Applied Mathematics E-Notes, 24(2024), 443-456 © ISSN 1607-2510
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Positive Solutions Of Fourth-Order Problem Under Nonlocal Boundary Value Conditions*

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Received 8 July 2023

Abstract

The current study intends to delve into the existence of positive solutions to fourth-order differential equations dependent on all derivatives in nonlinearity under the Stieltjes integral boundary conditions

$$\begin{cases} u^{(4)}(t) = g(t, u(t), u'(t), u''(t), u'''(t)), t \in [0,1], \\ u(0) = \alpha_1[u],\ u'(1) = \alpha_2[u],\ u''(0) + \alpha_3[u] = 0,\ u'''(1) + \alpha_4[u] = 0, \end{cases}$$

where $g : [0,1] \times {\mathbb{R}_+}^2 \times {\mathbb{R}_-}^2 \to \mathbb{R}_+$ is continuous, $\alpha_i[u] = \int_0^1 u(t)\, \mathrm{d}\mathcal{A}_i(t) \geq 0$ is Stieltjes integral represented by the bounded variation function $\mathcal{A}_i(t)$ $(i = 1, 2, 3, 4)$. Some growth conditions are posed on nonlinearity g meanwhile the spectral radius of corresponding linear operators are restricted. On the cones in $C^3[0,1]$ we apply the theory of fixed point index, thus verifying the existence of positive solutions. Some specific examples concerning fourth-order nonlinear differential equations are given under mixed boundary conditions involving multi-point with sign-changing coefficients and integral with sign-changing kernel.

1 Introduction

In this paper we investigate the existence of positive solutions for fourth-order boundary value problem (BVP) with dependence on all derivatives in nonlinearity subject to boundary conditions of Stieltjes integral type

$$\begin{cases} u^{(4)}(t) = g(t, u(t), u'(t), u''(t), u'''(t)),\ t \in [0,1] \\ u(0) = \alpha_1[u],\ u'(1) = \alpha_2[u],\ u''(0) + \alpha_3[u] = 0,\ u'''(1) + \alpha_4[u] = 0. \end{cases} \tag{1}$$

Based on fixed point index method and global bifurcation technique respectively, Li [1] and Ma [2] explored the beam equations involving the bending moment with the hinged ends

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)),\ t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

Recently, the authors in [3] investigated the existence of positive solutions to the following problems:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t)),\ t \in [0,1], \\ u(0) = \beta_1[u],\ u'(1) = \beta_2[u],\ u''(0) + \beta_3[u] = 0,\ u''(1) + \beta_4[u] = 0 \end{cases}$$

and

$$\begin{cases} -u^{(4)}(t) = g(t, u(t), u'(t), u''(t)),\ t \in [0,1], \\ u(0) = \alpha_1[u],\ u'(0) = \alpha_2[u],\ u''(0) = \alpha_3[u],\ u''(1) = \alpha_4[u], \end{cases}$$

*Mathematics Subject Classifications: 34B10, 34B15, 34B18.

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where $\alpha_i[u]$, $\beta_i[u]$ $(i = 1, 2, 3, 4)$ are Stieltjes integrals of signed measures. They modelled the deflection of beam equations with the nonlinearities including the slope u' and bending moment u''. The boundary conditions of Stieltjes integrals indicated that the mechanism at the end points hangs on the feedback along parts of the beam for the control of displacement.

The authors in [4] studied the existence of positive solutions to the boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)),\ t \in (0, 1), \\ au(0) - bu'(0) = \alpha[u],\ cu(1) + du'(1) = \beta[u] + \lambda[u'], \end{cases}$$

where $\lambda[u'] = \int_0^1 u'(t)\,\mathrm{d}\Lambda(t)$, Λ is a bounded variation function, a, b, c, d are nonnegative constants with $\rho = ac + ad + bc > 0$.

Lemma 1 ([5],[6]) *Let Ω be a bounded open subset of Banach space X with $0 \in \Omega$ and P be a cone in X. If $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator and $\mu Au \neq u$ for $u \in K \cap \partial\Omega$ and $\mu \in [0, 1]$, then the fixed point index $i(A, P \cap \partial\Omega, P) = 1$.*

Lemma 2 ([5],[6]) *Let Ω be a bounded open subset of Banach space X and P be a cone in X. If $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator and there exists $v_0 \in P \backslash \{0\}$ such that $u - Au \neq v\varphi_0$ for $u \in K \cap \partial\Omega$ and $v \geq 0$, then the fixed point index $i(A, P \cap \partial\Omega, P) = 0$.*

Lemma 3 ([5]) *Let P be a reproducing cone,i.e., $X = P - P$, in Banach space X and $L : X \to X$ be a completely continuous linear operator with $L(P) \subset P$. If the spectral radius $r(L) > 0$, then there exists $\varphi \in P \backslash \{0\}$ such that $L\varphi = r(L)\varphi$.*

Lemma 4 ([7]) *Let P be a cone in Banach space X and $L : X \to X$ be a completely continuous linear operator with $L(P) \subset P$. If there exist $v_0 \in P \backslash \{0\}$ and $\lambda_0 > 0$ such that $Lv_0 \geq \lambda_0 v_0$ in the sense of partial ordering induced by P, then there exist $u_0 \in P \backslash \{0\}$ and $\lambda_1 \geq \lambda_0$ such that $Lu_0 = \lambda_1 u_0$.*

2 Preliminaries

Let $E = C^3[0, 1]$ be the Banach space consisting of all third-order continuously differentiable functions on $[0, 1]$ with the norm

$$\|u\|_{C^3} = \max\left\{\|u\|_C, \|u'\|_C, \|u''\|_C, \|u'''\|_C\right\},$$

where $\|u\|_C = \max\{|u(t)| : t \in [0, 1]\}$ for $u \in C[0, 1]$.

