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The Average Directional Distance To The Boundary Of A Ball Or Disk*

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Abstract

For a point P in the interior of the unit disk and a given direction, one can compute the directional distance from P to the boundary of the disk in the prescribed direction. We compute the average directional distance over all points and all directions. The analogous computation is then carried out for any n-dimensional ball. The average distance depends on the dimension n and produces a sequence which is shown to be a scale of the Wallis integral sequence. This work was motivated by research on solar energy collection devices.

1 Introduction

1.1 The Average Directional Distance in a Unit Disk

Consider B^2 the unit disk $x^2+y^2 \leq 1$. Let P be any point in the interior of the disk. Define $d(P,\phi)$ to be the distance from P to the boundary of the disk in the direction of ϕ, where the ϕ is a directed angle measured from P with $\phi = 0$ corresponding to the ray starting at P parallel to the positive x-axis. See Figure 1 (Left).

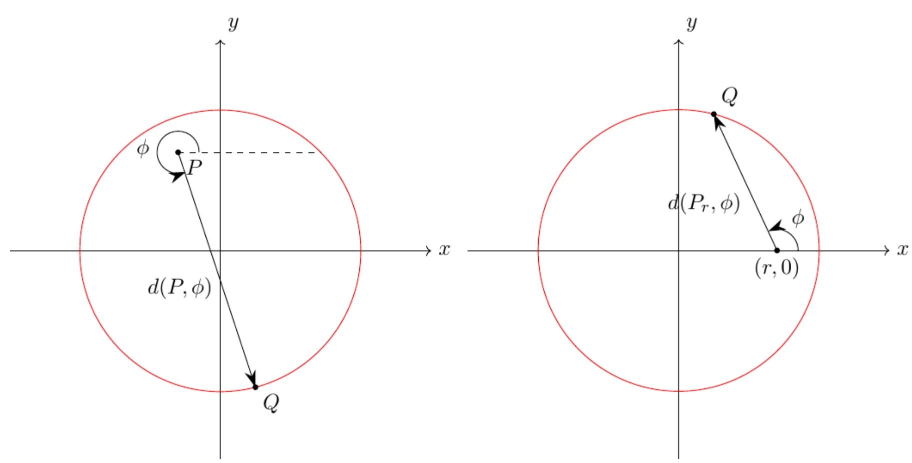

Figure 1: $d(P,\phi)$ for a given P and ϕ.

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The average directional distance to the boundary at P is

$$\bar{d}(P) = \frac{1}{2\pi}\int_0^{2\pi} d(P,\phi)\ d\phi.$$

By symmetry, the function $\bar{d}(P)$ only depends upon distance from P to the origin. We then average $\bar{d}(P)$ over all points P in the disk (with the uniform distribution) to obtain the average directional distance to the boundary

$$\bar{\bar{d}}_2 = \frac{1}{Area(B^2)}\iint_{B^2} \bar{d}(P)\, dA.$$

In this section, we prove the following Theorem.

Theorem 1 *The average directional distance for the unit disk is* $\bar{\bar{d}}_2 = \frac{8}{3\pi}$.

Proof. We start with a point on the positive x-axis of the form $P_r = (r, 0)$ with $0 \le r < 1$ and create a ray starting at P_r and directed angle ϕ, where $\phi = 0$ corresponds to the positive x-axis. Let $Q = (x, y)$ be the intersection of this ray and the unit circle and set

$$D = d(P_r, \phi) = dist(P_r, Q).$$

See Figure 1 (Right). The x-coordinate of Q is $x = r + D\cos\phi$ and $D = \sqrt{(x-r)^2 + y^2}$. Since $x^2 + y^2 = 1$, we have

$$D^2 = 1 - 2xr + r^2 = 1 - 2(r + D\cos\phi)r + r^2,$$

which is a quadratic in D having positive solution

$$d(P_r, \phi) = -r\cos\phi + \sqrt{r^2\cos^2\phi - r^2 + 1}.$$

Therefore,

$$\bar{d}(P_r) = \frac{\int_0^{2\pi}\left(-r\cos\phi + \sqrt{1 - r^2\sin^2\phi}\right)\ d\phi}{2\pi} = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{1 - r^2\sin^2\phi}\ d\phi,$$

which clearly depends only on r.

