Applied Mathematics E-Notes, 24(2024), 424-442 © ISSN 1607-2510
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On Implicit Neutral Caputo Tempered Fractional Differential Equations With Delay*

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Received 23 September 2023

Abstract

This article is a subject about some results of the existence and Ulam stability results for four classes of implicit neutral fractional differential equations involving the Caputo tempered fractional derivative with delay. The results are based on Krasnoselskii's fixed point theorem in Banach spaces, and the notion of the stability of Ulam kind. To illustrate our results, we will give some examples.

1 Introduction

The study of differential equations plays a fundamental role in understanding and modeling various natural phenomena and engineering systems [2, 9, 39–41]. In recent years, there has been a growing interest in extending the classical theory of differential equations to encompass more complex scenarios, we recommend consulting monographs such as [10, 22], as well as papers like [1, 3–6, 11, 12, 17]. One such extension is the introduction of tempered fractional derivatives. Buschman's earlier work [13] was the first to disclose the definitions of fractional integration with weak singular and exponential kernels.

The notion of tempered fractional derivatives extends the classical concept of fractional derivatives by considering functions with exponentially decaying tails. This extension is particularly relevant in applications where memory effects are essential, such as in viscoelastic materials, nonlocal models in physics, and fractional-order control systems. The tempered fractional derivative allows for a more accurate description of the underlying dynamics, capturing both long range memory and fast decaying behaviors. Further elaboration on this topic can be found in [24, 26–28, 35, 36].

Implicit neutral problems represent a class of differential equations where the equation involves both the dependent variable and its derivatives. These problems arise in various fields, including biology, physics, and engineering, and pose significant challenges in terms of mathematical analysis and numerical solutions we refer to the monographs of Hale [16], Hale and Verduyn Lunel [15], Hino et al. [19], Kolmanovskii and Myshkis [23], and the references therein, see also [25]. The incorporation of tempered fractional derivatives in implicit neutral problems offers a novel perspective, providing a deeper understanding of their behavior and characteristics.

The Ulam stability of ordinary and fractional differential equations has recently been studied in [8, 20, 29–33].

In this paper, we study existence, uniqueness and Ulam stability results for the Cauchy problem of implicit neutral fractional differential equation involving the Caputo tempered fractional derivative with finite delay

$$ {}_0^C\mathfrak{D}_t^{\zeta,\omega}\left(y(t)-\mathfrak{h}(t,y_t)\right)=\Psi\left(t,y_t,\ {}_0^C\mathfrak{D}_t^{\zeta,\omega}\left(y(t)-\mathfrak{h}(t,y_t)\right)\right);t\in J:=[0,\varkappa], \tag{1}$$

$$ y(t)=\phi(t);\qquad t\in[-\varsigma,0], \tag{2}$$

*Mathematics Subject Classifications: 26A33, 34A08, 34K37.

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where $0 < \zeta < 1$, $\omega \geq 0$, $\varkappa, \varsigma > 0$, ${}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}$ is the Caputo tempered fractional derivative, $\mathfrak{h} : J \times E_{\varsigma} \to \mathbb{R}, \Psi : J \times E_{\varsigma} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\phi \in E_{\varsigma}$ and $E_{\varsigma} := C([-\varsigma, 0], \mathbb{R})$.

For any $t \in I$, we defined $y_t \in E_{\varsigma}$ by

$$y_t(\theta) = y(t + \theta); \quad \text{for} \quad \theta \in [-\varsigma, 0].$$

Next we consider the Cauchy problem of implicit neutral fractional differential equation involving the Caputo tempered fractional derivative with infinite delay

$${}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right) = \Psi\left(t, y_t,\ {}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right)\right); t \in J := [0, \varkappa], \tag{3}$$

$$y(t) = \phi(t); \qquad t \in (-\infty, 0], \tag{4}$$

where $\mathfrak{h} : J \times \mathcal{B} \to \mathbb{R}, \Psi : J \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\phi \in \mathcal{B}$ and $\mathcal{B}$ is the phase space to be specified later.

For any $t \in I$, we defined $y_t \in \mathcal{B}$ by

$$y_t(\theta) = y(t + \theta); \quad \text{for} \quad \theta \in (-\infty, 0].$$

In addition to that, we study the Cauchy problem of implicit neutral fractional differential equation involving the Caputo tempered fractional derivative with state-dependent delay (the finite delay case)

$${}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_{\varrho(t,y_t)})\right) = \Psi\left(t, y_{\varrho(t,y_t)},\ {}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_{\varrho(t,y_t)})\right)\right); t \in J := [0, \varkappa], \tag{5}$$

$$y(t) = \phi(t); \qquad t \in [-\varsigma, 0], \tag{6}$$

where $\varrho : J \times E_{\varsigma} \to \mathbb{R}, \mathfrak{h} : J \times E_{\varsigma} \to \mathbb{R}, \Psi : J \times E_{\varsigma} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\phi \in E_{\varsigma}$.

Finally, we treat the last Cauchy problem of implicit neutral fractional differential equation involving the Caputo tempered fractional derivative with state-dependent delay (the infinite delay case)

$${}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_{\varrho(t,y_t)})\right) = \Psi\left(t, y_{\varrho(t,y_t)},\ {}_{0}^{C}\mathfrak{D}_{t}^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_{\varrho(t,y_t)})\right)\right); t \in J := [0, \varkappa], \tag{7}$$

$$y(t) = \phi(t); \qquad t \in (-\infty, 0], \tag{8}$$

where $\varrho : J \times \mathcal{B} \to \mathbb{R}, \mathfrak{h} : J \times \mathcal{B} \to \mathbb{R}, \Psi : J \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\phi \in \mathcal{B}$.

This paper is arranged as follows: Section 2 introduces some preliminaries, definitions, lemmas and auxiliary results that are used throughout this work. In section 3, we give some existence results for the problem (1)–(2) that are based on Krasnoselskii fixed point theorem, moreover we establish the Ulam stability of this problem. The same study applied to problem (1)–(2) in last section, we apply in section 4 to problem (3)–(4) and in section 5 to problems (5)–(6) and (7)–(8). Finally we present some examples to show the validity of our results.

2 Preliminaries

We denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$, with the following norm

$$\|y\|_{\infty} = \sup_{t \in J} |y(t)|.$$

Consider $E_{\varsigma} := C([-\varsigma, 0], \mathbb{R})$ the Banach space with the norm

$$\|y\|_{E_{\varsigma}} = \sup_{t \in [-\varsigma, 0]} |y(t)|.$$

Let $\mathcal{B}$ the phase space introduced by Hale and Kato in [14] and follow the terminology used in [19]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following axioms :

(A_1) If $y:(-\infty,\varkappa]\to\mathbb{R}$ is continuous on J and $y_0\in\mathcal{B}$, then for every $t\in J$ the following conditions hold:

- (i) $y_t\in\mathcal{B}$;
- (ii) there exists a positive constant $\widetilde{F}$ such that $|y(t)|\le\widetilde{F}\|y_t\|_{\mathcal{B}}$;
- (iii) there exists two functions $F(\cdot),\widehat{F}(\cdot):\mathbb{R}_+\to\mathbb{R}_+$ independent of y with F continuous and bounded and $\widehat{F}$ locally bounded such that

$$\|y_t\|_{\mathcal{B}}\le F(t)\sup_{\kappa\in[0,t]}|y(\kappa)|+\widehat{F}(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote $F^*=\sup_{t\in J}F(t)$, $\widehat{F}^*=\sup_{t\in J}\widehat{F}(t)$.