We can calculate $\delta_1(t) = 1$, $\delta_2(t) = t$, $\delta_3(t) = t - \frac{1}{2}t^2$ and $\delta_4(t) = \frac{1}{2}t - \frac{1}{6}t^3$, which are solutions to $u^{(4)}(t) = 0$ satisfying the following boundary conditions:

$$u(0) = 1,\ u'(1) = 0,\ u''(0) = 0,\ u'''(1) = 0;$$

$$u(0) = 0,\ u'(1) = 1,\ u''(0) = 0,\ u'''(1) = 0;$$

$$u(0) = 0,\ u'(1) = 0,\ u''(0) + 1 = 0,\ u'''(1) = 0;$$

$$u(0) = 0,\ u'(1) = 0,\ u''(0) = 0,\ u'''(1) + 1 = 0.$$

Let

$$k_0(t, s) = \begin{cases} st - \frac{1}{2}st^2 - \frac{1}{6}s^3, & 0 \leq s \leq t \leq 1, \\ st - \frac{1}{2}ts^2 - \frac{1}{6}t^3, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2}$$

$k_0(t, s)$ is the Green's function associated with

$$\begin{cases} u^{(4)}(t) = 0,\ t \in [0, 1], \\ u(0) = u'(1) = u''(0) = u'''(1) = 0. \end{cases}$$

Define an operator in $C^3[0,1]$ as

$$(Tu)(t) =: \sum_{i=1}^{4} \alpha_i[u]\delta_i(t) + \int_0^1 k_0(t,s)g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s, \tag{3}$$

where $\alpha_i[u] = \int_0^1 u(t)\,\mathrm{d}\mathcal{A}_i(t) \geq 0$, $(i = 1,2,3,4)$. We set

$$(Au)(t) =: \sum_{i=1}^{4} \alpha_i[u]\delta_i(t), (Fu)(t) =: \int_0^1 k_0(t,s)g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s,$$

so $(Tu)(t) = (Au)(t) + (Fu)(t)$. We impose the following hypotheses:

(C_1) $g : [0,1] \times {\mathbb{R}_+}^2 \times {\mathbb{R}_-}^2 \to \mathbb{R}_+$ is continuous, here $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

(C_2) For each $i \in \{1,2,3,4\}$, $\mathcal{A}_i$ is of bounded variation and

$$\mathcal{K}_i(s) := \int_0^1 k_0(t,s)\,\mathrm{d}\mathcal{A}_i(t) \geq 0, \ \forall s \in [0,1].$$

(C_3) $\alpha_i[\delta_j] \geq 0$ $(i,j = 1,2,3,4)$ and moreover for the 4×4 matrix

$$[A] = \begin{pmatrix} \alpha_1[\delta_1] & \alpha_1[\delta_2] & \alpha_1[\delta_3] & \alpha_1[\delta_4] \\ \alpha_2[\delta_1] & \alpha_2[\delta_2] & \alpha_2[\delta_3] & \alpha_2[\delta_4] \\ \alpha_3[\delta_1] & \alpha_3[\delta_2] & \alpha_3[\delta_3] & \alpha_3[\delta_4] \\ \alpha_4[\delta_1] & \alpha_4[\delta_2] & \alpha_4[\delta_3] & \alpha_4[\delta_4] \end{pmatrix},$$

its spectral radius $r([A]) < 1$.

Writing $\langle \alpha, \delta \rangle = \sum_{i=1}^{4} \alpha_i \delta_i$ for the inner product in $\mathbb{R}^4$, we define the operator S in $C^3[0,1]$ as

$$(Su)(t) = \langle (I - [A])^{-1}\alpha[Fu], \delta(t) \rangle + (Fu)(t),$$

where $\alpha[Fu] = (\alpha_1[Fu], \alpha_2[Fu], \alpha_3[Fu], \alpha_4[Fu])^T$ is the transpose of vector.

Lemma 5 *Suppose that (C_1) holds. Then BVP (1) has a solution if and only if there exists a fixed point of T in $C^3[0,1]$.*

Lemma 6 *Suppose that (C_1)–(C_3) hold. Then S can be written in the form as*

$$\begin{aligned} (Su)(t) &= \langle (I - [A])^{-1}\alpha[Fu], \delta(t) \rangle + (Fu)(t) \\ &= \int_0^1 (\langle (I - [A])^{-1}\mathcal{K}(s), \delta(t) \rangle + k_0(t,s))g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s \\ &=: \int_0^1 k_S(t,s)g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s, \end{aligned} \tag{4}$$

where $\mathcal{K}(s) = (\mathcal{K}_1(s), \mathcal{K}_2(s), \mathcal{K}_3(s), \mathcal{K}_4(s))^T$, i.e.,

$$k_S(t,s) = \langle (I - [A])^{-1}\mathcal{K}(s), \delta(t) \rangle + k_0(t,s) = \sum_{i=1}^{4} \kappa_i(s)\delta_i(t) + k_0(t,s) \tag{5}$$

and $\kappa_i(s)$ is the ith component of $(I - [A])^{-1}\mathcal{K}(s)$.

Lemma 7 *If (C_2) and (C_3) hold, then $\kappa_i \geq 0$ $(i = 1, 2, 3, 4)$ and for $s, t \in [0, 1]$,*

$$c_0(t)\Psi_0(s) \leq k_S(t, s) \leq \Psi_0(s), \tag{6}$$

where

$$\Psi_0(s) = \sum_{i=1}^{4} \kappa_i(s) + \frac{1}{6}s(3 - s^2), c_0(t) = \frac{1}{2}t - \frac{1}{6}t^3,$$

and

$$c_1(t)\Psi_1(s) \leq \frac{\partial k_S(t, s)}{\partial t} \leq \Psi_1(s), c_2(t)\Psi_2(s) \leq -\frac{\partial^2 k_S(t, s)}{\partial t^2} \leq \Psi_2(s), \tag{7}$$

where

$$\Psi_1(s) = \sum_{i=2}^{4} \kappa_i(s) + \frac{1}{2}s(2 - s), c_1(t) = \frac{1}{2} - \frac{1}{2}t^2,$$

$$\Psi_2(s) = \sum_{i=3}^{4} \kappa_i(s) + s, c_2(t) = t.$$

Proof. For $s \in [0, 1]$, $\kappa_i(s) \geq 0$ $(i = 1, 2, 3, 4)$ are due to [8]. (6) and (7) come directly from the inequalities

$$(\frac{1}{2}t - \frac{1}{6}t^3) \sum_{i=1}^{4} \kappa_i(s) \leq \sum_{i=1}^{4} \kappa_i(s)\delta_i(t) \leq \sum_{i=1}^{4} \kappa_i(s),$$

$$(\frac{1}{2}t - \frac{1}{6}t^3)(\frac{1}{6}s(3 - s^2)) \leq \frac{1}{2}t(3 - t^2)(\frac{1}{6}s(3 - s^2)) \leq k_0(t, s) \leq \frac{1}{6}s(3 - s^2),$$

we have, for $s, t \in [0, 1]$,

$$c_0(t)\Psi_0(s) \leq k_S(t, s) = \sum_{i=1}^{4} \kappa_i(s)\delta_i(t) + k_0(t, s) \leq \Psi_0(s).$$

