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Existence Of Positive Periodic Solution Of Third-Order Neutral Differential Equations With Distributed Deviating Arguments*

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Abstract

This study focuses on deriving sufficient conditions for the existence of positive ω-periodic solutions in two types of third-order neutral differential equations with distributed deviating arguments. To establish our findings, we employ the Krasnoselskii fixed point theorem. The derived results are further supported by an illustrative example, demonstrating the practical applicability of the obtained theoretical results.

1 Introduction

Neutral differential equations exhibit variations in various fields, including Mechanics, Biology, and Physics [9, 10, 14]. The investigation of positive periodic solutions in first and second-order neutral differential equations has been extensively explored by numerous authors, as evidenced by references such as [1, 3, 4, 5, 6, 7, 8, 11, 12, 13, 16] and related works.

In recent research studies, the authors in [2] and [15] focused on studying the existence of positive periodic solutions in two distinct classes of third-order neutral differential equations. These equations take the following forms, respectively:

$$[x(t) - g(t, x(t - \tau(t)))]''' = a(t)x(t) - f(t, x(t - \tau(t)))$$

and

$$[x(t) - g(t, x(t - \tau(t)))]''' = -a(t)x(t) + f(t, x(t - \tau(t))),$$

where $a, \tau \in C(\mathbb{R}, (0, \infty))$, $g \in C(\mathbb{R} \times [0, \infty), \mathbb{R})$, $f \in C(\mathbb{R} \times [0, \infty), [0, \infty))$, and $a(t)$, $\tau(t)$, $g(t, x)$, $f(t, x)$ are ω-periodic in t, where ω is a positive constant and

$$[x(t) - cx(t - \tau(t))]''' + a(t)x(t) = f(t, x(t - \tau(t)))$$

and

$$[x(t) - cx(t - \tau(t))]''' - a(t)x(t) = f(t, x(t - \tau(t))),$$

where $\tau \in C(\mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}, (0, \infty))$, $f \in C(\mathbb{R} \times [0, \infty), [0, \infty))$, and $a(t), \tau(t), f(t, x)$ are ω-periodic in t, and ω and c are constants with $|c| < 1$.

Building upon the motivation and inspiration provided by these prior works, we address the problem of establishing the existence of positive ω-periodic solutions in the following two classes of third-order neutral differential equations:

$$\left[x(t) - p_1(t)\int_{a_1}^{b_1} x(t - \mu(t, \zeta))d\zeta\right]''' = p_2(t)x(t) - \int_{a_1}^{b_1} f(t, x(t - \mu(t, \zeta)))d\zeta \tag{1}$$

and

$$\left[x(t) - p_1(t)\int_{a_1}^{b_1} x(t - \mu(t, \zeta))d\zeta\right]''' = -p_2(t)x(t) + \int_{a_1}^{b_1} f(t, x(t - \mu(t, \zeta)))d\zeta, \tag{2}$$

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where $p_1 \in C(\mathbb{R},\mathbb{R})$ and $p_2 \in C(\mathbb{R},(0,\infty))$ are ω-periodic functions, $\mu \in C(\mathbb{R}\times[a_1,b_1],\mathbb{R})$ $(b_1 > a_1 \geqslant 0)$ and $f \in C(\mathbb{R}\times(0,\infty),(0,\infty))$ are ω- periodic function in t, where ω is a positive constant.

This article presents the sufficient conditions for the existence of positive ω-periodic solutions to (1) and (2) by utilizing the Krasnoselskii fixed point theorem. The remaining sections of this paper are structured as follows: Section 2 introduces the necessary notations and presents relevant Lemmas from [15]. Section 3 provides the proofs for the criteria regarding the existence of positive ω-periodic solutions to (1) and (2), accompanied by an illustrative example.