Remark 1 *1. $(A_1)(ii)$ is equivalent to $|\phi(0)|\le\widetilde{F}\|\phi\|_{\mathcal{B}}$ for every $\phi\in\mathcal{B}$.*

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi,\varpi\in\mathcal{B}$ can verify $\|\phi-\varpi\|_{\mathcal{B}}=0$ without necessarily $\phi(\theta)=\varpi(\theta)$ for all $\theta\le 0$.

3. From the equivalence in the first remark, we can see that, for all $\phi,\varpi\in\mathcal{B}$ such that $\|\phi-\varpi\|_{\mathcal{B}}=0$. We necessarily have that $\phi(0)=\varpi(0)$.

Definition 1 ([26, 34, 37]) *Suppose that the function $\Psi\in C([\varkappa_1,\varkappa_2],\mathbb{R})$, $\omega>0$. Then, the Riemann-Liouville tempered fractional integral of order ζ is defined by*

$$ {}_{\varkappa_1}I_t^{\zeta,\omega}\Psi(t)=e^{-\omega t}\ {}_{\varkappa_1}I_t^{\zeta}\left(e^{\omega t}\Psi(t)\right)=\frac{1}{\Gamma(\zeta)}\int_{\varkappa_1}^{t}\frac{e^{-\omega(t-\tau)}\Psi(\tau)}{(t-\tau)^{1-\zeta}}d\tau, \tag{9}$$

where ${}_{\varkappa_1}\mathcal{I}_t^{\zeta}$ denotes the Riemann-Liouville fractional integral [21], defined by

$$ {}_{\varkappa_1}I_t^{\zeta}\Psi(t)=\frac{1}{\Gamma(\zeta)}\int_{\varkappa_1}^{t}\frac{\Psi(\tau)}{(t-\tau)^{1-\zeta}}d\tau. \tag{10}$$

Obviously, the tempered fractional integral (9) reduces to the Riemann-Liouville fractional integral (10) if $\omega=0$.

Definition 2 ([26, 34]) *For $n-1<\zeta<n$; $n\in\mathbb{N}^+$, $\omega\ge 0$. The Riemann-Liouville tempered fractional derivative is defined by*

$$ {}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega}\Psi(t)=e^{-\omega t}\ {}_{\varkappa_1}\mathfrak{D}_t^{\zeta}\left(e^{\omega t}\Psi(t)\right)=\frac{e^{-\omega t}}{\Gamma(n-\zeta)}\frac{d^n}{dt^n}\int_{\varkappa_1}^{t}\frac{e^{\omega\tau}\Psi(\tau)}{(t-\tau)^{\zeta-n+1}}dt,$$

where ${}_{\varkappa_1}\mathfrak{D}_t^{\zeta}$ denotes the Riemann-Liouville fractional derivative [21], given by

$$ {}_{\varkappa_1}\mathfrak{D}_t^{\zeta}\Psi(t)=\frac{d^n}{dt^n}\left({}_{\varkappa_1}\mathcal{I}_t^{n-\zeta}\Psi(t)\right)=\frac{1}{\Gamma(n-\zeta)}\frac{d^n}{dt^n}\int_{\varkappa_1}^{t}\frac{\Psi(\tau)}{(t-\tau)^{\zeta-n+1}}d\tau.$$

Definition 3 ([26, 37]) *For $n-1<\zeta<n$; $n\in\mathbb{N}^+$, $\omega\ge 0$. The Caputo tempered fractional derivative is defined as*

$$ {}_{\varkappa_1}^{C}\mathfrak{D}_t^{\zeta,\omega}\Psi(t)=e^{-\omega t}\ {}_{\varkappa_1}^{C}\mathfrak{D}_t^{\zeta}\left(e^{\omega t}\Psi(t)\right)=\frac{e^{-\omega t}}{\Gamma(n-\zeta)}\int_{\varkappa_1}^{t}\frac{1}{(t-\tau)^{\zeta-n+1}}\frac{d^n}{d\tau^n}\left(e^{\omega\tau}\Psi(\tau)\right)d\tau,$$

where ${}_{\varkappa_1}^{C}\mathfrak{D}_t^{\zeta,\omega}$ denotes the Caputo fractional derivative [21], given by

$$ {}_{\varkappa_1}^{C}\mathfrak{D}_t^{\zeta}\Psi(t)=\frac{1}{\Gamma(n-\zeta)}\int_{\varkappa_1}^{t}\frac{1}{(t-\tau)^{\zeta-n+1}}\frac{d^n}{d\tau^n}\Psi(\tau)d\tau.$$

Lemma 1 ([26]) *For a constant C,*

$$ {}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega} C = Ce^{-\omega t}\ {}_{\varkappa_1}\mathfrak{D}_t^{\zeta} e^{\omega t}, \qquad {}^{C}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega} C = Ce^{-\omega t}\ {}^{C}_{\varkappa_1}\mathfrak{D}_t^{\zeta} e^{\omega t}. $$

Obviously, ${}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega}(C) \neq {}^{C}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega}(C)$. And, ${}^{C}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega}(C)$ is no longer equal to zero, being different from ${}^{C}_{\varkappa_1}\mathfrak{D}_t^{\alpha}(C) \equiv 0$.

Lemma 2 ([26,37]) *Let $\Psi \in C^n([\varkappa_1, \varkappa_2], \mathbb{R})$, $\omega \geq 0$ and $n-1 < \zeta < n$. Then we have:*

$$ {}_{\varkappa_1}I_t^{\zeta,\omega}\left[{}^{C}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega}\Psi(t)\right] = \Psi(t) - \sum_{k=0}^{n-1} e^{-\omega t}\frac{(t-\varkappa_1)^k}{k!}\left[\left.\frac{d^k\left(e^{\omega t}\Psi(t)\right)}{dt^k}\right|_{t=\varkappa_1}\right], $$

and

$$ {}^{C}_{\varkappa_1}\mathfrak{D}_t^{\zeta,\omega}\left[{}_{a}I_t^{\zeta,\omega}\Psi(t)\right] = \Psi(t), \ \text{for } \zeta \in (0,1). $$

Now, we consider the Ulam stability for the problem (1)–(2). Let $\varepsilon > 0$ and $\aleph : J \to \mathbb{R}_+$ be a continuous and positive function. We put the following inequalities

$$ \left| {}^{C}_{0}\mathfrak{D}_t^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right) - \Psi\left(t, y_t,\ {}^{C}_{0}\mathfrak{D}_t^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right)\right)\right| \leq \varepsilon; \quad t \in J, \tag{11} $$

$$ \left| {}^{C}_{0}\mathfrak{D}_t^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right) - \Psi\left(t, y_t,\ {}^{C}_{0}\mathfrak{D}_t^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right)\right)\right| \leq \aleph(t); \quad t \in J, \tag{12} $$

$$ \left| {}^{C}_{0}\mathfrak{D}_t^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right) - \Psi\left(t, y_t,\ {}^{C}_{0}\mathfrak{D}_t^{\zeta,\omega}\left(y(t) - \mathfrak{h}(t, y_t)\right)\right)\right| \leq \varepsilon\aleph(t); \quad t \in J. \tag{13} $$

