Applied Mathematics E-Notes, 24(2024), 113-117 © ISSN 1607-2510
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Analogue Of The Cauchy-Schwarz Inequality For Determinants: A Simple Proof*

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Received 28 February, 2024

Abstract

In this note, we present a simple proof of an analogue of the Cauchy-Schwarz inequality relevant to products of determinants. Specifically, we show that

$$|\det(A^*MB)|^2 \le \det(A^*MA)\cdot\det(B^*MB), \quad A,B\in\mathbb{C}^{m\times n},$$

where $M\in\mathbb{C}^{m\times m}$ is hermitian positive definite. Here m and n are arbitrary. In case $m\le n$, equality holds trivially. Equality holds when $m>n$ and $\operatorname{rank}(A)=\operatorname{rank}(B)=n$ if and only if the columns of A and the columns of B span the same subspace of $\mathbb{C}^m$.

1 Introduction

The Cauchy–Schwarz inequality for vectors in $\mathbb{C}^n$ states that, if $x,y\in\mathbb{C}^n$, and (a,b) is the inner product in $\mathbb{C}^n$, then

$$|(x,y)|^2 \le (x,x)\cdot(y,y),$$

with equality if and only if x and y are linearly dependent. There are many analogues of this theorem in different settings, and these can be found in many books and papers on linear algebra and related subjects. In this work, we prove an analogue of this inequality that concerns determinants. Our result here also follows from Marcus and Moore [2]. (See the remarks at the end of this note.)

We begin with the following lemma:

Lemma 1 *Let U and V be two rectangular unitary matrices in $\mathbb{C}^{m\times n}$ with $m>n$, in the sense*

$$U^*U=I_n, \quad V^*V=I_n.$$

Then

$$|\det(U^*V)|\le 1. \tag{1}$$

Equality holds if and only if the columns of U and the columns of V span the same subspace of $\mathbb{C}^m$.

Proof. The matrices U and V have the following columnwise partitionings:

$$U=[u_1|u_2|\cdots|u_n], \quad u_i^*u_j=\delta_{i,j}; \quad V=[v_1|v_2|\cdots|v_n], \quad v_i^*v_j=\delta_{i,j}.$$

Then $W=U^*V\in\mathbb{C}^{n\times n}$ and the (i,j) element of W is $u_i^*v_j$. Therefore, U^*V has the columnwise partitioning

$$U^*V=W=[w_1|w_2|\cdots|w_n]; \quad w_j=[u_1^*v_j,u_2^*v_j,\ldots,u_n^*v_j]^T, \quad j=1,2,\ldots,n, \tag{2}$$

and

$$\|w_j\|^2=\sum_{i=1}^n|u_i^*v_j|^2, \quad j=1,2,\ldots,n.^1 \tag{3}$$

*Mathematics Subject Classifications: 15A15.

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[1] We denote by $\|z\|$, $z\in\mathbb{C}^q$, the standard l_2 norm of z, in every dimension q. Thus $\|z\|=\sqrt{z^*z}$ for all q.

Thus, by Hadamard's inequality[2], there holds

$$|\det(U^*V)| = |\det W| \le \prod_{j=1}^{n} \|w_j\|. \tag{4}$$

We add to the set of the n (orthonormal) vectors $u_1, u_2, \dots, u_n$ the $m-n$ (orthonormal) vectors $u_{n+1}, u_{n+2}, \dots, u_m$, such that $u_1, \dots, u_m$ is an orthonormal basis for $\mathbb{C}^m$, hence for every vector $x \in \mathbb{C}^m$, we have

$$x = \sum_{i=1}^{m} (u_i^* x) u_i \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{m} |u_i^* x|^2.$$

Next, we define

$$S_U = \operatorname{span}\{u_1, u_2, \dots, u_n\} \quad \text{and} \quad S_U^\perp = \operatorname{span}\{u_{n+1}, u_{n+2}, \dots, u_m\}.$$

Here S_U is the column space of U and $S_U^\perp$ is the orthogonal complement of S_U, and every vector $x \in \mathbb{C}^m$ is of the form

$$x = \hat{x} + \tilde{x}; \quad \hat{x} = \sum_{i=1}^{n} (u_i^* x) u_i \in S_U, \quad \tilde{x} = \sum_{i=n+1}^{m} (u_i^* x) u_i \in S_U^\perp; \quad \hat{x}^* \tilde{x} = 0,$$

therefore,

$$\|x\|^2 = \|\hat{x}\|^2 + \|\tilde{x}\|^2; \quad \|\hat{x}\|^2 = \sum_{i=1}^{n} |u_i^* x|^2, \quad \|\tilde{x}\|^2 = \sum_{i=n+1}^{m} |u_i^* x|^2.$$