To obtain $\bar{\bar{d}}_2$ one needs to vary $\bar{d}(P)$ over all P. But $\bar{d}(P) = \bar{d}(P_r)$, where r is the the distance from P to the origin. So the average $\bar{\bar{d}}_2$ over all points $P \in B^2$ is given by a polar integral where $dA = r\ dr\ d\theta$:

$$\begin{aligned}
\bar{\bar{d}}_2 &= \frac{1}{Area(B^2)}\iint_{B^2}\bar{d}(P)\, dA \\
&= \frac{\int_0^{2\pi}\int_0^1 \frac{2}{\pi}\int_0^{\pi/2}\sqrt{1 - r^2\sin^2\phi}\ d\phi\ r\ dr\ d\theta}{\int_0^{2\pi}\int_0^1 r\ dr\ d\theta} \\
&= \frac{4}{\pi}\int_0^1\int_0^{\pi/2} r\sqrt{1 - r^2\sin^2\phi}\ d\phi\ \ dr.
\end{aligned}$$

This resolves to $\frac{8}{3\pi}$. ∎

We note from the computation above that if we were to allow $r = 1$, $\bar{d}(P_1)$ is not zero, but $\frac{2}{\pi}$. This makes sense since, as r is near 1, it can be shown that $\bar{d}(P_r)$ is bounded away from zero.

2 The Average Distance in an n-Dimensional Unit Ball

Consider B^n the unit ball $x_1^2 + x_2^2 + ... + x_n^2 \le 1$. Let P be any point in the interior of the unit ball. Define $d(P, \mathbf{u})$ to be the distance from P to the boundary of the ball in the direction of $\mathbf{u}$, where $\mathbf{u}$ is a unit vector (which can be identified with a point on the sphere S^{n-1} or $x_1^2 + x_2^2 + ... + x_n^2 = 1$).

Define the average directional distance over all $\mathbf{u}$ to be the surface integral

$$\bar{d}(P) = \frac{1}{SurfaceArea(S^{n-1})} \int \cdots \int_{S^{n-1}} d(P, \mathbf{u})\ dS,$$

where dS is the generalized surface element from S^{n-1} and $SurfaceArea(S^{n-1})$ is the $n-1$ dimensional surface area of the $n-1$ dimensional sphere.

Finally, define the average distance over all points P in B^n to be

$$\bar{\bar{d}}_n = \frac{1}{Volume(B^n)} \int \cdots \int_{B_n} \bar{d}(P)\ dV.$$

In this section, we shall prove the following Theorem.

Theorem 2 *The average directional distance for the n dimensional unit ball is*

$$\bar{\bar{d}}_n = \frac{4}{\pi} \int_0^{\pi/2} \sin^{n+1}(t)\, dt.$$

Proof. For dimension $n = 1$, B^1 is simply the interval $[-1, 1]$ and for any point x inside $(-1, 1)$ the two distances to the boundary are $1 - x$ and $1 + x$. So, for all x, the average directional distance is 1. This implies that $\bar{\bar{d}}_1 = 1$, which agrees with