Definition 4 ([9,33]) *The problem (1)–(2) is Ulam-Hyers stable if there exists a real number $c_\Psi > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(J, \mathbb{R})$ of the inequality (11) there exists a solution $\bar{y} \in C(J, \mathbb{R})$ of (1)–(2) with*

$$ |y(t) - \bar{y}(t)| \leq \varepsilon c_\Psi; \quad t \in J. $$

Definition 5 ([9,33]) *The problem (1)–(2) is generalized Ulam-Hyers stable if there exists $c_\Psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_\Psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(J, \mathbb{R})$ of the inequality (11) there exists a solution $\bar{y} \in C(J, \mathbb{R})$ of (1)–(2) with*

$$ |y(t) - \bar{y}(t)| \leq c_\Psi(\varepsilon); \quad t \in J. $$

Definition 6 ([9,33]) *The problem (1)–(2) is Ulam-Hyers-Rassias stable with respect to $\aleph$ if there exists a real number $c_{M,\aleph} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(J, \mathbb{R})$ of the inequality (13) there exists a solution $\bar{y} \in C(J, \mathbb{R})$ of (1)–(2) with*

$$ |y(t) - \bar{y}(t)| \leq \varepsilon c_{M,\aleph}\aleph(t); \quad t \in J. $$

Definition 7 ([9,33]) *The problem (1)–(2) is generalized Ulam-Hyers-Rassias stable with respect to $\aleph$ if there exists a real number $c_{M,\aleph} > 0$ such that for each solution $y \in C(J, \mathbb{R})$ of the inequality (12) there exists a solution $\bar{y} \in C(J, \mathbb{R})$ of (1)–(2) with*

$$ |y(t) - \bar{y}(t)| \leq c_{M,\aleph}\aleph(t); \quad t \in J. $$

Notice that

(i) Definition 4 $\Longrightarrow$ Definition 5.

(ii) Definition 6 $\Longrightarrow$ Definition 7.

(iiI) Definition 6 for $\Phi(\cdot) = 1 \Longrightarrow$ Definition 4.

One can have similar remarks for the inequalities (11) and (13).

3 Implicit Neutral Caputo Tempered Fractional Differential Equations with Finite Delay

In this section, we study the existence, uniqueness and Ulam stability results for the problem (1)–(2) in a Banach space. Let

$$E = \{y : [-\varsigma, \varkappa] \to \mathbb{R};\ y\,|_{[-\varsigma,0]} \in E_\varsigma,\ y\,|_{[0,\varkappa]} \in C(J,\mathbb{R})\}.$$

We note that E is Banach space with the norm

$$\|y\|_E = \sup_{t\in[-\varsigma,\varkappa]} |y(t)|.$$

Lemma 3 *Let $0 < \zeta < 1$, and $\mu : J \to \mathbb{R}$, $\mathfrak{g} : J \to \mathbb{R}$ be two continuous functions. Then the problem*

$$ {}_0^C\mathfrak{D}_t^{\zeta,\omega}\,(y(t) - \mu(t)) = \mathfrak{g}(t); \quad t \in J := [0, \varkappa], \tag{14}$$

$$y(t) = \phi(t); \qquad t \in [-\varsigma, 0], \tag{15}$$

has a unique solution defined by

$$y(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mu(0)] + \mu(t) + \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\mathfrak{g}(\kappa)d\kappa; & t \in J, \\ y(t) = \phi(t), \quad t \in [-\varsigma, 0]. \end{cases} \tag{16}$$

Proof. Applying The Riemann-Liouville tempered fractional integral of order ζ to both sides the equation (14), and by using Lemma 2 and if $t \in J$, we get

$$y(t) - \mu(t) - e^{-\omega t}[y(0) - \mu(0)] = \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\mathfrak{g}(\kappa)d\kappa.$$

From the condition (15), we get

$$y(t) = e^{-\omega t}[\phi(0) - \mu(0)] + \mu(t) + \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\mathfrak{g}(\kappa)d\kappa, \quad t \in J.$$

Conversely, we can easily show by Definition 3, Lemma 1 and Lemma 2 that if y verifies (16), then it satisfies the problem (14)-(15). ■

Definition 8 *By a solution of the problem (1)–(2), we mean a function $y \in E$ that satisfies the equation (1) on J, and the initial condition (2) on $[-\varsigma, 0]$.*

As a consequence of Lemma 3, we give the following result.

Lemma 4 *Let $\Psi : J \times E_\varsigma \times \mathbb{R} \to \mathbb{R}$, $\mathfrak{h} : J \times E_\varsigma \to \mathbb{R}$ be two continuous functions. Then the problem (1)-(2) is equivalent to the following integral equation*

$$y(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0,\phi)] + \mathfrak{h}(t, y_t) \\ +\frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, y_s, \mathfrak{f}(\kappa))d\kappa, \quad t \in J, \\ \phi(t), \quad t \in [-\varsigma, 0], \end{cases} \tag{17}$$

where $\mathfrak{f} \in C(J,\mathbb{R})$ satisfies the following functional equation

$$\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t)).$$

Corollary 1 *The solutions of the problem (1)–(2) are the fixed points of the operator $M : E \to E$ defined by*

$$My(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0,\phi)] + \mathfrak{h}(t, y_t) \\ \\ +\frac{1}{\Gamma(\zeta)} \int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, y_s, \mathfrak{f}(\kappa))d\kappa, \quad t \in J, \\ \\ \phi(t), \quad t \in [-\varsigma, 0], \end{cases} \tag{18}$$

where $\mathfrak{f} \in C(J, \mathbb{R})$ satisfies the following functional equation

$$\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t)),$$

Remark 2 *Consider the operators $T, L : E \to E$ defined by*

$$Ty(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0,\phi)] \\ \\ +\frac{1}{\Gamma(\zeta)} \int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, y_s, \mathfrak{f}(\kappa))d\kappa, \quad t \in J, \\ \\ 0, \quad t \in [-\varsigma, 0], \end{cases} \tag{19}$$

where $\mathfrak{f} \in C(J, \mathbb{R})$ satisfies the following functional equation

$$\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t)),$$

and

$$Ly(t) = \begin{cases} \mathfrak{h}(t, y_t), \quad t \in J, \\ \\ \phi(t), \quad t \in [-\varsigma, 0]. \end{cases} \tag{20}$$

Then

$$My = Ty + Ly.$$

3.1 Existence of Solution

The existence result is based on Krasnoselskii's fixed point theorem. Let us introduce the following hypotheses:

(H_1) There exist constants $p_2^* > 0$, $0 < p_3^* < 1$ such that

$$|\Psi(t, y, \mu) - \Psi(t, \bar{y}, \bar{\mu})| \le p_2^* \|y - \bar{y}\|_{E_\varsigma} + p_3^* |\mu - \bar{\mu}|,$$

for any $y, \bar{y} \in E_\varsigma$, $\mu, \bar{\mu} \in \mathbb{R}$ and $t \in J$.

(H_2) The function $\mathfrak{h}$ satisfies the Lipschitz condition

$$|\mathfrak{h}(t, y) - \mathfrak{h}(t, \bar{y})| \le k\|y - \bar{y}\|_{E_\varsigma},$$

for $y, \bar{y} \in E_\varsigma$ and $t \in J$ where $0 < k < 1$.

Set

$$p_1^* = \sup_{t \in J} |\Psi(t, 0, 0)|, \quad \mathfrak{h}^* = \sup_{t \in J} |\mathfrak{h}(t, 0)|.$$

Remark 3 *The hypothesis* (H_1) *implies that*

$$|\Psi(t,y,\mu)| \le p_1^* + p_2^*\|y\|_{E_\varsigma} + p_3^*|\mu|,$$

for any $y \in E_\varsigma$, $\mu \in \mathbb{R}$ *and* $t \in J$.

