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A New Note On Factored Infinite Series And Trigonometric Fourier Series*

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Abstract

In this note we prove a general theorem dealing with $\varphi - |A, \beta; \delta|_k$ summability of infinite series and one pertaining to trigonometric Fourier series. This new theorem also includes several results previously published by others.

1 Introduction

The notion of absolute summability of an infinite series is in fact the generalization of the notion of the ordinary absolute convergence of an infinite series. Since its appearance, many authors have introduced various general methods of such kind of summability (see [6], the first organized book in which various methods of summability are treated systematically). Along many studies of absolute summability, some authors have studied the absolute summability of trigonometric Fourier series independently from infinite series, however some of them have studied the absolute summability of infinite series and trigonometric Fourier series simultaneously.

We freely can say that the most consistent and fruitful author, who have studied the absolute summability of an infinite series, and not only, is Bor (see [1]-[4] as examples). If the interested reader checks his published papers in ZbMath or in Mr Lookup, then she/he will find many documents which are attributed to him, mainly in the topic we are speaking. We will not mention all of his results, but for our intention we have to recall, later on, one of his result from [4]. The other authors we want to mention here are Sulaiman [18], Özarslan [10]–[17] with or without co-authors, Karakaş [8], Kartal [9], Chen [5], and Jena et al. [7].

Let $\sum a_n$ be an infinite series with partial sums $s = (s_n)$. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-m} = p_{-m} = 0, m \geq 1).$$

The sequence-to-sequence transformation $\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$ defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the terms of the sequence (p_n) (see [6]). Throughout this paper (X_n) is a positive increasing sequence tending to infinity as $n \to \infty$ such that $X_n := \sum_{v=0}^{n} \frac{p_v}{P_v}$. The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty.$$

Let $A = (a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries. Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, \beta; \delta|_k$, $k \geq 1$, $\delta \geq 0$ and β is a real number, if (see [14])

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} |\bar{\Delta} A_n(s)|^k < \infty,$$

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where

$$A_n(s) := \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \dots \quad \text{and} \quad \bar{\Delta} A_n(s) := A_n(s) - A_{n-1}(s).$$

Some special cases of $\varphi - |A, \beta; \delta|_k$ summability are presented as follows: For $\varphi_n = \frac{P_n}{p_n}$, $\varphi - |A, \beta; \delta|_k$ summability reduces to $|A, p_n, \beta; \delta|_k$ summability (see [16]). If we take $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 1$, then we obtain $|A, p_n; \delta|_k$ summability (see [10]). Also, for $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$ and $\delta = 0$, $\varphi - |A, \beta; \delta|_k$ summability reduce to $|A, p_n|_k$ summability (see [18]). Furthermore, by taking $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_n}{P_n}$, we get $|\bar{N}, p_n|_k$ summability.

Throughout this paper we write $u = O(v)$ if there exists a positive constant K, such that $u \leq Kv$.

Closing this section we need another notion pertaining to any sequence of real numbers. For any sequence (λ_n) we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$. The sequences (λ_n) is said to be of bounded variation, denoted by $\lambda_n \in BV$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.

As we have already seen above $|\bar{N}, p_n|_k$ summability is a particular case of $\varphi - |A, \beta; \delta|_k$ summability. This fact have motivated us to generalize Theorem 1 (see section 2) and to show its applicability to trigonometric Fourier series at a point.

2 Known Result

In [4] Bor has proved the following theorem dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 1 *If the sequences (X_n),(λ_n) and (p_n) satisfy the conditions*

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{1}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{4}$$

and

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty, \tag{5}$$

where $t_n = \frac{1}{n+1} \sum_{v=0}^{n} v a_v$, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 Main Result

There are many papers on absolute matrix summability of infinite series (see papers [8], [9], [11], [12], [13], [15]). In this research article we are going to prove a generalization of above mentioned theorem using $\varphi - |A, \beta; \delta|_k$ summability method under some suitable conditions, which is the main aim of it. To do this we need some notions and notations.

