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On Implicit Neutral Caputo Tempered Fractional Differential Equations With Retarded And Advanced Arguments*

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Abstract

The purpose of this article is to study the existence, uniqueness and Ulam stability results for a class of implicit neutral fractional differential equations involving the Caputo tempered fractional derivative with retarded and advanced arguments. The results are based on Banach's contraction principle, Schauder's and Darbo's fixed point theorems. To illustrate our results, we will give some examples.

1 Introduction

Fractional calculus has recently become an important tool in tackling complex issues across a variety of research fields. This approach involves extending differentiation and integration to non-integer orders, and has generated significant interest in both its theory and applications. To gain a comprehensive understanding of fractional calculus, we recommend consulting monographs such as [1–4, 13, 20, 35, 38], as well as papers like [5–7, 15, 16]. In recent years, there has been a notable increase in research on fractional calculus, with authors exploring diverse outcomes for different forms of fractional differential equations and inclusions, and under varying conditions. Further information can be found in papers like [8, 9, 18, 22–24, 33], and their respective references.

In [11], the authors considered the following fractional impulsive neutral integro-differential systems with infinite delay:

$$\begin{cases} D_\theta^q\left(y(\theta)-\chi\left(\theta, y_\theta\right)\right)=A(\theta, y)\left(y(\theta)-\chi\left(\theta, y_\theta\right)\right)+f\left(\theta, y_\theta, \int_0^\theta h\left(\theta, s, y_s\right) d s\right), \\ \qquad \theta \in[0, b], \quad \theta \neq \theta_\jmath, \\ \left.\Delta y\right|_{\theta=\theta_\jmath}=I_\jmath\left(y\left(\theta_\jmath^{-}\right)\right), \quad \theta=\theta_\jmath ; \jmath=1, \ldots, m, \\ y(0)+g(y)=\phi, \quad \phi \in B_\vartheta,\end{cases}$$

where $0<q<1$, D_θ^q is the Caputo fractional derivative and $y_\theta(.)$ denote $y_\theta(\kappa)=y(\theta+\kappa), \kappa \in(-\infty, 0]$. The results are obtained by a fixed point theorem.

In [27], the authors considered a class of problems for nonlinear Caputo tempered implicit fractional differential equations with boundary conditions and delay:

$${ }_0^C D_\theta^{\beta, \gamma} y(\theta)=\Psi\left(\theta, y_\theta,{ }_0^C D_\theta^{\beta, \gamma} y(\theta)\right), \quad \theta \in \Theta:=[0, T],$$

$$y(\theta)=\phi(\theta), \quad \theta \in[-\kappa, 0],$$

$$\delta_1 y(0)+\delta_2 y(T)=\delta_3,$$

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where $0 < \beta < 1$, $\gamma \geq 0$, ${}_{0}^{C}D_{\theta}^{\beta,\gamma}$ is the Caputo tempered fractional derivative, $\Psi : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $y \in C([-\kappa, 0], \mathbb{R})$, $0 < T < +\infty$, $\delta_1, \delta_2, \delta_3$ are real constants, and $\kappa > 0$ is the time delay. Their arguments are based on Banach, Schauder and Schaefer fixed point theorems.

Tempered fractional calculus has become a noteworthy class of fractional calculus operators in recent years, as it possesses analytic kernels and is capable of generalizing various forms of fractional calculus. This class is seen as an extension of fractional calculus, as it can describe the transition between normal and anomalous diffusion. Buschman's seminal work [17] established the definitions of fractional integration with weak singular and exponential kernels, and further elaboration on this topic can be found in [10, 26, 28–32, 34, 36]. Although the Caputo tempered fractional derivative has received little attention in the literature, it offers the potential to make a substantial contribution to this discipline. With the study of the Caputo tempered fractional derivative, we want to get a better grasp of the properties and possible uses of this unique mathematical notion, and in doing so, we hope to contribute to the advancement of fractional calculus.

In this paper, we investigate the existence, uniqueness and stability results of the following implicit neutral problem:

$$ {}_{0}^{C}D_{\theta}^{\sigma,\varrho}\left[y(\theta) - \Phi(\theta, y^{\theta})\right] = \Psi\left(\theta, y^{\theta},\ {}_{0}^{C}D_{\theta}^{\sigma,\varrho}y(\theta)\right), \quad \theta \in \Theta := [0, T], \tag{1} $$

$$ y(\theta) = \varpi(\theta), \quad \theta \in [-r, 0], \tag{2} $$

$$ y(\theta) = \widehat{\varpi}(\theta), \quad \theta \in [T, T + \delta], \tag{3} $$

where ${}_{0}^{C}D_{\theta}^{\sigma,\varrho}$ is the Caputo tempered fractional derivative of order $\sigma \in (0, 1)$, $\varrho \geq 0$, $r, \delta > 0$, $\Psi : \Theta \times C([-r, \delta], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, $\Phi : \Theta \times C([-r, \delta], \mathbb{R}) \to \mathbb{R}$ are given functions, $\varpi \in C([-r, 0], \mathbb{R})$, and $\widehat{\varpi} \in C([T, T + \delta], \mathbb{R})$. We denote by y^{θ} the element of $C([-r, \delta], \mathbb{R})$ defined by

$$ y^{\theta} = y(\theta + \kappa) : \kappa \in [-r, \delta]. $$

The paper is arranged as follows: In section 2, we give some notations, definitions and auxiliary results that are used throughout this work. Section 3 presents some existence and uniqueness results for the problem (1)–(3) that are based on Banach's contraction principle and Schauder's fixed point theorem. A similar problem to problem (1)–(3), albeit in Banach space, is presented in section 4, the results of this problem are based on Darbo's fixed point theorem coupled with the technique of measure of noncompactness. In section 5, we will establish the Ulam stability of the previous problem. In the last section, we give some demonstrative examples to prove the validity of our results.

2 Preliminaries

In this section, we give some notations, definitions and lemmas which are used throughout this paper. Let $(E, \|\cdot\|)$ be a Banach space and denote by $C(\Theta, E)$, where $\Theta := [0, T]$, the Banach space of all continuous functions from Θ into E with the norm

$$ \|y\|_{\infty} = \sup\{\|y(\theta)\| : \theta \in \Theta\}. $$

Let $C([-r, 0], E)$ the Banach space with the norm

$$ \|y\|_{[-r,0]} = \sup\{\|y(\theta)\| : \theta \in [-r, 0]\}. $$

Consider $C([T, T + \delta], E)$ the Banach space with the norm

$$ \|y\|_{[T,T+\delta]} = \sup\{\|y(\theta)\| : \theta \in [T, T + \delta]\}, $$

and $C([-r, \delta], E)$ the Banach space with the norm

$$ \|y\|_{[-r,\delta]} = \sup\{\|y(\theta)\| : \theta \in [-r, \delta]\}. $$

Let

$$\Upsilon = \big\{y : [-r, T+\delta] \to E : y|_{[0,T]} \in C(\Theta, E), y|_{[-r,0]} \in C([-r, 0], E) \\ \text{and } y|_{[T,T+\delta]} \in C([T, T+\delta], E)\big\}.$$

We note that Υ is a Banach space with the norm

$$\|y\|_\Upsilon = \sup_{\theta\in[-r,T+\delta]} \|y(\theta)\|.$$

Definition 1 (The Riemann-Liouville tempered fractional integral [28, 32, 36]) *Suppose that the function $\Psi \in C(\Theta, E)$, $\varrho \geq 0$. Then, the Riemann-Liouville tempered fractional integral of order σ is defined by*

$$ {}_0I_\theta^{\sigma,\varrho}\Psi(\theta) = e^{-\varrho\theta}\ {}_0I_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{1}{\Gamma(\sigma)}\int_0^\theta \frac{e^{-\varrho(\theta-\kappa)}\Psi(\kappa)}{(\theta-\kappa)^{1-\sigma}}d\kappa, \tag{4}$$

where ${}_0I_\theta^\sigma$ denotes the Riemann-Liouville fractional integral [25], defined by

$$ {}_0I_\theta^{\sigma}\Psi(\theta) = \frac{1}{\Gamma(\sigma)}\int_0^\theta \frac{\Psi(\kappa)}{(\theta-\kappa)^{1-\sigma}}d\kappa. \tag{5}$$

Obviously, the tempered fractional integral (4) reduces to the Riemann-Liouville fractional integral (5) if $\varrho = 0$.