Theorem 1 *Suppose that the hypotheses* $(H1)$, $(H2)$, *and the condition*

$$k + \frac{p_2^*\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)} < 1, \tag{21}$$

hold. Then the problem (1)–(2) has at least one solution defined on $[-\varsigma, \varkappa]$.

Proof. Let T, L be the operators defined in (19 and 20 respectively.). Set

$$R \ge \max\left\{\|\phi\|_{E_\varsigma}; \frac{C + \frac{p_1^*\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)}}{1 - k - \frac{p_2^*\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)}}\right\},$$

and let $\Omega_R = \{y \in E : \|y\|_E \le R\}$ be the closed and convex ball in E. The proof will be given in three steps.

Step 01. $Ty + L\mu \in \Omega_R$ whenever $y, \mu \in \Omega_R$.
Let $y, \mu \in \Omega_R$. Then for each $t \in [-\varsigma, 0]$, we have

$$|Ty(t) + L\mu(t)| = |\phi(t)| \le \|\phi\|_{E_\varsigma} \le R,$$

and for each $t \in J$, we have

$$\begin{aligned}
|Ty(t) + L\mu(t)| &\le e^{\omega t}|\phi(0) - \mathfrak{h}(0,\phi)| + |\mathfrak{h}(t,\mu_t)| + \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}|\mathfrak{f}(\kappa)|d\kappa \\
&\le |\phi(0)| + |\mathfrak{h}(0,\phi)| + |\mathfrak{h}(t,\mu_t)| + \frac{1}{\Gamma(\zeta)}\int_0^t (t-\kappa)^{\zeta-1}|\mathfrak{f}(\kappa)|d\kappa
\end{aligned}$$

where $\mathfrak{f} \in C(J,\mathbb{R})$ with $\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t))$. From Remark 3, for each $t \in J$, we get

$$\begin{aligned}
|\mathfrak{f}(t)| &= |\Psi(t, y_t, \mathfrak{f}(t))| \\
&\le p_1(t) + p_2(t)\|y_t\|_{E_\varsigma} + p_3(t)|\mathfrak{f}(t)| \\
&\le p_1^* + p_2^*\|y\|_E + p_3^*|\mathfrak{f}(t)| \\
&\le p_1^* + p_2^*R + p_3^*|\mathfrak{f}(t)|.
\end{aligned}$$

This gives

$$|\mathfrak{f}(t)| \le \frac{p_1^* + p_2^*R}{1 - p_3^*}.$$

Thus, by using (H_2), we get

$$\begin{aligned}
|Ty(t) + L\mu(t)| &\le |\phi(0)| + |\mathfrak{h}(0,\phi) - \mathfrak{h}(0,0)| + |\mathfrak{h}(0,0)| + |\mathfrak{h}(t,\mu_t) - \mathfrak{h}(t,0)| \\
&\quad + |\mathfrak{h}(t,0)| + \frac{(p_1^* + p_2^*R)\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)} \\
&\le \|\phi\|_{E_\varsigma} + k\|\phi\|_{E_\varsigma} + \mathfrak{h}^* + k\|\mu_t\|_{E_\varsigma} + \mathfrak{h}^* + \frac{(p_1^* + p_2^*R)\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)} \\
&\le (k+1)\|\phi\|_{E_\varsigma} + kR + 2\mathfrak{h}^* + \frac{(p_1^* + p_2^*R)\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)}.
\end{aligned}$$

Set $C = (k+1)\|\phi\|_{E_\varsigma} + 2\mathfrak{h}^*$. Thus

$$|Ty(t) + L\mu(t)| \leq C + kR + \frac{(p_1^* + p_2^* R)\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)} \leq R.$$

Hence, we get

$$\|Ty + L\mu\|_E \leq R.$$

This proves that $Ty + L\mu \in \Omega_R$ whenever $y, \mu \in \Omega_R$.

Step 02. $T : \Omega_R \to \Omega_R$ is compact and continuous.

Claim 1. $T : \Omega_R \to \Omega_R$ is compact.

From step 1, we have $T(\Omega_R) \subset \Omega_R$, and Ω_R is bounded, then $T(\Omega_R)$ is bounded. Let $t_1, t_2 \in J$, be such that $t_1 < t_2$ and let $y \in \Omega_R$. Then, there exist $\mathfrak{f} \in C(J, \mathbb{R})$ with $\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t))$, such that

$$\begin{aligned}
|Ty(t_2) - Ty(t_1)| &\leq \int_0^{t_1} \frac{|e^{-\omega(t_2-\kappa)}(t_2-\kappa)^{\zeta-1} - e^{-\omega(t_1-\kappa)}(t_1-\kappa)^{\zeta-1}|}{\Gamma(\zeta)} |\mathfrak{f}(\kappa)| d\kappa \\
&+ \int_{t_1}^{t_2} \frac{e^{-\omega(t_2-\kappa)}(t_2-\kappa)^{\zeta-1}}{\Gamma(\zeta)} |\mathfrak{f}(\kappa)| d\kappa \\
&\leq \frac{p_1^* + p_2^* R}{(1-p_3^*)} \left[\int_0^{t_1} \frac{|e^{-\omega(t_2-\kappa)}(t_2-\kappa)^{\zeta-1} - e^{-\omega(t_1-\kappa)}(t_1-\kappa)^{\zeta-1}| d\kappa}{\Gamma(\zeta)} \right. \\
&\left. + \frac{(t_2-t_1)^\zeta}{\Gamma(\zeta+1)} \right] \longrightarrow 0, \quad as \quad t_1 \to t_2.
\end{aligned}$$

Consequently, $T(\Omega_R)$ is equicontinuous. According the theorem of Arzela-Ascoli, $T : \Omega_R \to \Omega_R$ is compact.

Claim 2. T is continuous.

Let $\{y^n\}_{n\in\mathbb{N}}$ be a sequence such that $y^n \to y$ in Ω_R. Then, we have

$$|Ty^n(t) - Ty(t)| \leq \frac{1}{\Gamma(\zeta)} \int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1} |\mathfrak{f}_n(\kappa) - \mathfrak{f}(\kappa)| d\kappa,$$

where $\mathfrak{f}_n, \mathfrak{f} \in C(J, \mathbb{R})$ such that

$$\mathfrak{f}_n(t) = \Psi(t, y_t^n, \mathfrak{f}_n(t)),$$

and

$$\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t)).$$

From (H_1), we get

$$\begin{aligned}
|\mathfrak{f}_n(t) - \mathfrak{f}(t)| &= |\Psi(t, y_t^n, \mathfrak{f}_n(t)) - \Psi(t, y_t, \mathfrak{f}(t))| \\
&\leq p_2^* \|y_t^n - y_t\|_{E_\varsigma} + p_3^* |\mathfrak{f}_n(\kappa) - \mathfrak{f}(\kappa)|.
\end{aligned}$$

Then

$$|\mathfrak{f}_n(t) - \mathfrak{f}(t)| \leq \frac{p_2^*}{1-p_3^*} \|y_t^n - y_t\|_{E_\varsigma}.$$

Thus

$$|Ty^n(t) - Ty(t)| \leq \frac{p_2^*}{(1-p_3^*)\Gamma(\zeta)} \int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1} \|y_t^n - y_t\|_{E_\varsigma} d\kappa.$$

By the Lebesgue dominated convergence theorem, we get

$$\|Ty^n - Ty\|_\infty \longrightarrow 0, \quad as \quad n \to +\infty,$$

which implies that

$$\|Ty^n - Ty\|_E \longrightarrow 0, \quad as \quad n \to +\infty.$$

Hance, the operator T is continuous.