Moreover, the next two inequalities

$$(\frac{1}{2} - \frac{1}{2}t^2) \sum_{i=2}^{4} \kappa_i(s) \leq \sum_{i=1}^{4} \kappa_i(s)\delta_i^{'}(t) \leq \sum_{i=2}^{4} \kappa_i(s),$$

$$(\frac{1}{2} - \frac{1}{2}t^2)(\frac{1}{2}s(2 - s)) \leq (1 - t)(\frac{1}{2}s(2 - s)) \leq \frac{\partial k_0(t, s)}{\partial t} \leq \frac{1}{2}s(2 - s),$$

imply that, for $s, t \in [0, 1]$,

$$c_1(t)\Psi_1(s) \leq \frac{\partial k_S(t, s)}{\partial t} = \sum_{i=1}^{4} \kappa_i(s)\delta_i^{'}(t) + \frac{\partial k_0(t, s)}{\partial t} \leq \Psi_1(s).$$

Finally from the two inequalities

$$t \sum_{i=3}^{4} \kappa_i(s) \leq -\sum_{i=1}^{4} \kappa_i(s)\delta_i^{''}(t) = \kappa_3(s) + t\kappa_4(s) \leq \sum_{i=3}^{4} \kappa_i(s)$$

and

$$ts \leq -\frac{\partial^2 k_0(t, s)}{\partial t^2} \leq s,$$

we obtain that

$$c_2(t)\Psi_2(s) \le -\frac{\partial^2 k_S(t,s)}{\partial t^2} = -\sum_{i=1}^{4}\kappa_i(s)\delta_i^{''}(t) - \frac{\partial^2 k_0(t,s)}{\partial t^2} \le \Psi_2(s)$$

for $s, t \in [0,1]$. ■

Define two cones in $C^3[0,1]$ and several linear operators as follows.

$$P = \left\{u \in C^3[0,1] : u(t) \ge 0, u'(t) \ge 0, u''(t) \le 0, u'''(t) \le 0, \forall t \in [0,1]\right\}, \tag{8}$$

$$\begin{aligned} K = \{&u \in P : u(t) \ge c_0(t)\|u\|_C, u'(t) \ge c_1(t)\|u'\|_C, \\ &-u''(t) \ge c_2(t)\|u''\|_C,\ \forall t \in [0,1]; \alpha_i[u] \ge 0 (i = 1,2,3,4)\}, \end{aligned} \tag{9}$$

$$(L_i u)(t) = \int_0^1 k_S(t,s)(a_i u(s) + b_i u'(s) - c_i u''(s) - d_i u'''(s))\,\mathrm{d}s (i = 1,2), \tag{10}$$

$$(L_3 u)(t) = a_1 \int_0^1 k_S(t,s) u(s)\,\mathrm{d}s, \tag{11}$$

where $a_i, b_i, c_i, d_i (i = 1,2)$ are nonnegative constants.

It is clear from the analysis that P and K are cones, more specifically, P is a solid cone. Denote the cone ordering induced by P, $u \preceq v$ equivalently $v \succeq u$ if and only if $v - u \in P$.

Lemma 8 *Suppose that (C_1)–(C_3) hold. Then $S : P \to K$ and $L_i : C^3[0,1] \to C^3[0,1]$ are completely continuous operator with $L_i(P) \subset K$ $(i = 1,2,3)$.*

Proof. For $u \in P$ and $t \in [0,1]$, it is easy to see that $Su \in C^3[0,1]$, $(Su)(t) \ge 0$, $(Su)'(t) \ge 0$ and $(Su)''(t) \le 0$, by Lemma 7,

$$\begin{aligned} (Su)(t) &= \int_0^1 k_S(t,s) g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s \\ &\ge c_0(t)\int_0^1 \Psi_0(s) g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} (Su)'(t) &= \int_0^1 \frac{\partial k_S(t,s)}{\partial t} g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s \\ &\ge c_1(t)\int_0^1 \Psi_1(s) g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} -(Su)''(t) &= -\int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2} g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s \\ &\ge c_2(t)\int_0^1 \Psi_2(s) g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s, \end{aligned}$$

hence we have

$$\begin{aligned} (Su)(t) &= \int_0^1 k_S(t,s) g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s \\ &\le \int_0^1 \Psi_0(s) g(s, u(s), u'(s), u''(s), u'''(s))\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned}(Su)'(t) &= \int_0^1 \frac{\partial k_S(t,s)}{\partial t} g(s,u(s),u'(s),u''(s))\,\mathrm{d}s \\ &\le \int_0^1 \Psi_1(s) g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}-(Su)''(t) &= -\int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2} g(s,u(s),u'(s),u''(s))\,\mathrm{d}s \\ &\le \int_0^1 \Psi_2(s) g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s.\end{aligned}$$

Therefore, $(Su)(t) \ge c_0(t)\|Su\|_C$, $(Su)'(t) \ge c_1(t)\|(Su)'\|_C$ and $-(Su)''(t) \ge c_2(t)\|(Su)''\|_C$ for $t \in [0,1]$. Moreover, it follows from (C_2) that

$$\begin{aligned}\alpha_i[Su] &= \int_0^1 \Big(\int_0^1 k_S(t,s) g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s\Big)\,\mathrm{d}\mathcal{A}_i(t) \\ &= \int_0^1 \Big(\int_0^1 k_S(t,s)\,\mathrm{d}\mathcal{A}_i(t)\Big) g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s \\ &= \int_0^1 \mathcal{K}_i(s) g(s,u(s),u'(s),u''(s),u'''(s))\,\mathrm{d}s,\end{aligned}$$

that is, $Su \in K$. As for the complete continuity of S, it is obvious.

Drawing upon the same method, we can verify that $L_i : C^3[0,1] \to C^3[0,1]$ is completely continuous linear operator with $L_i(P) \subset K$ $(i = 1,2,3)$. ■

Lemma 9 *Suppose that (C_1)–(C_3) hold. Then S and T have the same fixed points in K. As a result, BVP (1) has a positive solution if and only if S has a fixed point in K.*

3 Main Results

Theorem 1 *Under hypotheses (C_1)–(C_3), suppose that*

(F_1) there exist constants $a_2, b_2, c_2, d_2 \ge 0$ and $r > 0$ such that

$$g(t,x_1,x_2,x_3,x_4) \le a_2x_1 + b_2x_2 - c_2x_3 - d_2x_4, \tag{12}$$

for all $(t,x_1,x_2,x_3,x_4) \in [0,1] \times [0,r]^2 \times [-r,0]^2$; moreover, the spectral radius $r(L_2) < 1$, where L_2 is defined by (10);