Definition 2 (The Riemann-Liouville tempered fractional derivative [28, 32]) *For $\beta - 1 < \sigma < \beta$; $\beta \in \mathbb{N}, \varrho \geq 0$, the Riemann-Liouville tempered fractional derivative is defined by*

$$ {}_0D_\theta^{\sigma,\varrho}\Psi(\theta) = e^{-\varrho\theta}\ {}_0D_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{e^{-\varrho\theta}}{\Gamma(\beta-\sigma)}\frac{d^\beta}{d\theta^\beta}\int_0^\theta \frac{e^{\varrho\kappa}\Psi(\kappa)}{(\theta-\kappa)^{\sigma-\beta+1}}d\kappa,$$

where ${}_0D_\theta^\sigma\left(e^{\varrho\theta}\Psi(\theta)\right)$ denotes the Riemann-Liouville fractional derivative [25], given by

$$ {}_0D_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{d^\beta}{d\theta^\beta}\Big(\ {}_0I_\theta^{\beta-\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right)\Big) = \frac{1}{\Gamma(\beta-\sigma)}\frac{d^\beta}{d\theta^\beta}\int_0^\theta \frac{(e^{\varrho\kappa}\Psi(\kappa))}{(\theta-\kappa)^{\sigma-\beta+1}}d\kappa.$$

Definition 3 (The Caputo tempered fractional derivative [28, 36]) *For $\beta-1 < \sigma < \beta$; $\beta \in \mathbb{N}, \varrho \geq 0$, the Caputo tempered fractional derivative is defined as*

$$ {}_0^CD_\theta^{\sigma,\varrho}\Psi(\theta) = e^{-\varrho\theta}\ {}_0^CD_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{e^{-\varrho\theta}}{\Gamma(\beta-\sigma)}\int_0^\theta \frac{1}{(\theta-\kappa)^{\sigma-\beta+1}}\frac{d^\beta\left(e^{\varrho\kappa}\Psi(\kappa)\right)}{d\kappa^\beta}d\kappa,$$

where ${}_0^CD_\theta^{\sigma,\varrho}\left(e^{\varrho\theta}\Psi(\theta)\right)$ denotes the Caputo fractional derivative [25], given by

$$ {}_0^CD_\theta^{\sigma}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{1}{\Gamma(\beta-\sigma)}\int_0^\theta \frac{1}{(\theta-\kappa)^{\sigma-\beta+1}}\frac{d^\beta\left(e^{\varrho\kappa}\Psi(\kappa)\right)}{d\kappa^\beta}d\kappa.$$

Lemma 1 ([28]) *For a constant C,*

$$ {}_0D_\theta^{\sigma,\varrho}C = Ce^{-\varrho\theta}{}_0D_\theta^\sigma e^{\varrho\theta}, \quad {}_0^CD_\theta^{\sigma,\varrho}C = Ce^{-\varrho\theta}\ {}_0^CD_\theta^\sigma e^{\varrho\theta}.$$

Obviously, ${}_0D_\theta^{\sigma,\varrho}(C) \neq {}_0^C D_\theta^{\sigma,\varrho}(C)$. And, ${}_0^CD_\theta^{\sigma,\varrho}(C)$ is no longer equal to zero, being different from ${}_0^CD_\theta^\sigma(C) \equiv 0$.

Lemma 2 ([28, 36]) *Let $\Psi \in C^\beta(\Theta, E)$ and $\beta - 1 < \sigma < \beta$; $\beta \in \mathbb{N}$. Then, the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the composite properties*

$$ {}_0I_\theta^{\sigma,\varrho}\left[{}_0^CD_\theta^{\sigma,\varrho}\Psi(\theta)\right] = \Psi(\theta) - \sum_{\jmath=0}^{\beta-1} e^{-\varrho\theta}\frac{\theta^\jmath}{\jmath!}\left[\left.\frac{d^\jmath\left(e^{\varrho\theta}\Psi(\theta)\right)}{d\theta^\jmath}\right|_{\theta=0}\right],$$

and

$$ {}_0^CD_\theta^{\sigma,\varrho}\left[{}_0I_\theta^{\sigma,\varrho}\Psi(\theta)\right] = \Psi(\theta), \ \textit{for } \sigma \in (0, 1).$$

2.1 Measure of Noncompactness

Definition 4 ([14]) *Let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \longrightarrow [0, \infty)$ defined by*

$$\alpha(M) = inf \left\{ \varepsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, \ \ diam(M_j) \leq \varepsilon \right\},$$

where $M \in \Omega_X$.

The map α satisfies the following properties:

- $\alpha(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact);
- $\alpha(M) = \alpha(\overline{M})$;
- $M_1 \subset M_2 \Rightarrow \alpha(M_1) \leq \alpha(M_2)$;
- $\alpha(M_1 + M_2) \leq \alpha(M_1) + \alpha(M_2)$;
- $\alpha(cM) = |c|\alpha(M)$, $c \in \mathbb{R}$;
- $\alpha(convM) = \alpha(M)$.

Lemma 3 ([21]) *Let $B \subset \Upsilon$ be a bounded and equicontinuous set. Then*

a) The function $\theta \to \alpha(B(\theta))$ is continuous on θ, and

$$\alpha_\Upsilon(B) = \sup_{\theta \in [-r, T+\delta]} \alpha(B(\theta)).$$

b) $\alpha\left(\left\{\int_0^T y(\kappa)dy : y \in B\right\}\right) \leq \int_0^T \alpha(B(\kappa))d\kappa$, where

$$B(\theta) = \{y(\theta) : y(\theta) \in B, \theta \in \Theta\}.$$

2.2 Some Fixed Point Theorems

Theorem 1 (Banach's fixed point theorem [37]) *Let X be a Banach space and $\mathcal{Y} : X \longrightarrow X$ a contraction, i.e. there exists $\jmath \in [0, 1)$ such that*

$$\|\mathcal{Y}(y_1) - \mathcal{Y}(y_2)\| \leq \jmath \|y_1 - y_2\|, \qquad for\ all\ y_1, y_2 \in X.$$

Then $\mathcal{Y}$ has a unique fixed point.

Theorem 2 (Schauder's fixed point theorem [37]) *Let X be a Banach space, D a bounded, closed, convex subset of X, and $\mathcal{Y} : D \longrightarrow D$ a compact and continuous map. Then $\mathcal{Y}$ has at least one fixed point in D.*

Theorem 3 (Darbo's fixed point Theorem [19]) *Let D be a non-empty, closed, bounded and convex subset of a Banach space X, and let $\mathcal{Y}$ be a continuous mapping of D into itself such that for any non-empty subset B of D,*

$$\alpha(\mathcal{Y}(B)) \leq \tau \alpha(B),$$

where $0 \leq \tau < 1$, and α is the Kuratowski measure of noncompactness on X. Then $\mathcal{Y}$ has a fixed point in D.

3 Existence Results

Consider the following fractional differential problem:

$$ {}_0^C D_\theta^{\sigma,\varrho}[y(\theta)-U(\theta)] = \mu(\theta), \quad if\ \theta\in\Theta,\ 0<\sigma<1, \tag{6}$$

$$ y(\theta) = \varpi(\theta), \quad if\ \theta\in[-r,0],\ r>0, \tag{7}$$

$$ y(\theta) = \widehat{\varpi}(\theta), \quad if\ \theta\in[T,T+\delta],\ \delta>0, \tag{8}$$

where $\mu:\Theta\to\mathbb{R}$ is a continuous function, $U:\Theta\to\mathbb{R}$, $\varpi\in C([-r,0],\mathbb{R})$, and $\widehat{\varpi}\in C([T,T+\delta],\mathbb{R})$.