Step 3. L is a contraction mapping.

Let $y, \mu \in \Omega_R$ for each $t \in J$, by using (H_2) we have

$$|Ly(t) - L\mu(t)| = |\mathfrak{h}(t, y_t) - \mathfrak{h}(t, \mu_t)| \leq k\|y_t - \mu_t\|_{E_\varsigma}.$$

Thus

$$\|Ly - L\mu\|_E \leq k\|y - \mu\|_E,$$

which implies that the operator L is a contraction.

As a consequence of the three above steps, from Krasnoselskii's fixed point theorem [7], the operator equation $(T + L)(y) = y$ has at least one solution which is a solution of the fractional problem (1)-(2). ■

3.2 Generalized Ulam-Hyers-Rassias Stability

Now, we prove a result about the Ulam-Hyers-Rassias stability of the problem (1)–(2). The following hypotheses will be used in the sequel.

(H_3) There exist continuous functions $q_1, q_2, q_3, \aleph : J \to \mathbb{R}_+$ with $q_3(t) < 1$ such that

$$(1 + \|y\|_{E_\varsigma})|\Psi(t, y, \mu)| \leq q_1(t)\aleph(t) + q_2(t)\aleph(t)\|y\|_{E_\varsigma} + q_3(t)|\mu|,$$

for any $y \in E_\varsigma$, $\mu \in \mathbb{R}$ and $t \in J$.

(H_4) There exists a continuous function $q_4 : J \to \mathbb{R}_+$ such that

$$(1 + \|y - \bar{y}\|_{E_\varsigma})|\mathfrak{h}(t, y) - \mathfrak{h}(t, \bar{y})| \leq q_4(t)\aleph(t)\|y - \bar{y}\|_{E_\varsigma},$$

for $y, \bar{y} \in E_\varsigma$ and $t \in J$.

(H_5) There exists $\lambda_\aleph > 0$ such that for each $t \in J$, we have

$$I_0^{\zeta,\omega}\aleph(t) \leq \lambda_\aleph \aleph(t).$$

Set

$$q_i^* = \sup_{t \in J} |q_i(t)|; \quad i \in \{1, 2, 3, 4\}, \quad \aleph^* = \sup_{t \in J} |\aleph(t)|.$$

Theorem 2 *Assume that the hypotheses* (H_1)*–*(H_5) *and the condition (21) hold. If* $q_4^*\aleph^* < 1$*, then the problem (1)–(2) is generalized Ulam-Hyers-Rassias stable.*

Proof. Let M be the operator defined in (18) and y be a solution of the inequality (12). For each $t \in J$, we have

$$|y(t) - e^{-\omega t}[\phi(0) - \mathfrak{h}(0, \phi)] - \mathfrak{h}(t, y_t) - I_0^{\zeta,\omega}\mathfrak{f}(t)| \leq \aleph(t),$$

where $\mathfrak{f} \in C(J, \mathbb{R})$ with $\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t))$, and let us assume that μ is a solution of problem (1)–(2). Thus, we have

$$\mu(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0, \phi)] + \mathfrak{h}(t, \mu_t) \\ \\ +\frac{1}{\Gamma(\zeta)} \int_0^t e^{-\omega(t-\kappa)}(t - \kappa)^{\zeta-1}\mathfrak{g}(\kappa)d\kappa; \quad t \in J, \\ \\ \phi(t); \quad t \in [-\varsigma, 0], \end{cases}$$

where $\mathfrak{g} \in C(J,\mathbb{R})$ with $\mathfrak{g}(t) = \Psi(t, \mu_t, \mathfrak{f}(t))$. From the hypotheses (H_3)–(H_5), for each $t \in J$, we have

$$\begin{aligned}
|y(t) - \mu(t)| &\leq |y(t) - e^{-\omega t}[\phi(0) - \mathfrak{h}(0,\phi)] - \mathfrak{h}(t, y_t) - I_0^{\zeta,\omega}\mathfrak{f}(t)| \\
&\quad + |\mathfrak{h}(t, y_t) - \mathfrak{h}(t, \mu_t)| + |I_0^{\zeta,\omega}(\mathfrak{f} - \mathfrak{g})(t)| \\
&\leq \aleph(t) + q_4^*\aleph(t) + \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}|\mathfrak{f}(\kappa) - \mathfrak{g}(\kappa)|d\kappa \\
&\leq \aleph(t) + q_4^*\aleph(t) + 2\left(\frac{q_1^* + q_2^*}{1 - q_3^*}\right) I_0^{\zeta,\omega}\aleph(t) \\
&\leq \aleph(t) + q_4^*\aleph(t) + 2\lambda_\aleph \frac{q_1^* + q_2^*}{1 - q_3^*}\aleph(t) \\
&\leq \left[1 + q_4^* + 2\lambda_\aleph \frac{q_1^* + q_2^*}{1 - q_3^*}\right]\aleph(t) \\
&=: c_{\Psi,\mathfrak{h},\aleph}\aleph(t).
\end{aligned}$$

Hence, we conclude the generalized Ulam-Hyers-Rassias stability of problem (1)–(2). ■

4 Implicit Neutral Caputo Tempered Fractional Differential Equations with Infinite Delay

In this section, we study the existence, uniqueness and Ulam stability results for the problem (3)–(4). Let

$$E^* = \{y : (-\infty, \varkappa] \to \mathbb{R};\quad y\,|_{(-\infty,0]} \in \mathcal{B},\quad y\,|_{[0,\varkappa]} \in C(J,\mathbb{R})\}.$$

4.1 Existence of Solution

The existence result is based on Krasnoselskii's fixed point theorem.

Definition 9 *By a solution of the problem (3)–(4), we mean a function $y \in E^*$ that satisfies the equation (3) on J, and the initial condition (4) on $(-\infty, 0]$.*

Let us introduce the following hypotheses:

(h_1) There exist constants $a_2^* > 0, 0 < a_3^* < 1$ such that

$$|\Psi(t, y, \mu) - \Psi(t, \bar{y}, \bar{\mu})| \leq a_2^*\|y - \bar{y}\|_{\mathcal{B}} + a_3^*|\mu - \bar{\mu}|,$$

for any $y, \bar{y} \in \mathcal{B}$, $\mu, \bar{\mu} \in \mathbb{R}$ and $t \in J$.

(h_2) The function $\mathfrak{h}$ satisfies the Lipschitz condition

$$|\mathfrak{h}(t, y) - \mathfrak{h}(t, \bar{y})| \leq k'\|y - \bar{y}\|_{\mathcal{B}},$$

for $y, \bar{y} \in \mathcal{B}$ and $t \in J$ where $0 < k' < \frac{1}{F^*}$.

Set

$$a_1^* = \sup_{t\in J}|\Psi(t, 0, 0)|, \quad \mathfrak{h}^* = \sup_{t\in J}|\mathfrak{h}(t, 0)|.$$

Remark 4 *The hypothesis (h_1) implies that*

$$|\Psi(t, y, \mu)| \leq a_1^* + a_2^*\|y\|_{\mathcal{B}} + a_3^*|\mu|,$$

for any $y \in \mathcal{B}$, $\mu \in \mathbb{R}$ and $t \in J$.