$$\frac{4}{\pi} \int_0^{\pi/2} \sin^2(t)\, dt.$$

For dimension $n = 2$,

$$\frac{4}{\pi} \int_0^{\pi/2} \sin^3(t)\, dt = \frac{8}{3\pi}.$$

For dimension $n \geq 3$ we will use hyperspherical coordinates (see [1]). Recall that for $(x_1, ..., x_n) \in \mathbb{R}^n$, the hyperspherical coordinates are $(\rho, \phi_1, \ldots, \phi_{n-1})$, where $\rho = \sqrt{\sum_{i=1}^n x_i^2}$ and $x_1 = \rho\cos(\phi_1)$, $x_k = \rho\cos(\phi_k)\prod_{i=1}^{k-1}\sin(\phi_i)$, for $1 \leq k \leq n-1$ and $x_n = \rho\prod_{i=1}^{n-1}\sin(\phi_i)$ where $0 \leq \phi_{n-1} < 2\pi$, and $0 \leq \phi_j \leq \pi$ for $1 \leq j \leq n-2$. Also recall that the volume element in hyperspherical coordinates is

$$dV = \rho^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\cdots\sin(\phi_{n-2})d\rho\, d\phi_1\ \cdots\ d\phi_{n-1},$$

and, for integrating on S^{n-1}, the surface element is

$$dS = \sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\cdots\sin(\phi_{n-2})d\phi_1\ \cdots\ d\phi_{n-1}.$$

As in the planar case, the function $\bar{d}(P)$ only depends upon ρ, the distance of P from the origin. So we will compute the distance for the conveniently located point $P_\rho = (\rho, 0, \cdots, 0)$. To specify a direction $\mathbf{v}$ (from P) amounts to identifying a point on the $n-1$ dimensional unit sphere. Such a direction then is given in hyperspherical coordinates by

$$(\cos(\psi_1),\ \sin(\psi_1)\cos(\psi_2),\ ...,\ \sin(\psi_1)\cdots\sin(\psi_{n-2})\cos(\psi_{n-1}),\ \sin(\psi_1)\cdots\sin(\psi_{n-2})\sin(\psi_{n-1})).$$

Taking $\psi_i = 0$ for all $i \geq 2$, then $\mathbf{v}$ is in the x_1-x_2 plane, and the picture is completely analogous to Figure 1 (Right) where ρ is r and ϕ is ψ_1. Therefore,

$$d(P_\rho, \mathbf{v}) = -\rho\cos(\psi_1) + \sqrt{1 - \rho^2\sin^2(\psi_1)}.$$

Now, by symmetry, all directions that make angle ψ_1 with the positive x_1-axis will have the same distance to the boundary. That is, the distance $d(P_\rho, \mathbf{u})$ (for any direction $\mathbf{u}$ from P_ρ) depends only on ψ_1 and ρ. So

$$\begin{aligned} & \bar{d}(P_\rho) \\ = \; & \frac{1}{SurfaceArea(S^{n-1})} \int \cdots \int_{S^{n-1}} d(P_\rho, \mathbf{u})\, dS \\ = \; & \frac{\int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(-\rho\cos(\psi_1) + \sqrt{1-\rho^2 \sin^2(\psi_1)} \right) \sin^{n-2}(\psi_1)\, \sin^{n-3}(\psi_2) \cdots \sin(\psi_{n-2})\; d\psi_1 \;\cdots\; d\psi_{n-1}}{\int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \sin^{n-2}(\psi_1) \sin^{n-3}(\psi_2) \cdots \sin(\psi_{n-2})\; d\psi_1 \cdots d\psi_{n-1}}, \end{aligned}$$

which simplifies to

$$\bar{d}(P_\rho) = \frac{\int_0^{\pi} \sqrt{1-\rho^2\sin^2(\psi_1)}\, \sin^{n-2}(\psi_1) d\psi_1}{\int_0^{\pi} \sin^{n-2}(\psi_1)\; d\psi_1}.$$

This formula depends only upon ρ. Any other point P with distance ρ to the origin will also have $\bar{d}(P) = \bar{d}(P_\rho)$ by symmetry.