Lemma 4 *Let $\sigma\in(0,1)$, and $\mu:\Theta\to\mathbb{R}$ be continuous. Then, the problem (6)–(8) has a unique solution given by:*

$$ y(\theta)=\begin{cases} \varpi(0)e^{-\varrho\theta}-U(0)e^{-\varrho\theta}+U(\theta)+\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\mu(\kappa)d\kappa, & \theta\in\Theta,\\ \varpi(\theta), & \theta\in[-r,0],\\ \widehat{\varpi}(\theta), & \theta\in[T,T+\delta]. \end{cases} \tag{9}$$

Proof. Suppose that y satisfies (6)–(8). From Lemma 2, we have

$$ y(\theta)-U(\theta)-e^{-\varrho\theta}[\varpi(0)-U(0)] = \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\mu(\kappa)d\kappa.$$

Then,

$$ y(\theta)-U(\theta) = e^{-\varrho\theta}[\varpi(0)-U(0)] + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\mu(\kappa)d\kappa.$$

Finally, we have

$$ y(\theta) = \varpi(0)e^{-\varrho\theta}-U(0)e^{-\varrho\theta}+U(\theta)+\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\mu(\kappa)d\kappa.$$

Conversely, we can easily show by Definition 3, Lemmas 1 and 2 that if y verifies (9), then it satisfies the problem (6)–(8). ■

Let

$$ \Upsilon_{\mathbb{R}} = \Big\{ y:[-r,T+\delta]\to\mathbb{R} : y|_{[0,T]}\in C(\Theta,\mathbb{R}), y|_{[-r,0]}\in C([-r,0],\mathbb{R}) \text{ and } y|_{[T,T+\delta]}\in C([T,T+\delta],\mathbb{R})\Big\}$$

be a Banach space with the norm

$$ \|y\|_{\Upsilon_{\mathbb{R}}} = \sup_{\theta\in[-r,T+\delta]}|y(\theta)|.$$

Definition 5 *By a solution of problem (1)–(3), we mean a function $y\in\Upsilon_{\mathbb{R}}$ that satisfies the equation (1) and the conditions (2)–(3).*

Lemma 5 *Let $\Psi:\Theta\times C([-r,\delta],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then, the problem (1)–(3) is equivalent to the following integral equation:*

$$ y(\theta)=\begin{cases} \varpi(0)e^{-\varrho\theta}-\Phi(0,\varpi(0))e^{-\varrho\theta}+\Phi(\theta,y^\theta) \\ +\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\Psi(\kappa,y^\kappa,h(\kappa))d\kappa, & if\ \theta\in\Theta,\\ \varpi(\theta), & if\ \theta\in[-r,0],\\ \widehat{\varpi}(\theta), & if\ \theta\in[T,T+\delta], \end{cases}$$

where $h\in C(\Theta,\mathbb{R})$ satisfies the following functional equation

$$ h(\theta) = \Psi(\theta,y^\theta,h(\theta)).$$

Let us put the following conditions:

(*A1*) The functions Ψ and Φ are continuous.

(*A2*) There exist constants $\lambda > 0$ and $0 < L < 1$ such that

$$|\Psi(\theta, \chi, \Im) - \Psi(\theta, \bar{\chi}, \bar{\Im})| \leq \lambda \|\chi - \bar{\chi}\|_{[-r,\delta]} + L|\Im - \bar{\Im}|,$$

for any $\chi, \bar{\chi} \in C([-r, \delta], \mathbb{R})$, $\Im, \bar{\Im} \in \mathbb{R}$ and $\theta \in \Theta$.

(*A3*) There exists constant $C > 0$ such that

$$|\Phi(\theta, \chi) - \Phi(\theta, \bar{\chi})| \leq C\|\chi - \bar{\chi}\|_{[-r,\delta]},$$

for any $\chi, \bar{\chi} \in C([-r, \delta], \mathbb{R})$, $\theta \in \Theta$.

We are now in a position to prove the existence result of the problem (1)–(3) based on the Banach's contraction principle.

Theorem 4 *Assume that the hypotheses (A1)–(A3) hold. If*

$$C + \frac{\lambda T^{\sigma}}{(1 - L)\Gamma(\sigma + 1)} < 1, \tag{10}$$

then the implicit fractional problem (1)–(3) has a unique solution.

Proof. Consider the operator $A : \Upsilon_{\mathbb{R}} \longrightarrow \Upsilon_{\mathbb{R}}$ defined by

$$Ay(\theta) = \begin{cases} \varpi(0)e^{-\varrho\theta} - \Phi(0, \varpi(0))e^{-\varrho\theta} + \Phi(\theta, y^{\theta}) \\ +\frac{1}{\Gamma(\sigma)} \int_0^{\theta} e^{-\varrho(\theta-\kappa)}(\theta - \kappa)^{\sigma-1}h(\kappa)d\kappa, & \text{if } \theta \in \Theta, \\ \varpi(\theta), \quad \text{if } \theta \in [-r, 0], \\ \widehat{\varpi}(\theta), \quad \text{if } \theta \in [T, T + \delta]. \end{cases}$$

Obviously, the fixed points of the operator A are solutions of the problem (1)–(3). Let $y, z \in \Upsilon_{\mathbb{R}}$. If $\theta \in [-r, 0]$ or $\theta \in [T, T + \delta]$, then

$$|Ay(\theta) - Az(\theta)| = 0.$$

If $\theta \in \Theta$, we have

$$|Ay(\theta) - Az(\theta)| \leq |\Phi(\theta, y^{\theta}) - \Phi(\theta, z^{\theta})| + \frac{1}{\Gamma(\sigma)} \int_0^{\theta} e^{-\varrho(\theta-\kappa)}(\theta - \kappa)^{\sigma-1}|h(\kappa) - k(\kappa)|d\kappa,$$

where h and k are two functions verifying

$$h(\theta) = \Psi(\theta, y^{\theta}, h(\theta)),$$

and

$$k(\theta) = \Psi(\theta, z^{\theta}, k(\theta)).$$

By condition $(A2)$, we find that

$$\begin{aligned} |h(\theta) - k(\theta)| &= |\Psi(\theta, y^{\theta}, h(\theta)) - \Psi(\theta, z^{\theta}, k(\theta))| \\ &\leq \lambda\|y^{\theta} - z^{\theta}\|_{[-r,\delta]} + L|h(\theta) - k(\theta)|, \end{aligned}$$

which implies that

$$|h(\theta) - k(\theta)| \leq \frac{\lambda}{1 - L}\|y^{\theta} - z^{\theta}\|_{[-r,\delta]}.$$

Then, for each $\theta \in \Theta$, we have

$$\begin{aligned}|Ay(\theta) - Az(\theta)| &\leq C\|y^\theta - z^\theta\|_{[-r,\delta]} \\ &\quad + \frac{\lambda}{(1-L)\Gamma(\sigma)} \int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|y^\kappa - z^\kappa\|_{[-r,\delta]} d\kappa \\ &\leq \left[C + \frac{\lambda T^\sigma}{(1-L)\Gamma(\sigma+1)}\right]\|y-z\|_{\Upsilon_\mathbb{R}}.\end{aligned}$$

Thus,

$$\|Ay - Az\|_{\Upsilon_\mathbb{R}} \leq \left[C + \frac{\lambda T^\sigma}{(1-L)\Gamma(\sigma+1)}\right]\|y-z\|_{\Upsilon_\mathbb{R}}.$$

Consequently, by the Banach's contraction principle, the operator A has a unique fixed point which is solution of the fractional problem (1)–(3). ■

Remark 1 *Let us put*

$$q_1(\theta) = |\Psi(\theta, 0, 0)|, \ \lambda = q_2^*, \ L = q_3^*.$$

Then, the condition $(A2)$ *implies that*

$$|\Psi(\theta, \chi, \Im)| \leq q_1(\theta) + q_2^*\|\chi\|_{[-r,\delta]} + q_3^*|\Im|,$$

for $\theta \in \Theta$, $\chi \in C([-r,\delta],\mathbb{R})$ *and* $\Im \in \mathbb{R}$ *with* $q_1 \in C(\Theta, \mathbb{R}_+)$, *such that*

$$q_1^* = \sup_{\theta\in\Theta} q_1(\theta).$$

Our second existence result for the problem (1)–(3) is based on Schauder's fixed point theorem.

Theorem 5 *Assume that in addition of (A1)–(A2), the following hypotheses hold:*

(A4) For each $\theta \in \Theta$ *and bounded set* $B \in C([-r,\delta],\mathbb{R})$, *the set*

$$\{\theta \longmapsto \Phi(\theta, y^\theta), y \in B\} \text{ is equicontinuous.}$$

(A5) There exist two functions $p_1, p_2 \in C(\Theta, \mathbb{R}_+)$, *such that*

$$|\Phi(\theta, \chi)| \leq p_1(\theta)\|\chi\|_{[-r,\delta]} + p_2(\theta),$$

for each $\chi \in C([-r,\delta],\mathbb{R})$, *where* $p_i^* = \sup_{\theta\in\Theta} p_i(\theta); i = 1, 2.$

If

$$p_1^* + \frac{q_2^* T^\sigma}{(1-q_3^*)\Gamma(\sigma+1)} < 1,$$

then the implicit fractional problem (1)–(3) has at least one solution.

Proof. The proof will be given in several steps.