Given a normal matrix $A := (a_{nv})$ be a normal matrix, two lower semimatrices $\bar{A} := (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} := \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \dots \tag{6}$$

with

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots, \tag{7}$$

$$A_n(s) := \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v, \tag{8}$$

and

$$\bar{\Delta} A_n(s) := \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{9}$$

Now, we are in able to prove the following main result.

Theorem 2 *Let A be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \tag{10}$$

$$a_{n-1,v} \geq a_{nv} \quad for \quad n \geq v + 1, \tag{11}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{12}$$

and

$$\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} = O(a_{nn}). \tag{13}$$

Let the sequence (X_n) and (λ_n) satisfy the conditions (1)–(3) of Theorem 1. If the conditions

$$\varphi_n p_n = O(P_n) \quad as \quad n \to \infty, \tag{14}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| = O\left(\varphi_v^{\beta(\delta k+k-1)-k}\right) \quad as \quad m \to \infty, \tag{15}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\hat{a}_{n,v+1}| = O(1) \quad as \quad m \to \infty, \tag{16}$$

and

$$\sum_{r=1}^{m} \varphi_r^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{17}$$

are satisfied, then the series $\sum a_n \lambda_n$ is $\varphi - |A, \beta; \delta|_k$ summable with $k \geq 1$, $\delta \geq 0$ and $k - \beta(\delta k + k - 1) > 0$, where $\Delta_v(\hat{a}_{nv}) := \hat{a}_{nv} - \hat{a}_{n,v+1}$.

We need the lemma for the proof of the main result.

Lemma 1 ([2]) *Under the conditions of Theorem 1, we get*

$$nX_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty, \tag{18}$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \tag{19}$$

$$X_n |\lambda_n| = O(1) \quad as \quad n \to \infty. \tag{20}$$

Now we pass to the proof of Theorem 2.

Proof of Theorem 2. Let (Θ_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (8) and (9), we have $\bar{\Delta}\Theta_n = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v$. By Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta}\Theta_n &= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r = \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)(v+1)t_v + \frac{\hat{a}_{nn}\lambda_n}{n}(n+1)t_n \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v}\Delta_v(\hat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v} + \frac{n+1}{n} a_{nn}\lambda_n t_n \\
&= \Theta_{n,1} + \Theta_{n,2} + \Theta_{n,3} + \Theta_{n,4}.
\end{aligned}$$

To prove Theorem 2, using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k+k-1)} \left|\Theta_{n,r}\right|^k < \infty \quad \text{for} \quad r = 1, 2, 3, 4.$$

Firstly, using Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left|\Theta_{n,1}\right|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right) \times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right)^{k-1}.
\end{aligned}$$

Since, by (6) and (7), $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = a_{nv} - a_{n-1,v}$, then, by using (6), (10) and (11), we obtain

$$\begin{aligned}
\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| &= \sum_{v=1}^{n-1} |a_{nv} - a_{n-1,v}| \\
&= \sum_{v=0}^{n-1} a_{n-1,v} - a_{n-1,0} - \sum_{v=0}^{n} a_{nv} + a_{n0} + a_{nn} \\
&= \bar{a}_{n-1,0} - a_{n-1,0} - \bar{a}_{n0} + a_{n0} + a_{nn} \leq a_{nn}.
\end{aligned} \tag{21}$$

Moreover, since $|\lambda_n| = O(1/X_n)$ by (20), we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |\Theta_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v||\lambda_v|^{k-1}|t_v|^k \right) \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v| \frac{|t_v|^k}{X_v^{k-1}} \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{|t_v|^k}{X_v^{k-1}} \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} |\lambda_v| \frac{|t_v|^k}{X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} \\
&\quad + O(1)|\lambda_m| \sum_{r=1}^{m} \varphi_r^{\beta(\delta k+k-1)-k} \frac{|t_r|^k}{X_r^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

taking into account (12), (14), (15) and Lemma 1.