(F_2) there exist positive constants a_1, b_1, c_1, C_0 satisfying

$$\min\left\{\frac{a_1}{3}\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s,\ \frac{b_1}{2}\int_0^1 c_1(s)\Psi_1(s)\,\mathrm{d}s,\ c_1\int_0^1 c_2(s)\Psi_2(s)\,\mathrm{d}s\right\} > 1 \tag{13}$$

such that

$$g(t,x_1,x_2,x_3,x_4) \ge a_1x_1 + b_1x_2 - c_1x_3 - C_0, \tag{14}$$

for all $(t,x_1,x_2,x_3,x_4) \in [0,1] \times {\mathbb{R}_+}^2 \times {\mathbb{R}_-}^2$;

(F_3) for any $M > 0$, there exists a positive continuous function $H_M(\rho)$ on $\mathbb{R}_+$ which satisfies

$$\int_0^{+\infty} \frac{\rho}{H_M(\rho)+1}\,\mathrm{d}\rho = +\infty \tag{15}$$

such that $\forall (t,x_1,x_2,x_3,x_4) \in [0,1] \times [0,M]^2 \times [-M,0] \times \mathbb{R}_-$,

$$g(t,x_1,x_2,x_3,x_4) \le H_M(|x_4|). \tag{16}$$

Then BVP (1) has one positive solution in K.

Proof. (i) Define $\Omega_r = \left\{u \in C^3[0,1] : \|u\|_{C^3} < r\right\}$. First we prove that $\mu S u \neq u$ for $u \in K \cap \partial\Omega_r$ and $\mu \in [0,1]$. In fact, if there exist $u_1 \in K \cap \partial\Omega_r$ and $\mu_0 \in [0,1]$ such that $u_1 = \mu_0 S u_1$, then we can deduce from

$$0 \le u_1(t), u_1'(t) \le r, 0 \le -u_1''(t), -u_1'''(t) \le r, \forall t \in [0,1]$$

and (12) that, for $t \in [0,1]$,

$$u_1(t) \le (L_2 u_1)(t), u_1'(t) \le (L_2 u_1)'(t),$$
$$u_1''(t) \ge (L_2 u_1)''(t), u_1'''(t) \ge (L_2 u_1)'''(t),$$

thus $(I - L_2)u_1 \preceq 0$. Due to the fact that the spectral radius $r(L_2) < 1$, we know that $I - L_2$ has a bounded inverse operator $(I - L_2)^{-1} : P \to P$ and $u_1 \preceq (I - L_2)^{-1} 0 = 0$ which contradicts $u_1 \in K \cap \partial\Omega_r$. Therefore, $i(S, K \cap \Omega_r, K) = 1$ follows from Lemma 1.

(ii) Let

$$M = \max\left\{\frac{3C_0 \int_0^1 \Psi_0(s)\,\mathrm{d}s}{a_1 \int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s - 3}, \frac{2C_0 \int_0^1 \Psi_1(s)\,\mathrm{d}s}{b_1 \int_0^1 c_1(s)\Psi_1(s)\,\mathrm{d}s - 2}, \frac{C_0 \int_0^1 \Psi_2(s)\,\mathrm{d}s}{c_1 \int_0^1 c_2(s)\Psi_2(s)\,\mathrm{d}s - 1}\right\}. \tag{17}$$

From formula (13), $M > 0$. By (15) it can easily be seen that

$$\int_M^{+\infty} \frac{\rho}{H_M(\rho) + C_0}\,\mathrm{d}\rho = +\infty,$$

and so there exists $M_1 > M$ such that

$$\int_M^{M_1} \frac{\rho}{H_M(\rho) + C_0}\,\mathrm{d}\rho > M. \tag{18}$$

(iii) In this step we construct a homotopy and find a subset Ω_R in order to compute the fixed point index later, where $\Omega_R = \left\{u \in C^3[0,1] : \|u\|_{C^3} < R\right\}$.

For $u \in K$ define the homotopy $H(\lambda, u) = Su + \lambda v$, where

$$v(t) = C_0 \int_0^1 k_S(t,s)\,\mathrm{d}s.$$

Then $v \in K$ and $H : [0,1] \times K \to K$ is completely continuous. Let $R > \{r, M_1\}$ and we will show that

$$H(\lambda, u) \neq u, \ \forall u \in K \cap \partial\Omega_R, \ \lambda \in [0,1]. \tag{19}$$

If it does not hold, there exist $u_2 \in K \cap \partial\Omega_R$ and $\lambda_0 \in [0,1]$ such that

$$H(\lambda_0, u_2) = u_2, \tag{20}$$

thus

$$\begin{aligned}
\|u_2\|_C &= u_2(1)\\
&= \int_0^1 k_S(1,s) g(s, u_2(s), u_2'(s), u_2''(s), u_2'''(s))\,\mathrm{d}s + \lambda_0 C_0 \int_0^1 k_S(1,s)\,\mathrm{d}s\\
&\ge \int_0^1 k_S(1,s)(a_1 u_2(s) + b_1 u_2'(s) - c_1 u_2''(s) - C_0 + \lambda_0 C_0)\,\mathrm{d}s\\
&\ge \int_0^1 k_S(1,s)(a_1 u_2(s) - C_0)\,\mathrm{d}s\\
&\ge \frac{a_1}{3}\|u_2\|_C \int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s - C_0 \int_0^1 \Psi_0(s)\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\|u_2'\|_C &= u_2'(0)\\
&= \int_0^1 \frac{\partial k_S(0,s)}{\partial t} g(s,u_2(s),u_2'(s),u_2''(s),u_2'''(s))\,\mathrm{d}s + \lambda_0 C_0 \int_0^1 \frac{\partial k_S(0,s)}{\partial t}\,\mathrm{d}s\\
&\geq \int_0^1 \frac{\partial k_S(0,s)}{\partial t}(a_1u_2(s)+b_1u_2'(s)+c_1u_2''(s)-C_0+\lambda_0C_0)\,\mathrm{d}s\\
&\geq \int_0^1 \frac{\partial k_S(0,s)}{\partial t}(b_1u_2'(s)-C_0)\,\mathrm{d}s\\
&\geq \frac{b_1}{2}\|u_2'\|_C \int_0^1 c_1(s)\Psi_1(s)\,\mathrm{d}s - C_0\int_0^1 \Psi_1(s)\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\|u_2''\|_C &= -u_2''(1)\\
&= -\int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2} g(s,u_2(s),u_2'(s),u_2''(s),u_2'''(s))\,\mathrm{d}s - \lambda_0 C_0 \int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2}\,\mathrm{d}s\\
&\geq -\int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2}(a_1u_2(s)+b_1u_2'(s)+c_1u_2''(s)-C_0+\lambda_0C_0)\,\mathrm{d}s\\
&\geq -\int_0^1 \frac{\partial^2 k_S(1,s)}{\partial t^2}(c_1u_2''(s)-C_0)\,\mathrm{d}s\\
&\geq c_1\|u_2''\|_C \int_0^1 c_2(s)\Psi_2(s)\,\mathrm{d}s - C_0\int_0^1 \Psi_2(s)\,\mathrm{d}s.
\end{aligned}$$