Then, for v_j, the j^{th} column of the matrix V, we have $v_j = \hat{v}_j + \tilde{v}_j$, where

$$\|\hat{v}_j\|^2 = \sum_{i=1}^{n} |u_i^* v_j|^2 = \|w_j\|^2 \quad \text{by (3)}, \quad \|\tilde{v}_j\|^2 = \sum_{i=n+1}^{m} |u_i^* v_j|^2.$$

Therefore,

$$1 = \|v_j\|^2 = \|w_j\|^2 + \|\tilde{v}_j\|^2 \quad \Rightarrow \quad \|w_j\| \le \|v_j\| = 1; \quad j = 1, 2, \dots, n. \tag{5}$$

This forces $\prod_{j=1}^{n} \|w_j\| \le 1$ in (4), thus proving (1).

We denote the column space of V by S_V. That is,

$$S_V = \operatorname{span}\{v_1, v_2, \dots, v_n\}.$$

Assume that $S_V \ne S_U$. Then, at least one of the vectors $v_1, v_2, \dots, v_n$, say v_p, does not belong to S_U, and this implies that

$$v_p = \hat{v}_p + \tilde{v}_p, \quad \tilde{v}_p \ne 0.$$

As a result,

$$1 = \|v_p\|^2 = \|w_p\|^2 + \|\tilde{v}_p\|^2 > \|w_p\|^2 \quad \Rightarrow \quad \|w_p\| < \|v_p\| = 1. \tag{6}$$

As a result of (5) and (6), we have $\prod_{j=1}^{n} \|w_j\| < 1$ in (4), which forces strict inequality in (1).

We now show that equality holds in (1) when $S_U = S_V$. In this case, each v_i is a linear combination of the u_j. That is,

$$V = U\Sigma, \quad \text{for some } \Sigma \in \mathbb{C}^{n \times n}.$$

Consequently,

$$I_n = V^*V = (U\Sigma)^*(U\Sigma) = \Sigma^*(U^*U)\Sigma = \Sigma^*\Sigma \quad \Rightarrow \quad \Sigma^*\Sigma = I_n,$$

[2] Hadamard's inequality: If H is an $s \times s$ matrix with columnwise partitioning $H = [h_1 | h_2 | \cdots | h_s]$, then

$$|\det H| \le \prod_{j=1}^{s} \|h_j\|, \quad \|h_j\| = \sqrt{h_j^* h_j}, \quad j = 1, \dots, s.$$

A proof of Hadamard's inequality can be found in Horn and Johnson [1, p. 477].

that is, Σ is a unitary matrix in $\mathbb{C}^{n\times n}$. Consequently,

$$U^*V = U^*(U\Sigma) = (U^*U)\Sigma = \Sigma \quad \Rightarrow \quad \det(U^*V) = \det\Sigma,$$

from which,

$$|\det(U^*V)| = |\det\Sigma| = 1.$$

This completes the proof. ∎

2 Main Result

Theorem 1 *Let $A, B \in \mathbb{C}^{m\times n}$, with m, n arbitrary. Then*

$$|\det(A^*B)|^2 \leq \det(A^*A)\cdot\det(B^*B). \tag{7}$$

1. *If $m < n$, equality holds in (7), both sides vanishing there.*
2. *If $m = n$, equality holds in (7).*
3. *(a) If $m > n$ and $\operatorname{rank}(A) < n$ or $\operatorname{rank}(B) < n$, equality holds in (7), both sides vanishing there.*

   (b) If $m > n$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, equality holds in (7) if and only if the columns of A and the columns of B span the same subspace of $\mathbb{C}^m$.

Proof. We start by noting that all three matrices A^*A, B^*B, and A^*B are $n\times n$.

1. If $m < n$, each one of the matrices A^*A, B^*B, and A^*B is of rank at most m, hence is singular. Therefore, (7) holds, both sides vanishing there.
2. If $m = n$, then A and B are square. Therefore,

   $$\det(A^*A) = (\det A^*)(\det A), \quad \det(B^*B) = (\det B^*)(\det B)$$

   and

   $$\det(A^*B) = (\det A^*)(\det B).$$

   The result in (7) now follows with equality there by invoking $\det C^* = \overline{\det C}$ for every square matrix C.
3. (a) If $m > n$, and either $\operatorname{rank}(A) < n$ (hence A^*A is singular) or $\operatorname{rank}(B) < n$ (hence B^*B is singular), we have that A^*B is singular as well. Therefore, (7) holds with equality, both sides vanishing there.