Step 1: The operator $A : \Upsilon_\mathbb{R} \longrightarrow \Upsilon_\mathbb{R}$ is continuous.
Let $\{y_\beta\}_{\beta\in\mathbb{N}}$ be a sequence such that $y_\beta \longrightarrow y$ in $\Upsilon_\mathbb{R}$. If $\theta \in [-r, 0]$ or $\theta \in [T, T+\delta]$, then

$$|Ay_\beta(\theta) - Ay(\theta)| = 0.$$

If $\theta \in \Theta$, we have

$$|Ay_\beta(\theta) - Ay(\theta)| \leq |\Phi(\theta, y_\beta^\theta) - \Phi(\theta, y^\theta)| + \frac{1}{\Gamma(\sigma)} \int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}|h_\beta(\kappa) - h(\kappa)|d\kappa,$$

where h_β and h are two functions satisfying the following functional equations:

$$h_\beta(\theta) = \Psi(\theta, y_\beta^\theta, h_\beta(\theta)),$$

and

$$h(\theta) = \Psi(\theta, y^\theta, h(\theta)).$$

By $(A2)$, we have

$$\begin{aligned} |h_\beta(\theta) - h(\theta)| &= |\Psi(\theta, y_\beta^\theta, h_\beta(\theta)) - \Psi(\theta, y^\theta, h(\theta))| \\ &\leq \lambda \|y_\beta^\theta - y^\theta\|_{[-r,\delta]} + L|h_\beta(\theta) - h(\theta)|. \end{aligned}$$

Then,

$$|h_\beta(\theta) - h(\theta)| \leq \frac{\lambda}{1-L}\|y_\beta^\theta - y^\theta\|_{[-r,\delta]}.$$

Thus,

$$\begin{aligned} |Ay_\beta(\theta) - Ay(\theta)| \leq\ & |\Phi(\theta, y_\beta^\theta) - \Phi(\theta, y^\theta)| \\ & + \frac{\lambda}{(1-L)\Gamma(\sigma)} \int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|y_\beta^\kappa - y^\kappa\|_{[-r,\delta]}d\kappa. \end{aligned}$$

By applying the Lebesgue dominated convergence theorem, we get

$$|Ay_\beta(\theta) - Ay(\theta)| \longrightarrow 0 \quad \text{as } \beta \longrightarrow \infty,$$

which implies that

$$\|Ay_\beta - Ay\|_{\Upsilon_{\mathbb{R}}} \longrightarrow 0 \quad \text{as } \beta \longrightarrow \infty.$$

Hence, the operator A is continuous. Let $R > 0$ such that

$$R \geq \max\left\{ \frac{|\varpi(0)| + 2p_2^* + \frac{q_1^* T^\sigma}{(1-q_3^*)\Gamma(\sigma+1)}}{1 - p_1^* - \frac{q_2^* T^\sigma}{(1-q_3^*)\Gamma(\sigma+1)}},\ \|\varpi\|_{[-r,0]},\ \|\widehat{\varpi}\|_{[T,T+\delta]} \right\}.$$

Define the ball

$$D_R = \{y \in \Upsilon_{\mathbb{R}} : \|y\|_{\Upsilon_{\mathbb{R}}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\Upsilon_{\mathbb{R}}$.

Step 2: $A(D_R) \subset D_R$.
Let $y \in D_R$. If $\theta \in [-r, 0]$. Then

$$|Ay(\theta)| \leq \|\varpi\|_{[-r,0]} \leq R,$$

and if $\theta \in [T, T+\delta]$, then

$$|Ay(\theta)| \leq \|\widehat{\varpi}\|_{[T,T+\delta]} \leq R.$$

For each $\theta \in \Theta$, we have

$$|Ay(\theta)| \leq |\varpi(0)|e^{-\varrho\theta} + |\Phi(0, \varpi(0))|e^{-\varrho\theta} + |\Phi(\theta, y^\theta)| + \frac{1}{\Gamma(\sigma)} \int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}|h(\kappa)|d\kappa.$$

From hypothesis $(A2)$, we have

$$\begin{aligned}
|h(\theta)| &= |\Psi(\theta, y^\theta, h(\theta))| \\
&\leq q_1(\theta) + q_2^*\|y^\theta\|_{[-r,\delta]} + q_3^*|h(\theta)| \\
&\leq q_1^* + q_2^*\|y\|_{\Upsilon_{\mathbb{R}}} + q_3^*|h(\theta)| \\
&\leq q_1^* + q_2^*R + q_3^*|h(\theta)|.
\end{aligned}$$

Then,

$$|h(\theta)| \leq \frac{q_1^* + q_2^*R}{1 - q_3^*}.$$

Finally, we get

$$|Ay(\theta)| \leq |\varpi(0)| + 2p_2^* + p_1^*R + \frac{(q_1^* + q_2^*R)T^\sigma}{(1 - q_3^*)\Gamma(\sigma + 1)} \leq R.$$

Thus, for each $\theta \in [-r, T + \delta]$,

$$|Ay(\theta)| \leq R,$$

which implies that

$$\|Ay\|_{\Upsilon_{\mathbb{R}}} \leq R.$$

Consequently, $A(D_R) \subset D_R$.

Step 3: $A(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $y \in D_R$. Then,

$$\begin{aligned}
|Ay(\theta_2) - Ay(\theta_1)| = \Bigg| &\varpi(0)e^{-\varrho\theta_2} - \Phi(0, \varpi(0))e^{-\varrho\theta_2} + \Phi(\theta_2, y^{\theta_2}) \\
&+ \frac{1}{\Gamma(\sigma)}\int_0^{\theta_2} e^{-\varrho(\theta_2-\kappa)}(\theta_2 - \kappa)^{\sigma-1}h(\kappa)d\kappa \\
&- \varpi(0)e^{-\varrho\theta_1} + \Phi(0, \varpi(0))e^{-\varrho\theta_1} - \Phi(\theta_2, y^{\theta_1}) \\
&- \frac{1}{\Gamma(\sigma)}\int_0^{\theta_1} e^{-\varrho(\theta_1-\kappa)}(\theta_1 - \kappa)^{\sigma-1}h(\kappa)d\kappa\Bigg| \\
\leq\ & |\varpi(0)|\left|e^{-\varrho\theta_2} - e^{-\varrho\theta_1}\right| + |\Phi(0, \varpi(0))|\left|e^{-\varrho\theta_2} - e^{-\varrho\theta_1}\right| \\
&+ |\Phi(\theta_2, y^{\theta_2}) - \Phi(\theta_1, y^{\theta_1})| \\
&+ \frac{1}{\Gamma(\sigma)}\int_0^{\theta_1}\left[e^{-\varrho(\theta_2-\kappa)}(\theta_2 - \kappa)^{\sigma-1} - e^{-\varrho(\theta_1-\kappa)}(\theta_1 - \kappa)^{\sigma-1}\right]|h(\kappa)|d\kappa \\
&+ \frac{1}{\Gamma(\sigma)}\int_{\theta_1}^{\theta_2} e^{-\varrho(\theta_2-\kappa)}(\theta_2 - \kappa)^{\sigma-1}|h(\kappa)|d\kappa \\
\leq\ & |\varpi(0)|\left|e^{-\varrho\theta_2} - e^{-\varrho\theta_1}\right| + |\Phi(0, \varpi(0))|\left|e^{-\varrho\theta_2} - e^{-\varrho\theta_1}\right| \\
&+ |\Phi(\theta_2, y^{\theta_2}) - \Phi(\theta_1, y^{\theta_1})| + \frac{(q_1^* + q_2^*R)(\theta_2^\sigma - \theta_1^\sigma)}{(1 - q_3^*)\Gamma(\sigma + 1)} \\
&+ \frac{(q_1^* + q_2^*R)(\theta_2 - \theta_1)^\sigma}{(1 - q_3^*)\Gamma(\sigma + 1)}.
\end{aligned}$$

As $\theta_1 \longrightarrow \theta_2$, the right-hand side of the aforementioned inequality approaches zero, implying that the operator A is equicontinuous. Equicontinuity in other intervals is easily demonstrated and therefore not discussed here. By virtue of Schauder's fixed point theorem, we can conclude that the problem (1)–(3) has at least one solution. ■