Next, we average $\bar{d}(P)$ over all P in B^n.

$$\begin{aligned} \bar{\bar{d}}_n \; &= \; \frac{1}{Volume(B^n)} \int \cdots \int_{B_n} \bar{d}(P)\; dV \\ &= \; \frac{\int_0^1 \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(\bar{d}(P)\right) \rho^{n-1} \sin^{n-2}(\phi_1)\; \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2})\; d\phi_1 \;\cdots\; d\phi_{n-1} d\rho}{\int_0^1 \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \rho^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2})\; d\phi_1 \;\cdots\; d\phi_{n-1}\; d\rho}. \end{aligned}$$

Noting that $\bar{d}(P) = \bar{d}(P_\rho)$ and after many cancellations, this reduces to

$$\frac{\int_0^1 \int_0^{\pi} \rho^{n-1} \sqrt{1-\rho^2\sin^2(\psi_1)}\, \sin^{n-2}(\psi_1)\; d\psi_1\; d\rho}{\left(\int_0^1 \rho^{n-1}\, d\rho\right) \left(\int_0^{\pi} \sin^{n-2}(\psi_1)\, d\psi_1\right)} = \frac{\int_0^1 \int_0^{\pi} \rho^{n-1} \sqrt{1-\rho^2\sin^2(\psi_1)}\, \sin^{n-2}(\psi_1)\; d\psi_1\; d\rho}{\frac{1}{n} \left(\int_0^{\pi} \sin^{n-2}(\psi_1)\, d\psi_1\right)}.$$

We now compute the numerator, which by symmetry is:

$$2 \int_0^1 \int_0^{\frac{\pi}{2}} \rho^{n-1} \sqrt{1-\rho^2\sin^2(\psi_1)}\, \sin^{n-2}(\psi_1)\; d\psi_1\; d\rho.$$

First, set $\rho\sin(\psi_1) = \sin(t)$, then $\rho\cos(\psi_1)\; d\psi_1 = \cos(t)\; dt$. This results in

$$\begin{aligned} & 2 \int_0^1 \int_{t=0}^{t=\arcsin(\rho)} \rho^{n-1} \cos(t) \left(\frac{\sin(t)}{\rho} \right)^{n-2} \frac{\cos(t)}{\sqrt{\rho^2 - \sin^2(t)}}\; dt\; d\rho \\ = \; & 2 \int_0^1 \int_{t=0}^{t=\arcsin(\rho)} \rho \sin^{n-2}(t)\; \frac{\cos^2(t)}{\sqrt{\rho^2 - \sin^2(t)}}\; dt\; d\rho \\ = \; & 2 \int_0^{\frac{\pi}{2}} \int_{\sin(t)}^1 \rho\cos(t)\sin^{n-2}(t)\; \frac{\cos(t)}{\sqrt{\rho^2 - \sin^2(t)}} d\rho\; dt \quad \text{(Switching the order of integration)} \\ = \; & 2 \int_0^{\frac{\pi}{2}} \sin^{n-2}(t)\; \cos^2(t) \sqrt{\rho^2 - \sin^2(t)}\; \Big|_{\sin(t)}^{1}\; dt \quad \text{(resolves)} \\ = \; & 2 \int_0^{\frac{\pi}{2}} \sin^{n-2}(t)\; \cos^3(t)\; dt = \frac{4}{n^2-1}. \end{aligned}$$

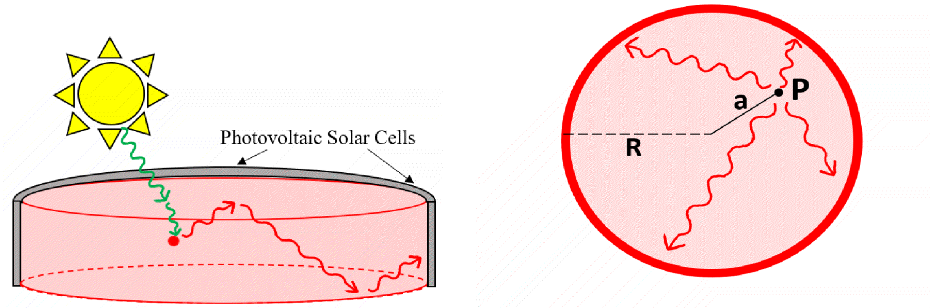

Figure 2: (Left) An LSC with exaggerated thickness. (Right) Flourescence translated to the solar cells at the boundary.