2 Preliminaries

Let

$$\Phi_\omega = \{x(t) : x(t) \in C(\mathbb{R},\mathbb{R}), \quad x(t+\omega) = x(t), \quad t \in \mathbb{R}\}$$

with the supremum norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. It is clear that $(\Phi_\omega, \|\cdot\|)$ is a Banach space. Define

$$C_\omega^+ = \{x(t) : x(t) \in C(\mathbb{R},(0,\infty)), \quad x(t+\omega) = x(t)\}$$

and

$$C_\omega^- = \{x(t) : x(t) \in C(\mathbb{R},(-\infty,0)), \quad x(t+\omega) = x(t)\}.$$

Let

$$m = \min\{p_2(t) : t \in [0,\omega]\}, \quad M = \max\{p_2(t) : t \in [0,\omega]\} \quad \text{and} \quad \beta = \sqrt[3]{M}.$$

Lemma 1 ([15]) *The equation*

$$y'''(t) - My(t) = h(t), \quad h \in C_\omega^-$$

has a unique ω-periodic solution

$$y(t) = \int_0^\omega G_1(t,s)(-h(s))ds,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{2\exp\left(\frac{\beta(s-t)}{2}\right)\left[\sin\left(\frac{\sqrt{3}\beta(t-s)}{2}+\frac{\pi}{6}\right)-\exp\left(-\frac{\beta\omega}{2}\right)\sin\left(\frac{\sqrt{3}\beta(t-s-\omega)}{2}+\frac{\pi}{6}\right)\right]}{3\beta^2\left[1+\exp(-\beta\omega)-2\exp\left(-\frac{\beta\omega}{2}\right)\cos\left(\frac{\sqrt{3}\beta\omega}{2}\right)\right]} + \dfrac{\exp\left(\beta(t-s)\right)}{3\beta^2(\exp(\beta\omega)-1)}, \\ \quad \text{for} \quad 0 \leqslant s \leqslant t \leqslant \omega \\ \dfrac{2\exp\left(\frac{\beta(s-t-\omega)}{2}\right)\left[\sin\left(\frac{\sqrt{3}\beta(t-s+\omega)}{2}+\frac{\pi}{6}\right)-\exp\left(-\frac{\beta\omega}{2}\right)\sin\left(\frac{\sqrt{3}\beta(t-s)}{2}+\frac{\pi}{6}\right)\right]}{3\beta^2\left[1+\exp(-\beta\omega)-2\exp\left(-\frac{\beta\omega}{2}\right)\cos\left(\frac{\sqrt{3}\beta\omega}{2}\right)\right]} + \dfrac{\exp\left(\beta(t+\omega-s)\right)}{3\beta^2(\exp(\beta\omega)-1)}, \\ \quad \text{for} \quad 0 \leqslant t \leqslant s \leqslant \omega. \end{cases}$$

Lemma 2 ([15]) $\int_0^\omega G_1(t,s)ds = \frac{1}{M}$ *and if* $\sqrt{3}\beta\omega < 4\pi/3$ *holds, then* $0 < A < G_1(t,s) \leq B$ *for all* $t \in [0,\omega]$ *and* $s \in [0,\omega]$.

Lemma 3 ([15]) *The equation*

$$y'''(t) - p_2(t)y(t) = h(t), \quad h \in C_\omega^-$$

has a unique positive ω-periodic solution

$$y(t) = (I - S_1B_1)^{-1}(S_1h)(t),$$

where $S_1, B_1 : \Phi_\omega \to \Phi_\omega$ *defined such that*

$$(S_1h)(t) = \int_0^\omega G_1(t,s)(-h(s))ds, \quad (B_1y)(t) = [-M + p_2(t)]y(t).$$

Define $T_1 : \Phi_\omega \to \Phi_\omega$ *by*

$$(T_1h)(t) = (I - S_1B_1)^{-1}(S_1h)(t).$$

If $\sqrt{3}\beta\omega < 4\pi/3$, *then* S_1 *and* B_1 *are completely continuous and therefore* T_1 *is completely continuous and satisfies*

$$0 < (S_1h)(t) \leqslant (T_1h)(t) \leqslant \frac{M}{m}\|S_1h\|, \quad h \in C_\omega^-.$$

