right\|_\infty, \qquad (44)
\end{aligned}
$$

and

$$
\begin{aligned}
\|(v-v^*)''\|_\infty \leq\ & \frac{\varsigma_2}{\Gamma(\beta-1)}+\frac{2k_2+2m_2|a_2|}{\Gamma(\beta-1)}\|v-v^*\|_\infty+\frac{2k_2}{\Gamma(\alpha-1)}\|u-u^*\|_\infty\\
&+\frac{2n_2|b_2|}{\Gamma(\beta-2)}\left\|(v-v^*)''\right\|_\infty. \qquad (45)
\end{aligned}
$$

So, it yields that

$$
\|u-u^*\|_E \leq \frac{\varsigma_1}{\Gamma(\alpha+1)}+\frac{\varsigma_1}{\Gamma(\alpha-1)}+M\|(u-u^*,v-v^*)\|_{E\times E} \qquad (46)
$$

and

$$
\|v-v^*\|_E \leq \frac{\varsigma_2}{\Gamma(\beta+1)}+\frac{\varsigma_2}{\Gamma(\beta-1)}+N\|(u-u^*,v-v^*)\|_{E\times E} \qquad (47)
$$

are satisfied.

It follows from (46) and (47) that

$$\|(u-u^*, v-v^*)\|_{E\times E} \leq 2\varsigma\Xi + (M+N)\|(u-u^*, v-v^*)\|_{E\times E},$$

where

$$\varsigma = \varsigma_1 + \varsigma_2,$$

$$\Xi = \max\left\{\frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha-1)}\ \frac{1}{\Gamma(\beta+1)}, \frac{1}{\Gamma(\beta-1)}\right\}.$$

Hence,

$$\|(u-u^*, v-v^*)\|_{E\times E} \leq \frac{2\varsigma\Xi}{1-(M+N)} := \varsigma\varpi_{f,g,h}, \varpi_{f,g,h} = \frac{2\Xi}{1-(M+N)}.$$

Thus, the solution of (1)–(2) is Ulam Hyers stable. ■

Remark 1 *If we consider the case*

$$\Delta_{f,g,h}(\varsigma) = \frac{2\varsigma\Xi}{1-(M+N)},$$

then the generalized Ulam Hyers stability is obtained for (1)–(2).

6 Conclusion

Using a general form of limiting differential equations that describe the deformation of an elastic beam supported at its endpoints, we have proposed a more general problem that involves fractional derivatives in the sense of Caputo. We have derived some sufficient conditions that guarantee the existence and uniqueness of solutions for the introduced system. We have also proved another main result: the existence of at least one solution for the same system. Several examples have been discussed in detail to show the applicability of the hypotheses. Using the existence and uniqueness results, the Ulam-Hyers and the generalized Ulam-Hyers stabilities have also been studied.

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