We use Hölder's inequality and (13) to obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |\Theta_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| v |\Delta\lambda_v| \frac{|t_v|}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| (v|\Delta\lambda_v|)^k \frac{|t_v|^k}{v} \right) \times \left(\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| (v|\Delta\lambda_v|)(v|\Delta\lambda_v|)^{k-1} \frac{|t_v|^k}{v} \right).
\end{aligned}$$

Since $v|\Delta\lambda_v| = O(1/X_v)$, the use of Lemma 1 implies

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |\Theta_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| (v|\Delta\lambda_v|) \frac{|t_v|^k}{v X_v^{k-1}} \right) \\
&= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| \frac{|t_v|^k}{v X_v^{k-1}} \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| \frac{|t_v|^k}{v X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \frac{|t_r|^k}{r X_r^{k-1}} + O(1) m|\Delta\lambda_m| \sum_{r=1}^{m} \frac{|t_r|^k}{r X_r^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta^2\lambda_v| X_v + \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_{v+1} + O(1) m|\Delta\lambda_m| X_m = O(1)
\end{aligned}$$

as $m \to \infty$, by means of (2), (3), (16), and (12).

Once more, applying Höder's inequality, we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left|\Theta_{n,3}\right|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \times \left(\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}||\lambda_{v+1}|^{k-1} \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v X_v^{k-1}} \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v X_v^{k-1}} \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| \sum_{r=1}^{v} \frac{|t_r|^k}{r X_r^{k-1}} + O(1) m |\Delta\lambda_m| \sum_{r=1}^{m} \frac{|t_r|^k}{r X_r^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_{v+1} + O(1) m |\Delta\lambda_m| X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by using (3), (12), (13), (14), (16) and Lemma 1.

Finally, by using Abel's formula and doing same operations as in $\Theta_{n,1}$ we get

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \left|\Theta_{n,4}\right|^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^k |\lambda_n||\lambda_n|^{k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k} |\lambda_n| \frac{|t_n|^k}{X_n^{k-1}} \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by using (12),(14),(15) and Lemma 1. Therefore, we obtain that

$$\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} |\Theta_{n,r}|^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 2. ■

4 An Application to Trigonometric Fourier Series

Let f be a 2π-periodic function and integrable over $(-\pi, \pi)$ in the sense of Lebesgue. The trigonometric Fourier series of f, at the point x, is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,$$

and

$$A_n(x) := a_n \cos nx + b_n \sin nx.$$

We write

$$\phi(t) := \frac{1}{2}\{f(x+t) + f(x-t)\} \ \text{ and } \phi_\alpha(t) := \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du \ \text{ for } \alpha > 0.$$

It is known (see [5]) that if $\phi_1(t) \in BV(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nA_n(x))$, i.e.

$$t_n(x) := \frac{1}{n}\sum_{k=1}^{n} kA_k(x).$$

Regarding to this well-known statement we recall the following.

Theorem 3 ([4]) *If $\phi_1(t) \in BV(0,\pi)$, and sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 1, then the series $\sum \lambda_n A_n(x)$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.*

Whence, we are able to generalize Theorem 3 for $\varphi - |A, \beta; \delta|_k$ summability method in the following form.

Theorem 4 *Let A be a positive normal matrix as in Theorem 2. If $\phi_1(t) \in BV(0,\pi)$, and sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 2, then the series $\sum \lambda_n A_n(x)$ is $\varphi - |A, \beta; \delta|_k$ summable with $k \geq 1$, $\delta \geq 0$, and $k - \beta(\delta k + k - 1) > 0$.*

5 Remarks on Some Cases of $\varphi - |A, \beta; \delta|_k$ Summability

Here we show most important cases of $\varphi - |A, \beta; \delta|_k$ summability. Namely, if we take $\varphi_n = \frac{P_n}{p_n}$ in Theorem 2, then we get a theorem on $|A, p_n, \beta; \delta|_k$ summability method which has been appeared in [17]. Also, if we take $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we obtain Theorem 1 dealing with $|\bar{N}, p_n|_k$ summability method. In this case, the condition (17) reduce to the condition (4). Finally, in the special cases of φ_n, β, δ and the matrix A, we can derive similar results from Theorem 4 for the trigonometric Fourier series.

6 Conclusion

Absolute summability of an infinite series as well as the trigonometric Fourier series has been studied deeply in many research articles. Here we have generalized several results previously done by other mathematicians. Namely, we have employed $\varphi - |A, \beta; \delta|_k$ summability to obtain a new and a quite general theorem. Finally, we have applied this result to the trigonometric Fourier series.

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