Lemma 4 ([15]) *The equation*

$$y'''(t) + My(t) = h(t), \quad h \in C_\omega^+$$

has a unique ω*-periodic solution*

$$y(t) = \int_0^\omega G_2(t,s)h(s)ds,$$

where

$$G_2(t,s) = \begin{cases} \dfrac{2\exp\left(\frac{\beta(t-s)}{2}\right)\left[\sin\left(\frac{\sqrt{3}\beta(t-s)}{2} - \frac{\pi}{6}\right) - \exp\left(\frac{\beta\omega}{2}\right)\sin\left(\frac{\sqrt{3}\beta(t-s-\omega)}{2} - \frac{\pi}{6}\right)\right]}{3\beta^2\left[1+\exp(\beta\omega) - 2\exp\left(\frac{\beta\omega}{2}\right)\cos\left(\frac{\sqrt{3}\beta\omega}{2}\right)\right]} + \dfrac{\exp\left(\beta(s-t)\right)}{3\beta^2(1-\exp(-\beta\omega))}, \\ \textit{for} \quad 0 \leqslant s \leqslant t \leqslant \omega \\ \\ \dfrac{2\exp\left(\frac{\beta(t+\omega-s)}{2}\right)\left[\sin\left(\frac{\sqrt{3}\beta(t+\omega-s)}{2} - \frac{\pi}{6}\right) - \exp\left(\frac{\beta\omega}{2}\right)\sin\left(\frac{\sqrt{3}\beta(t-s)}{2} - \frac{\pi}{6}\right)\right]}{3\beta^2\left[1+\exp(\beta\omega) - 2\exp\left(\frac{\beta\omega}{2}\right)\cos\left(\frac{\sqrt{3}\beta\omega}{2}\right)\right]} + \dfrac{\exp\left(\beta(s-t-\omega)\right)}{3\beta^2(1-\exp(-\beta\omega))}, \\ \textit{for} \quad 0 \leqslant t \leqslant s \leqslant \omega. \end{cases}$$

Lemma 5 ([15]) $\int_0^\omega G_2(t,s)ds = \frac{1}{M}$ *and if* $\sqrt{3}\beta\omega < 4\pi/3$ *holds, then* $0 < A < G_2(t,s) \leq B$ *for all* $t \in [0,\omega]$ *and* $s \in [0,\omega]$.

Lemma 6 ([15]) *The equation*

$$y'''(t) + p_2(t)y(t) = h(t), \quad h \in C_\omega^+$$

has a unique positive ω*-periodic solution*

$$y(t) = (I - S_2B_2)^{-1}(S_2h)(t),$$

where $S_2, B_2 : \Phi_\omega \to \Phi_\omega$ *are defined such that*

$$(S_2h)(t) = \int_0^\omega G_2(t,s)h(s)ds, \quad (B_2y)(t) = [M - p_2(t)]y(t).$$

Define $T_2 : \Phi_\omega \to \Phi_\omega$ *by*

$$(T_2h)(t) = (I - S_2B_2)^{-1}(S_2h)(t).$$

If $\sqrt{3}\beta\omega < 4\pi/3$, *then* S_2 *and* B_2 *are completely continuous and therefore* T_2 *is completely continuous and satisfies*

$$0 < (S_2h)(t) \leqslant (T_2h)(t) \leqslant \frac{M}{m}\|S_2h\|, \quad h \in C_\omega^+.$$

3 Main Results

Theorem 1 *Suppose that* $\sqrt{3}\beta\omega < 4\pi/3$, $0 < p_1(t)(b_1 - a_1) \leqslant q_1 < 1$. *Furthermore, assume that there exist two constants* k_1 *and* k_2 *with* $0 < k_1 < k_2$ *such that*

$$\frac{k_1M}{(b_1 - a_1)} \leqslant f(t,x) - p_2(t)p_1(t)x \leqslant \frac{mk_2(1-q_1)}{(b_1 - a_1)}, \quad \forall (t,x) \in [0,\omega] \times [k_1,k_2]. \tag{3}$$

Then (1) has at least one positive ω*-periodic solution* $x(t)$ *such that* $k_1 \leqslant x(t) \leqslant k_2$.