4 An Example

Consider the following implicit problem:

$$ {}_{0}^{C}D_{\theta}^{\frac{1}{3},1}\left[y(\theta)-\Phi(\theta,y^{\theta})\right]=\Psi\left(\theta,y^{\theta},\ {}_{0}^{C}D_{\theta}^{\frac{1}{3},1}y(\theta)\right),\ \theta\in[0,1], \tag{11} $$

$$ y(\theta)=\varpi(\theta),\quad \theta\in[-1,0], \tag{12} $$

$$ y(\theta)=\widehat{\varpi}(\theta),\quad \theta\in[1,2], \tag{13} $$

where $\varpi \in C([-1,0],\mathbb{R})$ and $\widehat{\varpi} \in C([1,2],\mathbb{R})$. Set

$$ \Psi\left(\theta,y^{\theta},\ {}_{0}^{C}D_{\theta}^{\frac{1}{3},1}y(\theta)\right)=\frac{ln(\theta)+2\|y^{\theta}\|_{[-r,\delta]}+\frac{2}{3}\left|{}_{0}^{C}D_{\theta}^{\frac{1}{3},1}y(\theta)\right|}{(15+e^{2t})\left(1+\|y^{\theta}\|_{[-r,\delta]}+\left|{}_{0}^{C}D_{\theta}^{\frac{1}{3},1}y(\theta)\right|\right)}, $$

and

$$ \Phi(\theta,y^{\theta})=\frac{\theta+\|y^{\theta}\|_{[-r,\delta]}}{100}. $$

Clearly, the functions Ψ and Φ are continuous, then the hypothesis $(A1)$ is satisfied. For any $\chi,\bar{\chi}\in C([-r,\delta],\mathbb{R})$, $\Im,\bar{\Im}\in\mathbb{R}$ and $\theta\in[0,1]$, we have

$$ |\Psi(\theta,\chi,\Im)-\Psi(\theta,\bar{\chi},\bar{\Im})|\leq\frac{1}{15}\left[2\|\chi-\bar{\chi}\|_{[-r,\delta]}+\frac{2}{3}|\Im-\bar{\Im}|\right]. $$

Then, the hypothesis $(A2)$ is satisfied with $\lambda=\frac{2}{15}$ and $L=\frac{2}{45}$. Also we have

$$ |\Phi(\theta,\chi)-\Phi(\theta,\bar{\chi})|\leq\frac{1}{100}\|\chi-\bar{\chi}\|_{[-r,\delta]}. $$

So, the condition $(A3)$ is satisfied with $C=\frac{1}{100}$. Furthermore, we have

$$ \begin{aligned} C+\frac{\lambda T^{\sigma}}{(1-L)\Gamma(\sigma+1)} &= \frac{1}{100}+\frac{6}{43\Gamma\left(\frac{4}{3}\right)} \\ &\approx 0.166257654193793 \\ &< 1. \end{aligned} $$

Since the condition of Theorem 4 is verified, the problem (11)–(13) has a unique solution.

5 Implicit Neutral Problem with Retarded and Advanced Arguments in Banach Spaces

This section is devoted to the study of existence and stability of a problem similar to problem (1)–(3) in a Banach space. This problem is given by

$$ {}_{0}^{C}D_{\theta}^{\sigma,\varrho}\left[y(\theta)-\Phi(\theta,y^{\theta})\right]=\Psi\left(\theta,y^{\theta},\ {}_{0}^{C}D_{\theta}^{\sigma,\varrho}y(\theta)\right),\quad \theta\in\Theta:=[0,T], \tag{14} $$

$$ y(\theta)=\varpi(\theta),\quad \theta\in[-r,0], \tag{15} $$

$$ y(\theta)=\widehat{\varpi}(\theta),\quad \theta\in[T,T+\delta], \tag{16} $$

where ${}_{0}^{C}D_{\theta}^{\sigma,\varrho}$ is the Caputo tempered fractional derivative of order $\sigma\in(0,1)$, $\varrho\geq 0$, $r,\delta>0$, $\Psi:\Theta\times C([-r,\delta],E)\times E\to E$, $\Phi:\Theta\times C([-r,\delta],E)\to E$ are given functions, $\varpi\in C([-r,0],E)$, and $\widehat{\varpi}\in C([T,T+\delta],E)$. We denote by y^{θ} the element of $C([-r,\delta],E)$ defined by

$$ y^{\theta}=y(\theta+\kappa):\kappa\in[-r,\delta]. $$

Definition 6 *By a solution of problem (14)–(16), we mean a function $y \in \Upsilon$ that satisfies the equation (14) and the conditions (15)–(16).*

Lemma 6 *Let $\Psi : \Theta \times C([-r,\delta],E) \times E \to E$ be a continuous function. Then, the problem (14)–(16) is equivalent to the following integral equation:*

$$y(\theta) = \begin{cases} \varpi(0)e^{-\varrho\theta} - \Phi(0,\varpi(0))e^{-\varrho\theta} + \Phi(\theta, y^{\theta}) \\ +\frac{1}{\Gamma(\sigma)}\int_0^{\theta} e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\Psi(\kappa, y^{\kappa}, h(\kappa))d\kappa, & if\ \theta \in \Theta, \\ \varpi(\theta), \qquad if\ \theta \in [-r,0], \\ \widehat{\varpi}(\theta), \qquad if\ \theta \in [T, T+\delta], \end{cases}$$

where $h \in C(\Theta, E)$ satisfies the following functional equation

$$h(\theta) = \Psi(\theta, y^{\theta}, h(\theta)).$$

Let us put the following conditions:

($A6$) The functions Ψ and Φ are continuous.

($A7$) There exist constants $\widehat{\lambda} > 0$ and $0 < \widehat{L} < 1$ such that

$$\|\Psi(\theta,\chi,\Im) - \Psi(\theta,\bar{\chi},\bar{\Im})\| \leq \widehat{\lambda}\|\chi - \bar{\chi}\|_{[-r,\delta]} + \widehat{L}\|\Im - \bar{\Im}\|,$$

for any $\chi, \bar{\chi} \in C([-r,\delta],E)$, $\Im, \bar{\Im} \in E$ and $\theta \in \Theta$.

($A8$) There exists constant $\widehat{C} > 0$ such that

$$\|\Phi(\theta,\chi) - \Phi(\theta,\bar{\chi})\| \leq \widehat{C}\|\chi - \bar{\chi}\|_{[-r,\delta]},$$

for any $\chi, \bar{\chi} \in C([-r,\delta],E)$, $\theta \in \Theta$.

($A9$) For each $\theta \in \Theta$ and bounded set $B \in C([-r,\delta],E)$, the set

$$\{\theta \longmapsto \Phi(\theta, y^{\theta}), y \in B\} \text{ is equicontinuous.}$$

($A10$) For each $\theta \in \Theta$ and bounded sets $B_1 \subseteq C([-r,\delta],E)$, $B_2 \subseteq E$, we have

$$\alpha(\Psi(\theta, B_1, B_2)) \leq \widehat{\lambda} \sup_{\kappa\in[-r,\delta]} \alpha(B_1(\kappa)) + \widehat{L}\alpha(B_2).$$

($A11$) For each $\theta \in \Theta$ and bounded set $B_1 \subseteq C([-r,\delta],E)$, we have

$$\alpha(\Phi(\theta, B_1)) \leq \widehat{C} \sup_{\kappa\in[-r,\delta]} \alpha(B_1(\kappa)).$$

Remark 2 ([12]) *It is worth noting that the hypotheses* ($A7$) *and* ($A10$) *are equivalent as well as the hypotheses* ($A8$) *and* ($A11$).

Remark 3 *Let us put*

$$\widehat{q_1}(\theta) = \|\Psi(\theta,0,0)\|, \ \widehat{\lambda} = \widehat{q_2}^*, \ \widehat{L} = \widehat{q_3}^*, \ \widehat{C} = \widehat{p_1}^*, \ \widehat{p_2}(\theta) = \|\Phi(\theta,0)\|.$$

Then, the condition ($A7$) *implies that*

$$\|\Psi(\theta,\chi,\Im)\| \leq \widehat{q_1}(\theta) + \widehat{q_2}^*\|\chi\|_{[-r,\delta]} + \widehat{q_3}^*\|\Im\|,$$

for $\theta \in \Theta$, $\chi \in C([-r,\delta],E)$ *and* $\Im \in \mathbb{R}$ *with* $\widehat{q_1} \in C(\Theta,\mathbb{R}_+)$, *such that*

$$\widehat{q_1}^* = \sup_{\theta\in\Theta} \widehat{q_1}(\theta).$$

And from hypothesis $(A8)$, *we have*

$$\|\Phi(\theta,\chi)\| \le \widehat{p_1}^* \|\chi\|_{[-r,\delta]} + \widehat{p_2}(\theta),$$

for each $\chi \in C([-r,\delta],E)$ *with* $\widehat{p_2} \in C(\Theta,\mathbb{R}_+)$ *such that*

$$\widehat{p_2}^* = \sup_{\theta\in\Theta} \widehat{p_2}(\theta).$$

Theorem 6 *Assume that the conditions* $(A6)$*–*$(A9)$ *are verified. If*

$$\widehat{p_1}^* + \frac{\widehat{q_2}^* T^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)} < 1,$$

then the implicit problem (14)–(16) has at least one solution.