These imply by (17) that

$$\|u_2\|_C \leq M,\ \|u_2'\|_C \leq M,\ \|u_2''\|_C \leq M. \tag{21}$$

From (16), (20) and (21), it follows that

$$\begin{aligned}
u_2^{(4)}(t) &= f(t,u_2(t),u_2'(t),u_2''(t),u_2'''(t)) + \lambda_0 C_0\\
&\leq f(t,u_2(t),u_2'(t),u_2''(t),u_2'''(t)) + C_0 \leq H_M(|u_2'''(t)|) + C_0
\end{aligned} \tag{22}$$

for $t \in [0,1]$, multiplying both sides of (22) by $-u_2'''(t) \geq 0$, we have that

$$\frac{-u_2'''(t)u_2^{(4)}(t)}{H_M(-u_2'''(t)) + C_0} \leq -u_2'''(t). \tag{23}$$

At this point, for $t \in [0,1]$, $-u_2'''(t) \leq M_1$. Otherwise, there is $t_0 \in [0,1]$ which makes $-u_2'''(t_0) > M_1$, and so by the mean value theorem, there is $\xi \in (0,1)$ which satisfies $-u_2'''(\xi) = u_2''(0) - u_2''(1) \leq M < M_1$. So there are $t_1, t_2 \in [0,1]$ that make $-u_2'''(t_1) = M_1$, $-u_2'''(t_2) = M$. Because $u_2^{(4)}(t) \geq 0$, so $t_1 < t_2$. Let $\rho = -u_2'''(t)$, integral (23) over $[t_1, t_2]$

$$\begin{aligned}
\int_M^{M_1} \frac{\rho}{H_M(\rho)+C_0}\,\mathrm{d}\rho &= \int_{-u_2'''(t_2)}^{-u_2'''(t_1)} \frac{\rho}{H_M(\rho)+C_0}\,\mathrm{d}\rho\\
&= \int_{-u_2'''(t_1)}^{-u_2'''(t_2)} \frac{-u_2'''(t)u_2^{(4)}(t)}{H_M(-u_2'''(t))+C_0}\,\mathrm{d}t\\
&\leq \int_{t_1}^{t_2} (-u_2'''(t))\,\mathrm{d}t \leq u_2''(t_1) - u_2''(t_2) \leq \|u_2''\|_C \leq M.
\end{aligned}$$

Hence by (18) we also have that $\|u_2'''\|_C \leq M_1$ and $\|u_2\|_{C^3} \leq M_1$, a contradiction to $\|u_2\|_{C^3} = R > M_1$.

From (19) it follows that

$$i(S, K\cap\Omega_R, K) = i(H(0,\cdot), K\cap\Omega_R, K) = i(H(1,\cdot), K\cap\Omega_R, K) \tag{24}$$

by the homotopy invariance property of fixed point index.

(iv) For the function $c_0(t)$, we have from (11) that

$$(L_3c_0)(t) = a_1 \int_0^1 k_S(t,s)c_0(s)\,\mathrm{d}s \geq (a_1 \int_0^1 c_0(s)\Phi_0(s)\,\mathrm{d}s)c_0(t).$$

From (13) it follows that

$$\lambda_0 =: a_1 \int_0^1 c_0(s)\Phi_0(s)\,\mathrm{d}s > 3.$$

Since L_3 is a completely continuous linear operator in $C[0,1]$, we consider the nonnegative cone $C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0, \forall \in [0,1]\}$ in Lemma 4.

There exist $\lambda_1 > \lambda_0$ and $\varphi_0 \in C^+[0,1]\setminus\{0\}$ such that $\varphi_0 = \lambda_1^{-1}L_3\varphi_0$. Obviously $\varphi_0 \in P$ can be directly verified, and thus $\varphi_0 \in K$ by Lemma 8.

(v) Now we prove that $u - H(1,u) \neq v\varphi_0$ for $u \in K \cap \partial\Omega_R$ and $v \geq 0$, where φ_0 is as in step (iii), and hence

$$i(H(1,\cdot), K \cap \Omega_R, K) = 0 \tag{25}$$

by Lemma 2.

If there exist $u_0 \in K \cap \partial\Omega_R$ and $v_0 \geq 0$ such that $u_0 - H(1,u_0) = v_0\varphi_0$, then it is clear that $v_0 > 0$ by (19) and thus

$$u_0(t) = (H(1,u_0))(t) + v_0\varphi_0(t) \geq v_0\varphi_0(t)$$

for $t \in [0,1]$. Set

$$v^* = \sup\{v > 0 : u_0(t) \geq v\varphi_0(t), \forall \in [0,1]\}.$$

Then $v_0 \leq v^* < +\infty$ and $u_0(t) \geq v^*\varphi_0(t)$ for $t \in [0,1]$. From (14) we have that, for $t \in [0,1]$,

$$\begin{aligned} u_0(t) &= (H(1,u_0))(t) + v_0\varphi_0(t) \geq (L_3u_0)(t) + v_0\varphi_0(t) \\ &\geq v^*(L_3\varphi_0)(t) + v_0\varphi_0(t) = \lambda_1 v^*\varphi_0(t) + v_0\varphi_0(t). \end{aligned}$$

Since $\lambda_1 > 1$, we have $\lambda_1 v^* + v_0 > v^*$, which contradicts the definition of v^*.

(vi) From (24) and (25), it follows that $i(S, K \cap \partial\Omega_R, K) = 0$ and

$$i(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = -1.$$

Hence S has one fixed point, i.e., BVP (1) has one positive solution in K. ∎

Theorem 2 *Under hypotheses* (C_1)–(C_3)*, suppose that*

(F_4) there exist constants $a_1, b_1, c_1, d_1, C_0 \geq 0$ such that

$$g(t,x_1,x_2,x_3,x_4) \leq a_1x_1 + b_1x_2 - c_1x_3 - d_1x_4 + C_0, \tag{26}$$

for all $(t,x_1,x_2,x_3,x_4) \in [0,1] \times {\mathbb{R}_+}^2 \times {\mathbb{R}_-}^2$, moreover the spectral radius $r(L_1) < 1$;

(F_5) there exist constants $a_2, b_2, c_2, d_2 \geq 0$ and $r > 0$ such that

$$g(t,x_1,x_2,x_3,x_4) \geq a_2x_1 + b_2x_2 - c_2x_3 - d_2x_4, \tag{27}$$

for all $(t,x_1,x_2,x_3,x_4) \in [0,1] \times [0,r]^2 \times [-r,0]^2$, moreover the spectral radius $r(L_2) \geq 1$; where $L_i : C^3[0,1] \to C^3[0,1]$ $(i = 1,2)$ are defined by (10)

Then BVP (1) has one positive solution in K.