Proof. Consider the set $\Omega_\omega = \{x \in \Phi_\omega : k_1 \leqslant x(t) \leqslant k_2, : t \in [0,\omega]\}$. It is evident that Ω_ω is a bounded, closed and convex subset of Φ_ω. We show that

$$x(t) = p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta + T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x(t-\mu(t,\zeta)))d\zeta\right)$$

is a solution of (1). It is clear that the equation

$$\begin{aligned} &\left[x(t) - p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta\right]''' - p_2(t)\left[x(t) - p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta\right] \\ =\;& p_2(t)p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x(t-\mu(t,\zeta)))d\zeta \end{aligned} \tag{4}$$

is equivalent to (1). Let $y(t) = x(t) - p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta$ in the equation (4), then we have

$$y'''(t) - p_2(t)y(t) = p_2(t)p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x(t-\mu(t,\zeta)))d\zeta.$$

Applying Lemma 3, we have

$$y(t) = T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x(t-\mu(t,\zeta)))d\zeta\right),$$

which verifies the desired claim.

Define the operators U_1 and U_2 on Ω_ω as follows:

$$\begin{aligned} (U_1x)(t) &= p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta \quad \text{and} \\ (U_2x)(t) &= T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x(t-\mu(t,\zeta)))d\zeta\right). \end{aligned}$$

Clearly, U_1x and U_2x are continuous and ω-periodic, i.e we have $U_1(\Omega_\omega) \subset \Phi_\omega$ and $U_2(\Omega_\omega) \subset \Phi_\omega$. For all $x_1, x_2 \in \Omega_\omega$ and $t \in \mathbb{R}$, from Lemma 2, Lemma 3 and (3), we obtain

$$\begin{aligned} (U_1x_1)(t) + (U_2x_2)(t) &= p_1(t)\int_{a_1}^{b_1} x_1(t-\mu(t,\zeta))d\zeta \\ &\quad + T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right) \\ &\leqslant q_1k_2 + \frac{M}{m}\left\| S_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta - \int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right)\right\| \\ &= q_1k_2 + \frac{M}{m} \\ &\quad \sup_{t\in[0,\omega]}\left|\int_0^\omega G_1(t,s)\int_{a_1}^{b_1}\Big[f(s,x_2(s-\mu(s,\zeta))) - p_2(s)p_1(s)x_2(s-\mu(s,\zeta))\Big]d\zeta ds\right| \\ &\leqslant q_1k_2 + \frac{M}{m}\int_0^\omega G_1(t,s)mk_2(1-q_1)ds = k_2 \end{aligned}$$

and similarly, we have

$$\begin{aligned}(U_1x_1)(t)+(U_2x_2)(t) &= p_1(t)\int_{a_1}^{b_1} x_1(t-\mu(t,\zeta))d\zeta\\ &\quad +T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta-\int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right)\\ &\geqslant S_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta-\int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right)\\ &= \int_0^{\omega} G_1(t,s)\int_{a_1}^{b_1}\Big[f(s,x_2(s-\mu(s,\zeta)))-p_2(s)p_1(s)x_2(s-\mu(s,\zeta))\Big]d\zeta ds\\ &\geqslant \int_0^{\omega} G_1(t,s)Mk_1 ds=k_1\end{aligned}$$

from this, we can conclude that $k_1 \leqslant (U_1x_1)(t)+(U_2x_2)(t)\leqslant k_2$ for all $x_1,x_2\in\Omega_\omega$ and $t\in\mathbb{R}$, i.e. we have $U_1x_1+U_2x_2\in\Omega_\omega$. Furthermore, for $x_1,x_2\in\Omega_\omega$, we have

$$\begin{aligned}|(U_1x_1)(t)-(U_1x_2)(t))| &= \left|p_1(t)\int_{a_1}^{b_1} x_1(t-\mu(t,\zeta))d\zeta-p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta\right|\\ &\leqslant p_1(t)\int_{a_1}^{b_1}|x_1(t-\mu(t,\zeta))-x_2(t-\mu(t,\zeta))|d\zeta.\end{aligned}$$

It follows that

$$\|U_1x_1-U_1x_2\|\leqslant q_1\|x_1-x_2\|.$$

Noting that $q_1<1$, it is clear that U_1 is a contraction mapping.