Theorem 3 *Suppose that the hypotheses (h1), (h2), and the condition*

$$k'F^* + \frac{\varkappa^\zeta a_2^* F^*}{(1-a_3^*)\Gamma(\zeta+1)} < 1, \tag{22}$$

hold. Then the problem (3)–(4) has at least one solution defined on $(-\infty, \varkappa]$.

Proof. Consider the operator $N : E^* \to E^*$ defined by

$$Ny(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0,\phi)] + \mathfrak{h}(t, y_t) \\ \\ +\frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, y_s, \mathfrak{f}(\kappa))d\kappa, \quad t \in J, \\ \\ \phi(t), \quad t \in (-\infty, 0], \end{cases}$$

where $\mathfrak{f} \in C(J, \mathbb{R})$ satisfies the following functional equation

$$\mathfrak{f}(t) = \Psi(t, y_t, \mathfrak{f}(t)).$$

Let $\mu : (-\infty, \varkappa] \to \mathbb{R}$ be a function defined by

$$\mu(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0,\phi)], \quad t \in J, \\ \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases}$$

Then $\mu_0 = \phi$ for all $t \in (-\infty, 0]$. For each $\nu \in E^*$ with $\nu(0) = 0$, for each $t \in (-\infty, \varkappa]$ we denote by $\bar{\nu}$ the function defined by

$$\bar{\nu}(t) = \begin{cases} \nu(t), \quad t \in J, \\ \\ 0, \quad t \in (-\infty, 0]. \end{cases}$$

If y satisfies $y(t) = Ny(t)$, we can decompose it as $y(t) = \nu(t) + \mu(t)$ for $t \in J$, which implies that $y_t = \nu_t + \mu_t$ for every $t \in J$, and the function $\nu(\cdot)$ satisfies

$$\nu(t) = \mathfrak{h}(t, \nu_t + \mu_t) + \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, \nu_t + \mu_t, \mathfrak{f}(\kappa))d\kappa,$$

where $\mathfrak{f} \in C(J, \mathbb{R})$ satisfies the following functional equation

$$\mathfrak{f}(t) = \Psi(t, \nu_t + \mu_t, \mathfrak{f}(t)).$$

Set

$$E_0 = \{\nu \in E^*, \quad \nu(0) = 0\},$$

and let $\|\cdot\|_{E_0}$ be the norm in E_0 defined by

$$\|\nu\|_{E_0} = \|\nu_0\|_{\mathcal{B}} + \sup_{t\in J}|\nu(t)| = \sup_{t\in J}|\nu(t)|; \quad \nu \in E_0,$$

where E_0 is a Banach space with the norm $\|\cdot\|_{E_0}$. Defined the operators $\mathcal{K}, \mathcal{P} : E_0 \to E_0$ by

$$\mathcal{K}\nu(t) = \mathfrak{h}(t, \nu_t + \mu_t),$$

and

$$\mathcal{P}\nu(t) = \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, \nu_t + \mu_t, \mathfrak{f}(\kappa))d\kappa.$$

Obviously, the operator N has a fixed point is equivalent to $\mathcal{K}+\mathcal{P}$ having a fixed point, and so we turn to proving that $\mathcal{K}+\mathcal{P}$ has a fixed point. Set

$$R^* \geq \frac{k'D + \mathfrak{h}^* + \frac{\varkappa^\zeta (a_1^* + a_2^* D)}{(1-a_3^*)\Gamma(\zeta+1)}}{1 - k' F^* - \frac{\varkappa^\zeta a_2^* F^*}{(1-a_3^*)\Gamma(\zeta+1)}},$$

with $D = \|\phi\|_{\mathcal{B}}(F^*(\widetilde{F} + k') + \widehat{F}^*) + F^* h^*$, and let $B_{R^*} = \{\nu \in E_0 : \|\nu\|_{E_0} \leq R^*\}$ be the closed and convex ball in E_0.

For each $\nu \in E_0$ and from (A_1), it follows that We can prove as in Theorem 1 that the operators $\mathcal{K}$ and $\mathcal{P}$ satisfy the conditions of Krasnoselskii's fixed point theorem [7]. This implies that the operator N has at least a fixed point which is a solution of problem (3)–(4). ∎

4.2 Generalized Ulam-Hyers-Rassias Stability

Now, we prove a result about the Ulam-Hyers-Rassias stability of the problem (3)–(4). The following hypotheses will be used in the sequel.

(h_3) There exist continuous functions $b_1, b_2, b_3, \tilde{\aleph} : J \to \mathbb{R}_+$ with $b_3(t) < 1$ such that

$$(1 + \|y\|_{\mathcal{B}})|\Psi(t, y, \mu)| \leq b_1(t)\tilde{\aleph}(t) + b_2(t)\tilde{\aleph}(t)\|y\|_{\mathcal{B}} + b_3(t)|\mu|,$$

for any $y \in \mathcal{B}$, $\mu \in \mathbb{R}$ and $t \in J$.

(h_4) There exists a continuous function $b_4 : J \to \mathbb{R}_+$ such that

$$(1 + \|y - \bar{y}\|_{\mathcal{B}})|\mathfrak{h}(t, y) - \mathfrak{h}(t, \bar{y})| \leq b_4(t)\tilde{\aleph}(t)\|y - \bar{y}\|_{\mathcal{B}},$$

for $y, \bar{y} \in \mathcal{B}$ and $t \in J$.

(h_5) There exists $\lambda_{\tilde{\aleph}} > 0$ such that for each $t \in J$, we have

$$I_0^{\zeta,\omega}\tilde{\aleph}(t) \leq \lambda_{\tilde{\aleph}}\tilde{\aleph}(t).$$

Set

$$b_i^* = \sup_{t \in J}|b_i(t)|; \quad i \in \{1, 2, 3, 4\}, \quad \aleph^* = \sup_{t \in J}|\aleph(t)|.$$

From Theorem 3, we can conclude the following result about the generalized Ulam-Hyers-Rassias stability of problem (3)–(4).

Theorem 4 *Assume that the hypotheses (h_1)–(h_5) and the condition (22) hold. If $b_4^*\aleph^* < 1$, then the problem (3)–(4) is generalized Ulam-Hyers-Rassias stable.*

5 Implicit Neutral Caputo Tempered Fractional Differential Equation with State-Dependent Delay

5.1 The Finite Delay Case

Set

$$\mathcal{R}(\varrho^-) = \{\varrho(t, \phi) : (t, \phi) \in J \times E_\varsigma, \varrho(t, \phi) \leq 0\}.$$

We always assume that $\varrho : J \times E_\varsigma \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) the function $t \to \phi_t$ is continuous from $\mathcal{R}(\varrho^-)$ into E_ς, and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\varrho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{E_\varsigma} \leq \mathcal{L}^\phi(t)\|\phi\|_{E_\varsigma}.$$

Remark 5 *The condition (H_ϕ) is frequently verified by functions continuous and bounded. For more details, see, for instance, [19].*

Lemma 5 ([18, Lemma 2.4]) *If $y : [-\varsigma, \varkappa] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_t\|_{E_\varsigma} \leq \mathcal{L}^\phi\|\phi\|_{E_\varsigma} + \sup_{\theta\in[0,max\{0,t\}]} |y(\theta)|, \quad t \in \mathcal{R}(\varrho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup_{t\in\mathcal{R}(\varrho^-)} |\mathcal{L}^\phi(t)|$.