To prove the existence of solution of the problem (14)–(16), we will use the concept of measures of noncompactness and Darbo's fixed point theorem.

Proof. Transform problem (14)–(16) into a fixed point problem by considering the operator $S : \Upsilon \longrightarrow \Upsilon$ by

$$Sy(\theta) = \begin{cases} \varpi(0)e^{-\varrho\theta} - \Phi(0,\varpi(0))e^{-\varrho\theta} + \Phi(\theta,y^\theta) \\ +\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}h(\kappa)d\kappa, & if\ \theta\in\Theta, \\ \varpi(\theta), \quad if\ \theta\in[-r,0], \\ \widehat{\varpi}(\theta), \quad if\ \theta\in[T,T+\delta]. \end{cases}$$

The proof will be given in several steps.

Step 1: The operator $S : \Upsilon \longrightarrow \Upsilon$ is continuous.
Let $\{y_\beta\}_{\beta\in\mathbb{N}}$ be a sequence such that $y_\beta \longrightarrow y$ in Υ. If $\theta \in [-r,0]$ or $\theta \in [T,T+\delta]$, then

$$\|Sy_\beta(\theta) - Sy(\theta)\| = 0.$$

If $\theta \in \Theta$, we have

$$\begin{aligned} \|Sy_\beta(\theta) - Sy(\theta)\| &\le \|\Phi(\theta,y_\beta^\theta) - \Phi(\theta,y^\theta)\| \\ &\quad + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|h_\beta(\kappa)-h(\kappa)\|d\kappa, \end{aligned}$$

where h_β and h are two functions satisfying the following functional equations:

$$h_\beta(\theta) = \Psi(\theta,y_\beta^\theta,h_\beta(\theta)),$$

and

$$h(\theta) = \Psi(\theta,y^\theta,h(\theta)).$$

From hypothesis $(A7)$, we have

$$\begin{aligned} \|h_\beta(\theta)-h(\theta)\| &= \|\Psi(\theta,y_\beta^\theta,h_\beta(\theta)) - \Psi(\theta,y^\theta,h(\theta))\| \\ &\le \widehat{\lambda}\|y_\beta^\theta - y^\theta\|_{[-r,\delta]} + \widehat{L}\|h_\beta(\theta)-h(\theta)\|. \end{aligned}$$

Then,

$$\|h_\beta(\theta) - h(\theta)\| \leq \frac{\widehat{\lambda}}{1-\widehat{L}}\|y_\beta^\theta - y^\theta\|_{[-r,\delta]}.$$

Thus,

$$\|Sy_\beta(\theta) - Sy(\theta)\| \leq C\|y_\beta^\theta - y^\theta\|_{[-r,\delta]} + \frac{\widehat{\lambda}}{(1-\widehat{L})\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|y_\beta^\kappa - y^\kappa\|_{[-r,\delta]}d\kappa.$$

By applying the Lebesgue dominated convergence theorem, we get

$$\|Sy_\beta(\theta) - Sy(\theta)\| \longrightarrow 0 \quad as \ \beta \longrightarrow \infty,$$

which implies that

$$\|Sy_\beta - Sy\|_\Upsilon \longrightarrow 0 \quad as \ \beta \longrightarrow \infty.$$

Consequently, the operator S is continuous.

Let $R > 0$ such that

$$R \geq \max\left\{\frac{\|\varpi(0)\| + 2\widehat{p_2}^* + \frac{\widehat{q_1}^* T^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)}}{1-\widehat{p_1}^* - \frac{\widehat{q_2}^* T^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)}}, \|\varpi\|_{[-r,0]}, \|\widehat{\varpi}\|_{[T,T+\delta]}\right\}.$$

Define the ball

$$D_R = \{y \in \Upsilon : \|y\|_\Upsilon \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of Υ.

Step 2: $S(D_R) \subset D_R$.

Let $y \in D_R$. If $\theta \in [-r, 0]$, then

$$\|Sy(\theta)\| \leq \|\varpi\|_{[-r,0]} \leq R,$$

and if $\theta \in [T, T+\delta]$, then

$$\|Sy(\theta)\| \leq \|\widehat{\varpi}\|_{[T,T+\delta]} \leq R.$$

For each $\theta \in \Theta$, we have

$$\|Sy(\theta)\| \leq \|\varpi(0)\|e^{-\varrho\theta} + \|\Phi(0,\varpi(0))\|e^{-\varrho\theta} + \|\Phi(\theta, y^\theta)\| + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|h(\kappa)\|d\kappa.$$

From hypothesis $(A7)$, we have

$$\|h(\theta)\| = \|\Psi(\theta, y^\theta, h(\theta))\| \leq \widehat{q_1}^* + \widehat{q_2}^* R + \widehat{q_3}^*\|h(\theta)\|.$$

Then,

$$\|h(\theta)\| \leq \frac{\widehat{q_1}^* + \widehat{q_2}^* R}{1-\widehat{q_3}^*}.$$

Finally, we get

$$\|Sy(\theta)\| \leq \|\varpi(0)\| + 2\widehat{p_2}^* + \widehat{p_1}^* R + \frac{(\widehat{q_1}^* + \widehat{q_2}^* R)T^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)} \leq R.$$

Thus, for each $\theta \in [-r, T+\delta]$, we have

$$\|Sy(\theta)\| \leq R,$$

which implies that

$$\|Sy\|_{\Upsilon} \leq R.$$

Consequently, $S(D_R) \subset D_R$.

Step 3: $S(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $y \in D_R$. Then,

$$\begin{aligned}
\|Sy(\theta_2) - Sy(\theta_1)\| = \Bigg\| & \varpi(0)e^{-\varrho\theta_2} - \Phi(0,\varpi(0))e^{-\varrho\theta_2} + \Phi(\theta_2, y^{\theta_2}) \\
& + \frac{1}{\Gamma(\sigma)} \int_0^{\theta_2} e^{-\varrho(\theta_2-\kappa)}(\theta_2-\kappa)^{\sigma-1} h(\kappa) d\kappa \\
& - \varpi(0)e^{-\varrho\theta_1} + \Phi(0,\varpi(0))e^{-\varrho\theta_1} - \Phi(\theta_2, y^{\theta_1}) \\
& - \frac{1}{\Gamma(\sigma)} \int_0^{\theta_1} e^{-\varrho(\theta_1-\kappa)}(\theta_1-\kappa)^{\sigma-1} h(\kappa) d\kappa \Bigg\| \\
\leq\ & \|\varpi(0)\| \left\| e^{-\varrho\theta_2} - e^{-\varrho\theta_1} \right\| + \|\Phi(0,\varpi(0))\| \left\| e^{-\varrho\theta_2} - e^{-\varrho\theta_1} \right\| \\
& + \|\Phi(\theta_2, y^{\theta_2}) - \Phi(\theta_1, y^{\theta_1})\| \\
& + \frac{1}{\Gamma(\sigma)} \int_0^{\theta_1} \left[e^{-\varrho(\theta_2-\kappa)}(\theta_2-\kappa)^{\sigma-1} - e^{-\varrho(\theta_1-\kappa)}(\theta_1-\kappa)^{\sigma-1} \right] \|h(\kappa)\| d\kappa \\
& + \frac{1}{\Gamma(\sigma)} \int_{\theta_1}^{\theta_2} e^{-\varrho(\theta_2-\kappa)}(\theta_2-\kappa)^{\sigma-1} \|h(\kappa)\| d\kappa \\
\leq\ & \|\varpi(0)\| \left\| e^{-\varrho\theta_2} - e^{-\varrho\theta_1} \right\| + \|\Phi(0,\varpi(0))\| \left\| e^{-\varrho\theta_2} - e^{-\varrho\theta_1} \right\| \\
& + \|\Phi(\theta_2, y^{\theta_2}) - \Phi(\theta_1, y^{\theta_1})\| + \frac{(\widehat{q_1}^* + \widehat{q_2}^* R)(\theta_2^\sigma - \theta_1^\sigma)}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)} \\
& + \frac{(\widehat{q_1}^* + \widehat{q_2}^* R)(\theta_2 - \theta_1)^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)}.
\end{aligned}$$

As $\theta_1 \longrightarrow \theta_2$, the right-hand side of the inequality above tend to zero. Thus, the operator S is equicontinuous. The equicontinuity in the other intervals is trivial. Thus we omit it.