From Lemma 3, we know that T_1 is completely continuous, so is U_2. By applying Krasnoselskii's Fixed Point Theorem, we can conclude that there exists $x\in\Omega_\omega$ such that $U_1x+U_2x=x$, which is a positive ω-periodic solution of (1). ■

Theorem 2 *Suppose that $\sqrt{3}\beta\omega<4\pi/3$, $-1<q_0\leqslant p_1(t)(b_1-a_1)<0$, $-q_0M<m$. Moreover, assume that there exist two constants k_1 and k_2 with $0<k_1<k_2$ such that*

$$\frac{(k_1-q_0k_2)M}{(b_1-a_1)}\leqslant f(t,x)-p_2(t)p_1(t)x\leqslant\frac{mk_2}{(b_1-a_1)},\quad \forall (t,x)\in[0,\omega]\times[k_1,k_2]. \tag{5}$$

Then (1) has at least one positive ω-periodic solution $x(t)$ such that $k_1\leqslant x(t)\leqslant k_2$.

Proof. Let $\Omega_\omega=\{x\in\Phi_\omega : k_1\leqslant x(t)\leqslant k_2,: t\in[0,\omega]\}$ and define U_1 and U_2 as in the proof of Theorem 1. It is clear that Ω_ω is a bounded, closed and convex subset of Φ_ω and $U_1(\Omega_\omega)\subset\Phi_\omega$ and $U_2(\Omega_\omega)\subset\Phi_\omega$. We show that $U_1x_1+U_2x_2\in\Omega_\omega$ for all $x_1,x_2\in\Omega_\omega$. For $x_1,x_2\in\Omega_\omega$ and $t\in\mathbb{R}$, we have from (5), Lemma 2 and Lemma 3 that

$$\begin{aligned}(U_1x_1)(t)+(U_2x_2)(t) &= p_1(t)\int_{a_1}^{b_1} x_1(t-\mu(t,\zeta))d\zeta\\ &\quad +T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta-\int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right)\\ &\leqslant \frac{M}{m}\left\|S_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta-\int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right)\right\|\\ &= \frac{M}{m}\sup_{t\in[0,\omega]}\left|\int_0^{\omega} G_1(t,s)\int_{a_1}^{b_1}\Big[f(s,x_2(s-\mu(s,\zeta)))-p_2(s)p_1(s)x_2(s-\mu(s,\zeta))\Big]d\zeta ds\right|\\ &\leqslant \frac{M}{m}\int_0^{\omega} G_1(t,s)mk_2 ds=k_2\end{aligned}$$

and

$$
\begin{aligned}
(U_1x_1)(t)+(U_2x_2)(t) &= p_1(t)\int_{a_1}^{b_1} x_1(t-\mu(t,\zeta))d\zeta \\
&\quad +T_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta-\int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right) \\
&\geqslant p_1(t)(b_1-a_1)k_2 \\
&\quad +S_1\left(p_2(t)p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta-\int_{a_1}^{b_1} f(t,x_2(t-\mu(t,\zeta)))d\zeta\right) \\
&\geqslant q_0k_2+\int_0^{\omega} G_1(t,s)\int_{a_1}^{b_1}\Big[f(s,x_2(s-\mu(s,\zeta)))-p_2(s)p_1(s)x_2(s-\mu(s,\zeta))\Big]d\zeta ds \\
&\geqslant q_0k_2+\int_0^{\omega} G_1(t,s)(k_1-q_0k_2)M ds=k_1.
\end{aligned}
$$

Thus, we have $U_1x_1+U_2x_2\in\Omega_\omega$ for all $x_1,x_2\in\Omega_\omega$. Now we show U_1 is a contraction operator on Ω_ω. In fact, for $x_1,x_2\in\Omega_\omega$, we have

$$
\begin{aligned}
|(U_1x_1)(t)-(U_1x_2)(t))| &= \left|p_1(t)\int_{a_1}^{b_1} x_1(t-\mu(t,\zeta))d\zeta-p_1(t)\int_{a_1}^{b_1} x_2(t-\mu(t,\zeta))d\zeta\right| \\
&\leqslant -p_1(t)\int_{a_1}^{b_1}|x_1(t-\mu(t,\zeta))-x_2(t-\mu(t,\zeta))|d\zeta.
\end{aligned}
$$

By using the sup norm, we see that

$$\|U_1x_1-U_1x_2\|\leqslant -q_0\|x_1-x_2\|.$$

Because of $-q_0<1$, U_1 is a contraction mapping.