Proof. Let $W = \{u \in K : u = \mu Su,\ \mu \in [0,1]\}$, where S and K are respectively defined in (4) and (9).

We first assert that W is a bounded set. In fact, if $u \in W$, then $u = \mu Su$ for some $\mu \in [0,1]$. From (10) and (26) we have that

$$\begin{aligned} u(t) &= \mu(Su)(t) = \mu \int_0^1 k_S(t,s) g(s, u(s), u'(s), u''(s), u'''(s))\, \mathrm{d}s \\ &\leq \int_0^1 k_S(t,s)(a_1 u(s) + b_1 u'(s) - c_1 u''(s) - d_1 u'''(s) + C_0)\, \mathrm{d}s \\ &= (L_1 u)(t) + C_0 \int_0^1 k_S(t,s)\, \mathrm{d}s \end{aligned}$$

and

$$((I - L_1)u)(t) \leq C_0 \int_0^1 k_S(t,s)\, \mathrm{d}s =: v(t), t \in [0,1].$$

Obviously $v \in P$ and it is easy to see from (26) that, for $t \in [0,1]$,

$$u'(t) \leq (L_1 u)'(t) + v'(t), \quad u''(t) \geq (L_1 u)''(t) + v''(t), \quad u'''(t) \geq (L_1 u)'''(t) + v'''(t),$$

thus $(I - L_1)u \preceq v$. Because of the spectral radius $r(L_1) < 1$, we know that $I - L_1$ has a bounded inverse operator $(I - L_1)^{-1}$, which can be written as

$$(I - L_1)^{-1} = I + L_1 + L_1^2 + \cdots + L_1^2 + \cdots.$$

Since $L_1(P) \subset K \subset P$ by Lemma 8, we have $(I - L_1)^{-1}(P) \subset P$, which implies the inequality $u \preceq (I - L_1)^{-1} v$, Therefore, for $t \in [0,1]$,

$$0 \leq u(t) \leq ((I - L_1)^{-1} v)(t), \quad 0 \leq u'(t) \leq ((I - L_1)^{-1} v)'(t),$$

$$0 \geq u''(t) \geq ((I - L_1)^{-1} v)''(t), \quad 0 \geq u'''(t) \geq ((I - L_1)^{-1} v)'''(t),$$

which imply that $\|u\|_{C^3} \leq \|(I - L_1)^{-1} v\|_{C^3}$, i.e., W is bounded.

Now select $R > \max\{r, \sup W\}$, then $\mu Su \neq u$ for $u \in K \cap \partial\Omega_R$ and $\mu \in [0,1]$, and $i(S, K \cap \partial\Omega_R, K) = 1$ follows from Lemma 1.

Since $L_2 : P \to K \subset P$ and $r(L_2) \geq 1$, it follows from Lemma 3 that there exists $\varphi_0 \in P \backslash \{0\}$ such that $L_2 \varphi_0 = r(L_2)\varphi_0$. Furthermore, $\varphi_0 = r(L_2)^{-1} L_2 \varphi_0 \in K$.

Suppose that S has no fixed points in $K \cap \partial\Omega_r$, and we will show that $u - Su \neq v\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $v \geq 0$.

Otherwise, there exist $u_0 \in K \cap \partial\Omega_r$ and $v_0 \geq 0$ such that $u_0 - Su_0 = v_0 \varphi_0$, and it is clear that $v_0 > 0$. Since $u_0 \in K \cap \partial\Omega_r$, we have

$$0 \leq u_0(t), u_0'(t) \leq r, \quad -r \leq u_0''(t), \quad u_0'''(t) \leq 0, \quad \forall t \in [0,1].$$

It follows from (5), (10),and (27) that $\forall t \in [0,1]$,

$$(Su_0)(t) \geq (L_2 u_0)(t), \quad (Su_0)'(t) \geq (L_2 u_0)'(t)$$

and

$$(Su_0)''(t) \leq (L_2 u_0)''(t), \quad (Su_0)''(t) \leq (L_2 u_0)''(t),$$

which imply that

$$u_0 = v_0 \varphi_0 + Su_0 \succeq v_0 \varphi_0 + L_2 u_0 \succeq v_0 \varphi_0. \tag{28}$$

Set $v^* = \sup\{v > 0 : u_0 \succeq v\varphi_0\}$. Then $v_0 \leq v^* < \infty$ and $u_0 \succeq v^* \varphi_0$. Hence from (28) it follows that

$$u_0 \succeq v_0 \varphi_0 + L_2 u_0 \succeq v_0 \varphi_0 + v^* L_2 \varphi_0 = v_0 \varphi_0 + v^* r(L_2) \varphi_0.$$

However, $r(L_2) \geq 1$, so $u_0 \succeq (v_0 + v^*)\varphi_0$, which is a contradiction to the definition of v^*. Therefore $u - Su \neq v\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $v \geq 0$.

Therefore it follows from Lemma 2 that $i(S, K \cap \partial\Omega_r, K) = 0$.