5.2 Existence of Solution

The existence result is based on Krasnoselskii's fixed point theorem.

Definition 10 *By a solution of the problem (5)–(6), we mean a function $y \in E$ that satisfies the equation (5) on J, and the initial condition (6) on $[-\varsigma, 0]$.*

Theorem 5 *Suppose that the hypotheses $(H1)$, $(H2)$, (H_ϕ), and the condition*

$$k + \frac{p_2^*\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)} < 1,$$

hold. Then the problem (5)–(6) has at least one solution defined on $[-\varsigma, \varkappa]$.

Proof. Consider the operators $\tilde{T}, \tilde{L} : E \to E$ defined by

$$\tilde{T}y(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0, y_{\varrho(0,\phi)})] \\ \\ +\frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, y_{\varrho(\kappa,y_s)}, \tilde{\mathfrak{f}}(\kappa))d\kappa, \quad t \in J, \\ \\ 0, \quad t \in [-\varsigma, 0], \end{cases}$$

where $\tilde{\mathfrak{f}} \in C(J, \mathbb{R})$ satisfies the following functional equation

$$\tilde{\mathfrak{f}}(t) = \Psi(t, y_{\varrho(t,y_t)}, \tilde{\mathfrak{f}}(t)),$$

and

$$\tilde{L}y(t) = \begin{cases} \mathfrak{h}(t, y_{\varrho(t,y_t)}), \quad t \in J, \\ \\ \phi(t), \quad t \in [-\varsigma, 0]. \end{cases}$$

Set

$$\tilde{R} \geq \max\left\{\|\phi\|_{E_\varsigma}; \frac{E + \frac{F\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)}}{1 - k - \frac{p_2^*\varkappa^\zeta}{(1-p_3^*)\Gamma(\zeta+1)}}\right\},$$

with $E = \|\phi\|_{E_\varsigma}(1 + K(2\mathcal{L}^\phi + 1)) + 2h^*$ and $F = p_1^* + p_2^*\mathcal{L}^\phi\|\phi\|_{E_\varsigma}$, and let $B_{\tilde{R}} = \left\{y \in E : \|y\|_E \leq \tilde{R}\right\}$ be the closed and convex ball in E. We can prove as in Theorem 1 that the operators $\tilde{T}$ and $\tilde{L}$ satisfy the conditions of Theorem Krasnoselskii's fixed point theorem [7]. This implies that the operator $\tilde{T} + \tilde{L}$ has at least a fixed point which is a solution of problem (5)–(6). ■

5.3 The Infinite Delay Case

Set

$$\Re(\varrho^-) = \{\varrho(t,\phi) : (t,\phi) \in J \times \mathcal{B}, \varrho(t,\phi) \leq 0\}.$$

We always assume that $\varrho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(h_ϕ) the function $t \to \phi_t$ is continuous from $\Re(\varrho^-)$ into $\mathcal{B}$, and there exists a continuous and bounded function $\tilde{\mathcal{L}}^\phi : \Re(\varrho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \tilde{\mathcal{L}}^\phi(t)\|\phi\|_{\mathcal{B}}.$$

Remark 6 *The condition (h_ϕ) is frequently verified by functions continuous and bounded. For more details, see, for instance, [19].*

Lemma 6 ([18], Lemma 2.4) *If $y : (-\infty, \varkappa] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_t\|_{\mathcal{B}} \leq (\widehat{F}^* + \tilde{\mathcal{L}}^\phi)\|\phi\|_{\mathcal{B}} + L^* \sup_{\theta\in[0,max\{0,t\}]} |y(\theta)|, \quad t \in \Re(\varrho^-) \cup J,$$

where $\tilde{\mathcal{L}}^\phi = \sup_{t\in\mathcal{R}(\varrho^-)} |\tilde{\mathcal{L}}^\phi(t)|$.

5.4 Existence of Solution

The existence result is based on Krasnoselskii's fixed point theorem.

Definition 11 *By a solution of the problem (7)–(8), we mean a function $y \in E^*$ that satisfies the equation (7) on J, and the initial condition (8) on $(-\infty, 0]$.*

Theorem 6 *Suppose that the hypotheses (h1), (h2), (h_ϕ), and the condition*

$$k'F^* + \frac{\varkappa^\zeta a_2^* F^*}{(1-a_3^*)\Gamma(\zeta+1)} < 1,$$

hold. Then the problem (7)–(8) has at least one solution defined on $(-\infty, \varkappa]$.

Proof. Consider the operator $\tilde{N} : E^* \to E^*$ defined by

$$\tilde{N}y(t) = \begin{cases} e^{-\omega t}[\phi(0) - \mathfrak{h}(0, y_{\varrho(0,\phi)})] + \mathfrak{h}(t, y_{\varrho(t,y_t)}) \\ +\frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, y_{\varrho(\kappa,y_s)}, \tilde{\mathfrak{f}}(\kappa))d\kappa; \quad t \in J, \\ \phi(t); \quad t \in (-\infty, 0], \end{cases}$$

where $\tilde{\mathfrak{f}} \in C(J,\mathbb{R})$ satisfies the following functional equation

$$\tilde{\mathfrak{f}}(t) = \Psi(t, y_{\varrho(t,y_t)}, \tilde{\mathfrak{f}}(t)).$$

If y satisfies $y(t) = \tilde{N}y(t)$, then there is similar transformation to that in the proof of Theorem 3, given the following decomposition $y(t) = \nu(t) + \mu(t)$ for $t \in J$, which implies that $y_t = \nu_t + \mu_t$ for every $t \in J$, and the function $\nu(\cdot)$ satisfies

$$\begin{aligned}\nu(t) =& \mathfrak{h}(t, \nu_{\varrho(\kappa,\nu_t+\mu_t)} + \mu_{\varrho(\kappa,\nu_t+\mu_t)}) \\ &+ \frac{1}{\Gamma(\zeta)}\int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, \nu_{\varrho(\kappa,\nu_s+\mu_s)} + \mu_{\varrho(\kappa,\nu_s+\mu_s)}, \tilde{\mathfrak{f}}(\kappa))d\kappa,\end{aligned}$$

where $\tilde{\mathfrak{f}} \in C(J, \mathbb{R})$ satisfies the following functional equation

$$\tilde{\mathfrak{f}}(t) = \Psi(t, \nu_{\varrho(\kappa,\nu_s+\mu_s)} + \mu_{\varrho(\kappa,\nu_s+\mu_s)}, \tilde{\mathfrak{f}}(t)).$$

Defined the operators $\tilde{\mathcal{K}}, \tilde{\mathcal{P}} : E_0 \to E_0$ by

$$\tilde{\mathcal{K}}\nu(t) = \mathfrak{h}(t, \nu_{\varrho(\kappa,\nu_t+\mu_t)} + \mu_{\varrho(\kappa,\nu_t+\mu_t)}),$$

and

$$\tilde{\mathcal{P}}\nu(t) = \frac{1}{\Gamma(\zeta)} \int_0^t e^{-\omega(t-\kappa)}(t-\kappa)^{\zeta-1}\Psi(\kappa, \nu_{\varrho(\kappa,\nu_s+\mu_s)} + \mu_{\varrho(\kappa,\nu_s+\mu_s)}, \tilde{\mathfrak{f}}(\kappa))d\kappa.$$

Obviously, the operator $\tilde{N}$ has a fixed point is equivalent to $\tilde{\mathcal{K}} + \tilde{\mathcal{P}}$ having a fixed point, and so we turn to proving that $\tilde{\mathcal{K}} + \tilde{\mathcal{P}}$ has a fixed point. Set

$$\tilde{R}^* \geq \frac{k'G + \mathfrak{h}^* + \frac{\varkappa^\zeta(a_1^*+a_2^*G)}{(1-a_3^*)\Gamma(\zeta+1)}}{1 - k'F^* - \frac{\varkappa^\zeta a_2^* F^*}{(1-a_3^*)\Gamma(\zeta+1)}},$$

with $G = Lh^*\|\phi\|_{\mathcal{B}}(1 + F^*k')(F^*\widetilde{F} + \widehat{F}^* + \tilde{\mathcal{L}}^\phi)$, and let $B_{\tilde{R}^*} = \left\{\nu \in E_0 : \|\nu\|_{E_0} \leq \tilde{R}^*\right\}$ be the closed and convex ball in E_0.