From Lemma 3, we know that T_1 is completely continuous, so is U_2. There is an $x\in\Omega_\omega$ such that $U_1x+U_2x=x$ because of Krasnoselskii's Fixed Point Theorem and therefore $x(t)$ is a positive ω-periodic solution of (1). ■

Theorem 3 *Suppose that* $\sqrt{3}\beta\omega<4\pi/3$, $0<p_1(t)(b_1-a_1)\leqslant q_1<1$. *Furthermore, assume that there exist two constants* k_1 *and* k_2 *with* $0<k_1<k_2$ *such that*

$$\frac{k_1M}{(b_1-a_1)}\leqslant f(t,x)-p_2(t)p_1(t)x\leqslant\frac{mk_2(1-q_1)}{(b_1-a_1)},\quad \forall(t,x)\in[0,\omega]\times[k_1,k_2].$$

Then (2) has at least one positive ω*-periodic solution* $x(t)$ *such that* $k_1\leqslant x(t)\leqslant k_2$.

Theorem 4 *Suppose that* $\sqrt{3}\beta\omega<4\pi/3$, $-1<q_0\leqslant p_1(t)(b_1-a_1)<0$, $-q_0M<m$. *In addition, assume that there exist two constants* k_1 *and* k_2 *with* $0<k_1<k_2$ *such that*

$$\frac{(k_1-q_0k_2)M}{(b_1-a_1)}\leqslant f(t,x)-p_2(t)p_1(t)x\leqslant\frac{mk_2}{(b_1-a_1)},\quad \forall(t,x)\in[0,\omega]\times[k_1,k_2].$$

Then (2) has at least one positive ω*-periodic solution* $x(t)$ *such that* $k_1\leqslant x(t)\leqslant k_2$.

Since the proofs for Theorem 3 and Theorem 4, respectively, are similar to those for Theorem 1 and Theorem 2, we omitted them to avoid unnecessary repetition.

Example 1 *Consider the equation*

$$\left[x(t) - \frac{\exp(\sin(2t))}{100} \int_{\pi/3}^{\pi/2} x(t - 10\zeta - \cos(2t)) d\zeta\right]'''$$
$$= \left(\frac{1}{4} + \frac{\sin(2t)}{100}\right) x(t) - \int_{\pi/3}^{\pi/2} (3.2 + \exp(\sin(2t)) + \sin(x^3(t - 10\zeta - \cos(2t)))) d\zeta. \tag{6}$$

It should be noted that (6) is of the form (1) with $\omega = \pi$, $p_1(t) = \frac{\exp(\sin(2t))}{100}$, $p_2(t) = \left(\frac{1}{4} + \frac{\sin(2t)}{100}\right)$, $f(t,x) = 3.2 + \exp(\sin(2t)) + \sin(x^3)$, $\mu(t,\zeta) = 10\zeta + \cos(2t)$, $a_1 = \pi/3$ *and* $b_1 = \pi/2$. *Furthermore,* $M = 0.26$, $\beta = \sqrt[3]{M} = \sqrt[3]{0.26}$. *Since* $\sqrt{3}\beta\pi = 3.4730 < \pi/3 = 4.1818$ *and*

$$p_1(t)(b_1 - a_1) = \frac{\exp(\sin(2t))}{100}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) \leqslant q_1 = 0.0142 < 1.$$

All conditions of Theorem 1 are met with $k_1 = 4$ *and* $k_2 = 16$. *As a result, (6) has at least one positive* π*-periodic solution.*

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