Making use of the properties of fixed point index, we have that

$$i(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(S, K \cap \partial\Omega_R, K) - i(S, K \cap \partial\Omega_r, K) = 1,$$

and hence S has one fixed point in K. Therefore, BVP (1) has one positive solution in K. ■

4 Examples

In order to illustrate clearly with examples, we take the fourth-order boundary problem into account by undertaking mixed multi-point and integral boundary conditions with sign-changing coefficients:

$$\begin{cases} u^{(4)}(t) = g(t, u(t), u'(t), u''(t), u'''(t)),\ t \in [0,1], \\ u(0) = \alpha_1[u],\ u'(1) = \alpha_2[u],\ u''(0) + \alpha_3[u] = 0,\ u'''(1) + \alpha_4[u] = 0. \end{cases} \tag{29}$$

that is, $\alpha_1[u] = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{6}u(\frac{3}{4})$, $\alpha_2[u] = \int_0^1 u(t)(t - \frac{1}{4})\,\mathrm{d}t$, $\alpha_3[u] = u(\frac{1}{4}) - \frac{1}{2}u(\frac{1}{2})$, and $\alpha_4[u] = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{8}u(\frac{1}{2})$.We estimate some coefficients and do some calculations.

$$\begin{aligned} 0 \ &\leq\ \mathcal{K}_1(s) = \int_0^1 k_0(t,s)\,\mathrm{d}\mathcal{A}_1(t) = \frac{1}{2}k_0(\frac{1}{4}, s) - \frac{1}{6}k_0(\frac{3}{4}, s) \\ &= \begin{cases} \frac{1}{32}s - \frac{1}{18}s^3, & 0 \leq s \leq \frac{1}{4}, \\ \frac{3}{64}s - \frac{1}{16}s^2 + \frac{1}{36}s^3 - \frac{1}{768}, & \frac{1}{4} \leq s \leq \frac{3}{4}, \\ \frac{1}{96}, & \frac{3}{4} \leq s \leq 1, \end{cases} \end{aligned}$$

and hence $0 \leq \mathcal{K}_1(s) \leq 0.0105$;

$$\mathcal{K}_2(s) := \int_0^1 k_0(t,s)(t - \frac{1}{4})\,\mathrm{d}t = \frac{1}{8}s - \frac{1}{24}s^3 + \frac{1}{120}s^5 - \frac{1}{96}s^4 \geq 0 \ \ (0 \leq s \leq 1),$$

and hence $0 \leq \mathcal{K}_2(s) \leq 0.0813$;

$$\begin{aligned} 0 \ &\leq\ \mathcal{K}_3(s) = \int_0^1 k_0(t,s)\,\mathrm{d}\mathcal{A}_3(t) = k_0(\frac{1}{4}, s) - \frac{1}{2}k_0(\frac{1}{2}, s) \\ &= \begin{cases} \frac{1}{32}s - \frac{1}{12}s^3, & 0 \leq s \leq \frac{1}{4}, \\ \frac{1}{12}s^3 - \frac{1}{8}s^2 + \frac{1}{16}s - \frac{1}{384}, & \frac{1}{4} \leq s \leq \frac{1}{2}, \\ \frac{1}{128}, & \frac{1}{2} \leq s \leq 1, \end{cases} \end{aligned}$$

and hence $0 \leq \mathcal{K}_3(s) \leq 0.0079$;

$$\begin{aligned} 0 \ &\leq\ \mathcal{K}_4(s) = \int_0^1 k_0(t,s)\,\mathrm{d}\mathcal{A}_4(t) = \frac{1}{2}k_0(\frac{1}{4}, s) - \frac{1}{8}k_0(\frac{1}{2}, s) \\ &= \begin{cases} \frac{1}{16}s - \frac{1}{16}s^3, & 0 \leq s \leq \frac{1}{4}, \\ \frac{1}{48}s^3 - \frac{1}{16}s^2 + \frac{5}{64}s - \frac{1}{768}, & \frac{1}{4} \leq s \leq \frac{1}{2}, \\ \frac{1}{16}s - \frac{1}{32}s^2 + \frac{1}{768}, & \frac{1}{2} \leq s \leq 1, \end{cases} \end{aligned}$$

and hence $0 \leq \mathcal{K}_4(s) \leq 0.0248$. The 4×4 matrix

$$[A] = \begin{pmatrix} \alpha_1[\delta_1] & \alpha_1[\delta_2] & \alpha_1[\delta_3] & \alpha_1[\delta_4] \\ \alpha_2[\delta_1] & \alpha_2[\delta_2] & \alpha_2[\delta_3] & \alpha_2[\delta_4] \\ \alpha_3[\delta_1] & \alpha_3[\delta_2] & \alpha_3[\delta_3] & \alpha_3[\delta_4] \\ \alpha_4[\delta_1] & \alpha_4[\delta_2] & \alpha_4[\delta_3] & \alpha_4[\delta_4] \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{32} & \frac{1}{96} \\ \frac{1}{4} & \frac{5}{24} & \frac{1}{8} & \frac{13}{160} \\ \frac{1}{2} & 0 & \frac{1}{32} & \frac{1}{128} \\ \frac{3}{8} & \frac{1}{16} & \frac{1}{16} & \frac{25}{768} \end{pmatrix},$$

and its spectral radius $r([A]) = 0.3961 < 1$. These mean that (C_2) and (C_3) are satisfied.

Now we take into account the constants in Theorem 1 and Theorem 2.

$$(I-[A])^{-1} < \begin{pmatrix} 1.5487 & 0.0014 & 0.0513 & 0.0173 \\ 0.6877 & 1.2724 & 0.1939 & 0.1159 \\ 0.8050 & 0.0014 & 1.0596 & 0.0174 \\ 0.6967 & 0.0829 & 0.1009 & 1.0490 \end{pmatrix}$$

and

$$(I-[A])^{-1}\mathcal{K}(s) < \begin{pmatrix} 0.0172 \\ 0.1151 \\ 0.0174 \\ 0.0409 \end{pmatrix},$$

thus

$$k_S(t,s) < 0.0172 + 0.1151t + 0.0174 \times (t - \frac{1}{2}t^2) + 0.0409 \times (\frac{1}{2}t - \frac{1}{6}t^3) + k_0(t,s) < 0.4880.$$

So, for $u \in C^3[0,1]$ and $t \in [0,1]$,

$$\begin{aligned} |(L_i u)(t)| &\le 0.4880 \int_0^1 (a_i|u(s)| + b_i|u'(s)| + c_i|u''(s)| + d_i|u'''(s)|)\,\mathrm{d}s \\ &\le 0.4880(a_i + b_i + c_i + d_i)\|u\|_{C^3} \ (i = 1, 2), \end{aligned}$$

where L_i $(i = 1, 2)$ are defined in (10). Since all the terms are nonnegative in the first derivative of $k_S(t,s)$ with respect to t, we also have

$$\frac{\partial k_S(t,s)}{\partial t} < 0.1151 + 0.0174(1-t) + 0.0409(\frac{1}{2} - \frac{1}{2}t^2) + \frac{\partial k_0(t,s)}{\partial t}$$

and

$$\left|\frac{\partial^2 k_S(t,s)}{\partial t^2}\right| < 0.0174 + 0.0409t + \left|\frac{\partial^2 k_0(t,s)}{\partial t^2}\right|$$