Step 4: S is a contraction.
Let B be a subset of D_R. If $\theta \in [-r, 0]$, then

$$\begin{aligned}
\alpha(S(B(\theta))) &= \alpha\big\{Sy(\theta), \quad y \in B\big\} \\
&= \alpha\big\{\varpi(\theta), \quad y \in B\big\} \\
&= 0,
\end{aligned}$$

and if $\theta \in [T, T+\delta]$, then

$$\begin{aligned}
\alpha(S(B(\theta))) &= \alpha\big\{Sy(\theta), \quad y \in B\big\} \\
&= \alpha\big\{\widehat{\varpi}(\theta), \quad y \in B\big\} \\
&= 0.
\end{aligned}$$

For each $\theta \in [0, T]$, we have

$$\begin{aligned}
\alpha(S(B(\theta))) &= \alpha\big\{Sy(\theta), \quad y \in B\big\} \\
&= \alpha\Bigg\{ \varpi(0)e^{-\varrho\theta} + \Phi(0,\varpi(0))e^{-\varrho\theta} + \Phi(\theta, y^\theta)
\end{aligned}$$

$$+\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}h(\kappa)d\kappa,\quad y\in B\Big\}$$
$$\leq \alpha\big\{\Phi(\theta,y^\theta),\quad y\in B\big\}+\alpha\bigg\{\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}h(\kappa)d\kappa,\quad y\in B\bigg\}.$$

By condition $(A10)$, we have

$$\begin{aligned}\alpha(h(\theta)) &= \alpha(\Psi(\theta,y^\theta,h(\theta))\\ &\leq \widehat{\lambda}\sup_{\theta\in[-r,\delta]}\alpha(y^\theta)+\widehat{L}\alpha(h(\theta))\\ &\leq \widehat{\lambda}\sup_{\theta\in[-r,T+\delta]}\alpha(y(\theta))+\widehat{L}\alpha(h(\theta)).\end{aligned}$$

Thus,

$$\alpha(h(\theta))\leq\frac{\widehat{\lambda}}{1-\widehat{L}}\sup_{\theta\in[-r,T+\delta]}\alpha(y(\theta)),$$

and

$$\begin{aligned}\alpha(S(B(\theta)))&\leq\widehat{C}\sup_{\theta\in[-r,\delta]}\alpha(y^\theta)+\frac{\widehat{\lambda}}{1-\widehat{L}}\int_0^\theta(\theta-\kappa)^{\sigma-1}\{\sup_{\kappa\in[-r,T+\delta]}\alpha(y(\kappa))d\kappa,\quad y\in B\}\\ &\leq\widehat{C}\alpha_\Upsilon(B)+\frac{\widehat{\lambda}T^\sigma}{(1-\widehat{L})\Gamma(\sigma+1)}\alpha_\Upsilon(B).\end{aligned}$$

Therefore,

$$\alpha_\Upsilon(S(B))\leq\left[\widehat{C}+\frac{\widehat{\lambda}T^\sigma}{(1-\widehat{L})\Gamma(\sigma+1)}\right]\alpha_\Upsilon(B),$$

and by Remark 3 , we have

$$\alpha_\Upsilon(S(B))\leq\left[\widehat{p_1}^*+\frac{\widehat{q_2}^*T^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)}\right]\alpha_\Upsilon(B).$$

Thus, by Theorem (6), the operator S is a contraction. As a consequence of Darbo's fixed point theorem, we deduce that the operator S has a fixed point that is a solution of the problem (14)–(16). ■

5.1 Ulam-Hyers-Rassias Stability

In this part, we will establish the Ulam stability for the problem (14)–(16). Let $\omega\in C(\Theta,\mathbb{R}_+)$.

Definition 7 ([1]) *The problem (14)–(16) is Ulam-Hyers-Rassias stable with respect to ω if there exists $C_{\Psi,\omega}>0$ such that for each $\varepsilon>0$ and for each solution $y\in\Upsilon$ of the inequality*

$$\left\|{}^C_0D^\sigma_\theta\left[y(\theta)-\Phi(\theta,y^\theta)\right]-\Psi\left(\theta,y_\theta,\ {}^C_0D^\sigma_\theta y(\theta)\right)\right\|\leq\varepsilon\omega(\theta),\quad\theta\in\Theta,\tag{17}$$

there exists a solution $\bar{y}\in\Upsilon$ of the problem (14)–(16) with

$$\|y(\theta)-\bar{y}(\theta)\|\leq\varepsilon C_{\Psi,\omega}\ \omega(\theta),\quad\theta\in\Theta.$$

Remark 4 *A function $y\in\Upsilon$ is a solution of the inequality (17) if and only if there exists a function $\ell\in C(\Theta,E)$ (which depend on y) such that*

1. $\|\ell(\theta)\|\leq\varepsilon\omega(\theta),$ *for each $\theta\in\Theta$.*

2. ${}_0^C D_\theta^{\sigma,\varrho} y(\theta) = \Psi\left(\theta, y^\theta, {}_0^C D_\theta^{\sigma,\varrho} y(\theta)\right) + \ell(\theta),$ *for each* $\theta \in \Theta$.

Lemma 7 *The solution of the following perturbed problem*

$$ {}_0^C D_\theta^{\sigma,\varrho}\left[y(\theta) - \Phi(\theta, y^\theta)\right] = \Psi\left(\theta, y^\theta, {}_0^C D_\theta^{\sigma,\varrho} y(\theta)\right) + \ell(\theta), \quad \theta \in \Theta := [0,T], $$

$$ y(\theta) = \varpi(\theta), \quad \theta \in [-r, 0], $$

$$ y(\theta) = \widehat{\varpi}(\theta), \quad \theta \in [T, T+\delta], $$

is given by

$$ y(\theta) = \begin{cases} \varpi(0)e^{-\varrho\theta} + \Phi(0,\varpi(0))e^{-\varrho\theta} + \Phi(\theta, y^\theta) + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}h(\kappa)d\kappa \\ +\frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\ell(\kappa)d\kappa \qquad if\ \theta \in \Theta, \\ \varpi(\theta), \qquad if\ \theta \in [-r,0], \\ \widehat{\varpi}(\theta), \qquad if\ \theta \in [T, T+\delta]. \end{cases} $$

Moreover, the solution satisfies the following inequality

$$ \left\| y(\theta) - \left[\varpi(0)e^{-\varrho\theta} + \Phi(0,\varpi(0))e^{-\varrho\theta} + \Phi(\theta, y^\theta) + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}h(\kappa)d\kappa\right]\right\| $$
$$ \leq \varepsilon\lambda_\omega\omega(\theta), \quad for\ each\ \theta \in \Theta. $$

Theorem 7 *Assume that the conditions* $(A6)$*–*$(A9)$ *hold and that the condition (10) is verified. If*

(A12) *there exists a nondecreasing function* $\omega \in C(\Theta, \mathbb{R}_+)$, *and* $\lambda_\omega > 0$ *such that, for any* $\theta \in \Theta$

$$ {}_0I_\theta^{\sigma,\varrho}\omega(\theta) \leq \lambda_\omega \omega(\theta), $$

hold, then the problem (14)–(16) is Ulam-Hyers-Rassias stable.