Therefore, it can be shown that
$$\bar{\bar{d}}_n = 2\frac{n}{(n+1)(n-1)W_{n-2}},$$
where W_{n-2} is the Wallis integral $\int_0^{\frac{\pi}{2}} \sin^{n-2}(\psi_1)\, d\psi_1$ for $n \geq 3$.

It is well known that the Wallis integrals have the property $W_n = \left(\frac{n-1}{n}\right) W_{n-2}$ for $n \geq 2$, therefore $W_{n+3} = \left(\frac{n+2}{n+3}\right) W_{n+1}$ for $n \geq 0$. So for $n \geq 2$,

$$\bar{\bar{d}}_{n+2} = 2\frac{n+2}{(n+3)(n+1)W_n} = 2\frac{n+2}{(n+3)(n+1)\frac{(n-1)}{n}W_{n-2}} = \frac{(n+2)}{(n+3)}\bar{\bar{d}}_n.$$

This implies that the sequences W_{n+1} and $\bar{\bar{d}}_n$ have the same recursive relation. Since $\bar{\bar{d}}_1 = 1 = \frac{4}{\pi}W_2$ and $\bar{\bar{d}}_2 = \frac{8}{3\pi} = \frac{4}{\pi}W_3$. We must have $\bar{\bar{d}}_n = \frac{4}{\pi}W_{n+1}$ for all $n \geq 1$, which proves the theorem. ■

Below is a table showing the values of $\bar{\bar{d}}_n$ for $n = 1, \ldots, 5$.

n	$\bar{\bar{d}}_n$	Decimal/Approximation
1	1	1
2	$8/(3\pi)$	0.8488263632
3	$3/4$	0.75
4	$32/(15\pi)$	0.6790610905
5	$5/8$	0.625

Note that as the dimension n increases, $\bar{\bar{d}}_n$ decreases. In fact,
$$\lim_{n\to\infty} \bar{\bar{d}}_n = 0.$$

These results can be generalized to any n-ball of radius R, in particular, the average directional distance to the boundary of an n-dimensional ball with radius R is
$$\bar{\bar{d}}_n^R = R \cdot \bar{\bar{d}}_n.$$

3 Motivation, Application, and Open Problems

This article arose from an undergraduate physics research project that involved modeling luminescent solar concentrators (LSCs) used in solar panels.

An LSC is a solar energy device, often in the shape of a disk (see Figure 2), that absorbs sunlight and channels the resulting fluorescence to its edge where it is then converted into electricity by photovoltaic solar cells [2]. The efficiency of the LSC depends on the average distance the fluorescence must travel from any point inside the LSC to any point on its edge. Thus, measuring the average distance from any point inside the panel to any point on the panel's edge is critical in the effort to understand energy loss. An approximated average distance for a disk appears in [4] using a Monte Carlo method, which closely approximates our exact result. We also generated Monte Carlo simulations which well-approximated the theoretical findings for dimensions $n = 2$ through $n = 5$. Finally, we anticipate that these results may play a role in wave propogation, acoustics, or resonance in disks or spheres.

The authors have solved the problem below, which is relevant to LSCs of differing shapes.

Problem 1 *Compute the average directional distance to the boundary for the following regular polygons: an equilateral triangle, a square, and a regular hexagon.*

However, the following problems remain open:

Problem 2 *Calculate the average directional distance to the boundary for any triangle and other non-regular planar figures.*

Problem 3 *Calculate the average directional distance to the boundary for a cube or for the other Platonic solids.*

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