for $u \in C^3[0,1]$ and $t \in [0,1]$,

$$\begin{aligned} |(L_i u)'(t)| &\le 1.1151 \int_0^1 (a_i|u(s)| + b_i|u'(s)| + c_i|u''(s)| + d_i|u'''(s)|)\,\mathrm{d}s \\ &\le 1.1151(a_i + b_i + c_i + d_i)\|u\|_{C^3} \ (i = 1, 2), \end{aligned}$$

$$\begin{aligned} |(L_i u)''(t)| &\le 1.0583 \int_0^1 (a_i|u(s)| + b_i|u'(s)| + c_i|u''(s)| + d_i|u'''(s)|)\,\mathrm{d}s \\ &\le 1.0583(a_i + b_i + c_i + d_i)\|u\|_{C^3} \ (i = 1, 2), \end{aligned}$$

$$\begin{aligned} |(L_i u)'''(t)| &\le \int_0^1 (a_i|u(s)| + b_i|u'(s)| + c_i|u''(s)| + d_i|u'''(s)|)\,\mathrm{d}s \\ &\le (a_i + b_i + c_i + d_i)\|u\|_{C^3} \ (i = 1, 2). \end{aligned}$$

Therefore the radius $r(L_i) \le \|L_i\| \le 1.1151(a_i + b_i + c_i + d_i) < 1$ if

$$a_i + b_i + c_i + d_i < 1.1151^{-1} \ (i = 1, 2). \tag{30}$$

On the other hand, we have from Lemmas 7 and 8 that, for $u \in K \setminus \{0\}$ and $t \in [0,1]$,

$$\begin{aligned}(L_2u)(t) &\geq \int_0^1 k_S(t,s)a_2u(s)\,\mathrm{d}s \geq a_2c_0(t)\int_0^1 \Psi_0(s)u(s)\,\mathrm{d}s \\ &\geq a_2c_0(t)\int_0^1 \Psi_0(s)c_0(s)\|u\|_C\,\mathrm{d}s = a_2c_0(t)\|u\|_C\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s\end{aligned}$$

and

$$\|(L_2u)\|_C = (L_2u)(1) \geq \frac{1}{3}a_2\|u\|_C\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s$$

hence

$$\begin{aligned}(L_2^2u)(t) &\geq a_2\int_0^1 k_S(t,s)(L_2u)(s)\,\mathrm{d}s \\ &\geq a_2c_0(t)\int_0^1 \Psi_0(s)(L_2u)(s)\,\mathrm{d}s \geq a_2c_0(t)\int_0^1 \Psi_0(s)c_0(s)\|(L_2u)\|_C\,\mathrm{d}s \\ &\geq \frac{1}{3}a_2^2c_0(t)\|u\|_C(\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s)^2\end{aligned}$$

and

$$\|(L_2^2u)\|_C = (L_2^2u)(1) \geq \frac{1}{9}a_2^2\|u\|_C(\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s)^2.$$

By induction,

$$\|(L_2^nu)\|_C = (L_2^nu)(1) \geq (\frac{a_2}{3})^n\|u\|_C(\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s)^n.$$

As a result, it follows that, for $u \in K \setminus \{0\}$,

$$\|L_2^n\|\|u\|_{C^3} \geq \|L_2^nu\|_{C^3} \geq \|L_2^nu\|_C \geq (\frac{a_2}{3})^n\|u\|_C(\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s)^n$$

and according to Gelfand's formula, the spectral radius

$$\begin{aligned}r(L_2) &= \lim_{n\to\infty}\|L_2^n\|^{\frac{1}{n}} \\ &\geq \frac{a_2}{3}(\int_0^1 \Psi_0(s)c_0(s)\,\mathrm{d}s)\lim_{n\to\infty}(\frac{\|u\|_C}{\|u\|_{C^3}})^{\frac{1}{n}} \\ &= \frac{a_2}{3}(\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s)\end{aligned}$$

which implies that $r(L_2) \geq 1$ when

$$a_2 \geq \frac{945}{17} = \frac{3}{\int_0^1(\frac{1}{2}s - \frac{1}{6}s^3)\times\frac{1}{6}s(3-s^2)\,\mathrm{d}s} \geq \frac{3}{\int_0^1 c_0(s)\Psi_0(s)\,\mathrm{d}s}. \tag{31}$$

Example 1 *If*

$$g(t,x_1,x_2,x_3,x_4) = \frac{\frac{1}{2}x_1^2 + \frac{1}{8}x_2^2 + \frac{1}{8}x_3^2 + \frac{1}{9}x_4^2}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

then BVP (29) has a positive solution.

Proof. Take $a_2 = \frac{1}{2}$, $b_2 = c_2 = \frac{1}{8}$, $d_2 = \frac{1}{9}$, $r < 1$, it is easy to check that (12) and (30) for $i = 2$ are satisfied. Now take $a_1 = 56$, $b_1 = 22$, $c_1 = 4$, it is clear that

$$\frac{a_1}{3}\int_0^1 (\frac{1}{2}s - \frac{1}{6}s^3)\Psi_0(s)\,\mathrm{ds} > \frac{a_1}{3}\int_0^1 (\frac{1}{2}s - \frac{1}{6}s^3)\frac{1}{6}s(3-s^2)\,\mathrm{ds} > 1,$$

$$\frac{b_1}{2}\int_0^1 (\frac{1}{2} - \frac{1}{2}s^2)\Psi_1(s)\,\mathrm{ds} > \frac{b_1}{2}\int_0^1 (\frac{1}{2} - \frac{1}{2}s^2)\frac{1}{2}s(2-s)\,\mathrm{ds} > 1,$$

$$c_1\int_0^1 s\Psi_2(s)\,\mathrm{ds} > c_1\int_0^1 s^2\,\mathrm{ds} > 1.$$

so (13) is valid. It can be seen that (14) is satisfied for C_0 large enough. Let $H_M(\rho) = M^2 + \rho^2$ for (F_3). Then BVP (29) has a positive solution by Theorem 1. ■

Example 2 *If $g(t, x_1, x_2, x_3, x_4) = \sqrt[3]{x_1} - \sqrt[3]{x_3}$, then BVP (29) has a positive solution.*

Proof. Take $a_1 = \frac{1}{6}$, $c_1 = \frac{1}{4}$, $b_1 = d_1 = 0$, $C_0 = 2$ and $a_2 = 56$, $c_2 = 1$, $b_2 = d_2 = 0$, $r = \frac{1}{2500}$. Obviously, (30) and (31) are satisfied, meanwhile conditions (26) and (27) are fulfilled. Then BVP (29) has a positive solution by Theorem 2. ■

Acknowledgment. The authors express their sincere gratitude to the referees for their valuable comments and suggestions that improved the original manuscript.

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