Proof. Let $y \in \Upsilon$ be a solution of the inequality (17) and $\bar{y} \in \Upsilon$ the solution of the problem (14)–(16). Then

$$ \|y(\theta) - \bar{y}(\theta)\| \leq \lambda_\omega \varepsilon\omega(\theta) + \|\Phi(\theta, y^\theta) - \Phi(\theta, \bar{y}^\theta)\| $$
$$ + \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|h(\kappa) - \bar{h}(\kappa)\|d\kappa, $$

where h and $\bar{h}$ are two functions satisfying the following functional equations:

$$ h(\theta) = \Psi(\theta, y^\theta, h(\theta)), $$

and

$$ \bar{h}(\theta) = \Psi(\theta, \bar{y}^\theta, \bar{h}(\theta)). $$

From hypothesis $(A7)$, we have

$$ \begin{aligned} \|h(\theta) - \bar{h}(\theta)\| &= \|\Psi(\theta, y^\theta, h(\theta)) - \Psi(\theta, \bar{y}^\theta, \bar{h}(\theta))\| \\ &\leq \widehat{\lambda}\|y^\theta - \bar{y}^\theta\|_{[-r,\delta]} + \widehat{L}\|h(\theta) - \bar{h}(\theta)\|, \end{aligned} $$

which implies that

$$ \|h(\theta) - \bar{h}(\theta)\| \leq \frac{\widehat{\lambda}}{1-\widehat{L}}\|y^\theta - \bar{y}^\theta\|_{[-r,\delta]}. $$

Then,

$$\|y(\theta)-\bar{y}(\theta)\| \leq \lambda_\omega \varepsilon\omega(\theta)+\widehat{C}\|y-\bar{y}\|_\Upsilon+\frac{\widehat{\lambda}}{(1-\widehat{L})\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\|y^\kappa-\bar{y}^\kappa\|_{[-r,\delta]}d\kappa.$$

Thus,

$$\|y-\bar{y}\|_\Upsilon \leq \lambda_\omega \varepsilon\omega(\theta)+\widehat{C}\|y-\bar{y}\|_\Upsilon+\frac{\widehat{\lambda}T^\sigma}{(1-\widehat{L})\Gamma(\sigma+1)}\|y-\bar{y}\|_\Upsilon.$$

Finally, we get

$$\|y-\bar{y}\|_\Upsilon \leq \frac{\lambda_\omega \varepsilon\omega(\theta)}{1-\widehat{C}-\frac{\widehat{\lambda}T^\sigma}{(1-\widehat{L})\Gamma(\sigma+1)}}.$$

Then, for each $\theta\in\Theta$, we have

$$\|y-\bar{y}\|_\Upsilon \leq C_{\Psi,w}\ \varepsilon\omega(\theta),$$

where

$$C_{\Psi,w}=\frac{\lambda_\omega}{1-\widehat{C}-\frac{\widehat{\lambda}T^\sigma}{(1-\widehat{L})\Gamma(\sigma+1)}}.$$

Consequently, the problem (14)–(16) is Ulam-Hyers-Rassias stable with respect to ω. ■

6 An Example

Set

$$E=l^1=\left\{y=(y_1,y_2,\ldots,y_\beta,\ldots),\sum_{\beta=1}^{\infty}|y_\beta|<\infty\right\},$$

where E is a Banach space with the norm $\|y\|=\sum_{\beta=1}^{\infty}|y_\beta|$. Consider the following implicit problem:

$$ {}_0^CD_\theta^{\frac{1}{2},3}\left[y(\theta)-\Phi(\theta,y^\theta)\right]=\Psi\left(\theta,y^\theta,\ {}_0^CD_\theta^{\frac{1}{2},3}y(\theta)\right),\ \theta\in[0,1], \tag{18}$$

$$y(\theta)=\varpi(\theta),\quad \theta\in[-1,0], \tag{19}$$

$$y(\theta)=\widehat{\varpi}(\theta),\quad \theta\in[1,2], \tag{20}$$

where $\varpi\in C([-1,0],E)$ and $\widehat{\varpi}\in C([1,2],E)$. Set

$$\Psi_\beta\left(\theta,y_\beta^\theta,\ {}_0^CD_\theta^{\frac{1}{2},3}y_\beta(\theta)\right)=\frac{9e^{7t}+\cos\left(\|y^\theta\|_{[-r,\delta]}\right)+\cos\left(\left|{}_0^CD_\theta^{\frac{1}{2},1}y_\beta(\theta)\right|\right)}{100e^{\theta+11}\left(1+\|y^\theta\|_{[-r,\delta]}+\left\|{}_0^CD_\theta^{\frac{1}{2},1}y(\theta)\right\|\right)},$$

and

$$\Phi_\beta(\theta,y_\beta^\theta)=\frac{\sqrt{2\pi}\|y^\theta\|_{[-r,\delta]}+ln(e+\theta)}{77},$$

for $\theta\in[0,1],\ y\in C([-r,\delta],E)$, where

$$y=(y_1,y_2,\ldots,y_\beta,\ldots),$$
$$\Psi=(\Psi_1,\Psi_2,\ldots,\Psi_\beta,\ldots),$$
$$\Phi=(\Phi_1,\Phi_2,\ldots,\Phi_\beta,\ldots),$$

and

$${}_0^CD_\theta^{\frac{1}{2},3}y=\left({}_0^CD_\theta^{\frac{1}{2},3}y_1\ ,\ {}_0^CD_\theta^{\frac{1}{2},3}y_2\ ,\ldots,\ {}_0^CD_\theta^{\frac{1}{2},3}y_\beta,\ldots\right).$$

Clearly, Ψ and Φ are continuous functions, then the hypothesis $(A6)$ is satisfied.

For any $\chi, \bar{\chi} \in C([-r,\delta], E)$, $\Im, \bar{\Im} \in E$ and $\theta \in [0,1]$, we have

$$\|\Psi(\theta,\chi,\Im) - \Psi(\theta,\bar{\chi},\bar{\Im})\| \leq \frac{1}{100e^{11}}\left[\|\chi-\bar{\chi}\|_{[-r,\delta]} + \|\Im-\bar{\Im}\|\right],$$

and

$$\|\Phi(\theta,\chi) - \Phi(\theta,\bar{\chi})\| \leq \frac{\sqrt{2\pi}}{77}\,\|\chi-\bar{\chi}\|_{[-r,\delta]}.$$

Then, the assumptions $(A7)$, $(A8)$ are satisfied with $\widehat{\lambda} = \widehat{L} = \frac{1}{100e^{11}}$ and $\widehat{C} = \frac{\sqrt{2\pi}}{77}$. Also we have

$$\|\Psi(\theta,\chi,\Im)\| \leq \frac{1}{100e^{\theta+11}}\left[9e^{7t} + \|\chi\|_{[-r,\delta]} + \|\Im\|\right],$$

and

$$\|\Phi(\theta,\chi)\| \leq \frac{\sqrt{2\pi}}{77}\|\chi\|_{[-r,\delta]} + \frac{ln(e+\theta)}{77}.$$

So $\widehat{q_1}(\theta) = \frac{9e^{7t}}{100e^{\theta+11}}$, $\widehat{q_2}^* = \widehat{q_3}^* = \frac{1}{100e^{11}}$, $\widehat{p_1}^* = \frac{\sqrt{2\pi}}{77}$ and $\widehat{p_2}(\theta) = \frac{ln(e+\theta)}{77}$. Moreover, we have

$$\lim_{\theta_1 \longrightarrow \theta_2}\left(\Phi(\theta_2, y^{\theta_2}) - \Phi(\theta_1, y^{\theta_1})\right) \longrightarrow 0 \quad as\ \theta_1 \longrightarrow \theta_2.$$

Thus, the hypothesis $(A9)$ is verified, and since the condition of Theorem 6

$$\begin{aligned}\widehat{p_1}^* + \frac{\widehat{q_2}^* T^\sigma}{(1-\widehat{q_3}^*)\Gamma(\sigma+1)} &= \frac{\sqrt{2\pi}}{77} + \frac{2}{(100e^{11}-1)\sqrt{\pi}} \\ &\approx 0.0325538024147905 \\ &< 1,\end{aligned}$$

is satisfied, then the problem (18)–(20) has at least one solution.

For any $\theta \in [0,1]$, we take $\omega(\theta) = e^{\sqrt{8}}$, then

$${}_0I_\theta^{\sigma,\varrho}\omega(\theta) = \frac{1}{\Gamma(\sigma)}\int_0^\theta e^{-\varrho(\theta-\kappa)}(\theta-\kappa)^{\sigma-1}\omega(\kappa)d\kappa.$$

So

$${}_0I_\theta^{\frac{1}{2},3}(e^{\sqrt{8}}) \leq \frac{2}{\Gamma\left(\frac{3}{2}\right)}(e^{\sqrt{8}}) = \frac{2}{\sqrt{\pi}}(e^{\sqrt{8}}) := \frac{2}{\sqrt{\pi}}\omega(\theta).$$

Hence, the condition $(A12)$ is satisfied with $\lambda_\omega = \frac{2}{\sqrt{\pi}}$. It follows from Theorem 7 that the problem (18)–(20) is Ulam-Hyers-Rassias stable with respect to ω.

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