We can prove as in Theorem 1 that the operators $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{P}}$ satisfy the conditions of Krasnoselskii's fixed point theorem [7]. This implies that the operator $\tilde{N}$ has at least a fixed point which is a solution of problem (7)–(8). ■

6 Some Examples

Example 1 *Consider the implicit neutral Caputo tempered fractional differential equations with delay*

$$ {}_0^C\mathfrak{D}_t^{\frac{1}{2},2}\left(y(t) - \mathfrak{h}(t, y_t)\right) = \Psi\left(t, y_t,\ {}_0^C\mathfrak{D}_t^{\frac{1}{2},2}\left(y(t) - \mathfrak{h}(t, y_t)\right)\right); t \in [0,1], \tag{23}$$

$$y(t) = t^2; \qquad t \in \Xi, \tag{24}$$

where

$$\Psi(t, u, v) = \frac{t}{e^{t+6}}(t^2 + tu + v); \quad t \in [0,1], u \in \Upsilon, v \in \mathbb{R},$$

and

$$\mathfrak{h}(t, u) = \frac{t}{1 + \|u\|_\Upsilon}\left(e^{-2} + \frac{1}{e^{t^2+1}}\right); \quad t \in [0,1], u \in \Upsilon.$$

Case 01. *We put $\Xi = [-5, 0]$ and $\Upsilon = C([-5, 0], \mathbb{R})$.*
For each $t \in [0,1], u \in C([-5,0], \mathbb{R})$ and $v \in \mathbb{R}$, we get

$$|\Psi(t, u, v)| \leq \frac{t^3}{e^{t+6}} + \frac{t^2}{e^{t+6}}\|u\|_\Upsilon + \frac{t}{e^{t+6}}|v|,$$

and

$$|\Psi(t, u, v) - \Psi(t, \bar{u}, \bar{v})| \leq e^{-6}\|u - \bar{u}\|_\Upsilon + e^{-6}|v - \bar{v}|.$$

Then the hypothesis (H_1) is satisfied with

$$p_1(t) = \frac{t^3}{e^{t+6}}, \quad p_2(t) = \frac{t^2}{e^{t+6}}, \quad p_3(t) = \frac{t}{e^{t+6}},$$

and $p_1^* = p_1^* = p_1^* = e^{-6}$. The hypothesis (H_2) is satisfied by $k = 2e^{-1}$. Take $u, \bar{u} \in C([-5,0], \mathbb{R})$, Then we have

$$|\mathfrak{h}(t, u) - \mathfrak{h}(t, \bar{u})| \leq 2e^{-1}\|u - \bar{u}\|_\Upsilon.$$

The hypothesis (H_3)*–*(H_5) *are satisfied by*

$$\aleph(t) = t, \quad q_1(t) = \frac{t^2}{e^{t+6}}, \quad q_2(t) = \frac{t}{e^{t+6}}, \quad q_3(t) = \frac{1}{e^{t+6}}, \quad q_4(t) = \left(e^{-2} + \frac{1}{e^{t^2+1}}\right),$$

and $\lambda_\phi = \sup_{t\in[0,1]} \frac{2e^{2-2t}}{\sqrt{t\pi}}$. *Since the conditions of Theorem 1, and Theorem 2 are satisfied. Hence, our problem (23)–(24) has at least a solution defined on* $[-5, 1]$*, and it is generalized Ulam-Hyers-Rassias stable.*

Case 02. *We put* $\Xi = (-\infty, 0]$ *and* $\Upsilon = \mathcal{B}$.
The phase space $\mathcal{B}$ *be* C_γ*, the space for any real positive constant* γ*, defined by*

$$C_\gamma = \{\phi \in C((-\infty, 0], \mathbb{R}) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \quad \text{exist in } \mathbb{R}\}$$

endowed with the norm

$$\|\phi\|_{C_\gamma} = \sup_{\theta\in(-\infty,0]} |\phi(\theta)|.$$

Then in the space C_γ *axioms* (A_1)*–*(A_3) *are satisfied, with* $\widetilde{F} = 1, \quad F(t) = \widehat{F}(t) = 1$*, (see [2]).*
Simple computations show that all conditions of Theorem 3, and Theorem 4 are satisfied. Hence, our problem (23)–(24) has at least a solution defined on $(-\infty, 1]$*, and it is generalized Ulam-Hyers-Rassias stable.*

Example 2 *Consider the implicit neutral Caputo tempered fractional differential equations with stat-dependent delay*

$$ {}^C_0\mathfrak{D}^{\frac{1}{2},2}_t (y(t) - \mathfrak{h}(t, y_t)) = \Psi\left(t, y_{\varrho(t,y_t)}, {}^C_0\mathfrak{D}^{\frac{1}{2},2}_t \left(y(t) - \mathfrak{h}(t, y_{\varrho(t,y_t)})\right)\right); t \in [0, 1], \tag{25}$$

$$y(t) = t + 1; \qquad t \in \Xi, \tag{26}$$

where

$$\Psi(t, u, v) = \frac{t}{e^{t+6}}(t^2 + tu(t + \sigma(u(t))) + v); \quad t \in [0, 1], u \in \Upsilon, v \in \mathbb{R},$$

and

$$\mathfrak{h}(t, u) = \frac{t}{1 + \|u(t + \sigma(u(t)))\|_\Upsilon}\left(e^{-2} + \frac{1}{e^{t^2+1}}\right); \quad t \in [0, 1], u \in \Upsilon,$$

where $\sigma \in \Gamma$,

$$\varrho(t, \phi) = t + \sigma(\phi(t)), \quad (t, \phi) \in J \times \Upsilon.$$

Case 01. *We put* $\Xi = [-5, 0], \quad \Gamma = C(\mathbb{R}, [0, 5])$ *and* $\Upsilon = C([-5, 0], \mathbb{R})$.

Simple computations show that all conditions of Theorem 5, and Theorem 2 are satisfied. Hence, our problem (25)–(26) has at least a solution defined on $[-5, 1]$*, and it is generalized Ulam-Hyers-Rassias stable.*

Case 02. *We put* $\Xi = (-\infty, 0], \quad \Gamma = C(\mathbb{R}, [0, +\infty))$ *and* $\Upsilon = \mathcal{B}$.
$\mathcal{B} = C_\gamma$ *is the phase space defined in Example 1.*

Simple computations show that all conditions of Theorem 6 and Theorem 4 are satisfied. Hence, our problem (25)–(26) has at least a solution defined on $(-\infty, 1]$*, and it is generalized Ulam-Hyers-Rassias stable.*

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