   (b) If $m > n$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, we proceed as follows:
   Consider the QR factorizations of A and B, namely,

   $$A = Q_AR_A, \quad B = Q_BR_B; \quad Q_A, Q_B \in \mathbb{C}^{m\times n}, \quad R_A, R_B \in \mathbb{C}^{n\times n},$$

   where Q_A and Q_B are unitary in the sense

   $$Q_A^*Q_A = I_n, \quad Q_B^*Q_B = I_n,$$

   and R_A and R_B are upper triangular square matrices with nonzero diagonal elements. Now, it is easy to verify that

   $$A^*A = R_A^*R_A, \quad B^*B = R_B^*R_B, \quad A^*B = R_A^*(Q_A^*Q_B)R_B.$$

   Note that $Q_A^*Q_B \in \mathbb{C}^{n\times n}$, just as R_A, R_B. Therefore,

   $$\det(A^*A) = (\det R_A^*)(\det R_A) = |\det R_A|^2,$$

and

$$\det(B^*B) = (\det R_B^*)(\det R_B) = |\det R_B|^2.$$

Next,

$$\det(A^*B) = (\det(Q_A^*Q_B))(\det R_A^*)(\det R_B).$$

Therefore,

$$|\det(A^*B)| = |\det(Q_A^*Q_B)|\,|\det R_A|\,|\det R_B|,$$

which implies that

$$|\det(A^*B)|^2 = |\det(Q_A^*Q_B)|^2 \det(A^*A)\det(B^*B).$$

The rest of the proof can now be achieved (i) by realizing that the column spaces of A and Q_A are the same, and so are the column spaces of B and Q_B, and (ii) by applying Lemma 1 to $|\det(Q_A^*Q_B)|$ since $m > n$. We leave the details to the reader.

This completes the proof. ∎

By applying Theorem 1 to the matrices $\tilde{A} = M^{1/2}A$ and $\tilde{B} = M^{1/2}B$, where $M \in \mathbb{C}^{m\times m}$ is hermitian positive definite, we obtain the following general form of Theorem 1.

Theorem 2 *Let $A, B \in \mathbb{C}^{m\times n}$, with m, n arbitrary, and let $M \in \mathbb{C}^{m\times m}$ be hermitian positive definite. Then*

$$|\det(A^*MB)|^2 \leq \det(A^*MA)\cdot\det(B^*MB). \tag{8}$$

1. *If $m < n$, equality holds in (8), both sides vanishing there.*
2. *If $m = n$, equality holds in (8).*
3. *(a) If $m > n$ and $\operatorname{rank}(A) < n$ or $\operatorname{rank}(B) < n$, equality holds in (8), both sides vanishing there.*

   (b) If $m > n$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, equality holds in (8) if and only if the columns of A and the columns of B span the same subspace of $\mathbb{C}^m$.

Proof. We first note that, because $M^{1/2}$ is nonsingular, $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(A)$ and $\operatorname{rank}(\tilde{B}) = \operatorname{rank}(B)$. Therefore, parts 1, 2, and 3(a) of the theorem follow immediately from parts 1, 2, and 3(a) of Theorem 1.

As for part 3(b) of the theorem concerning the case $m > n$ and $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, we first observe that $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(\tilde{B}) = n$ as well, and, by part 3(b) of Theorem 1, equality holds in (8) if and only if the columns of $\tilde{A} = M^{1/2}A$ and the columns of $\tilde{B} = M^{1/2}B$ span the same n-dimensional subspace $\tilde{\mathbb{X}}$ of $\mathbb{C}^m$, that is, if and only if

$$\tilde{B} = \tilde{A}F \quad \text{for some nonsingular } F \in \mathbb{C}^{n\times n}. \tag{9}$$

We now need only to show that this is possible if and only if the columns of A and the columns of B span the same n-dimensional subspace $\mathbb{X}$ of $\mathbb{C}^m$. Upon multiplying both sides of the equality in (9) by $M^{-1/2}$, we also obtain

$$M^{-1/2}\tilde{B} = M^{-1/2}\tilde{A}F \quad \Rightarrow \quad B = AF,$$

which implies that the columns of A and the columns of B span the same n-dimensional subspace $\mathbb{X}$ of $\mathbb{C}^m$. We have thus shown that $\tilde{B} = \tilde{A}F \Leftrightarrow B = AF$. This completes the proof. ∎

Remark 1 *The proof of the Cauchy–Schwarz inequality in $\mathbb{C}^n$ makes use of the fact that the inner product (x, y) in $\mathbb{C}^n$ is bilinear in x and y. Because X^*MY is bilinear in X and Y, one might think that the proof of the Cauchy–Schwarz inequality in $\mathbb{C}^n$ can be applied to $\det(A^*MB)$, $\det(A^*MA)$, and $\det(B^*MB)$ to obtain (8). This is not possible, however, since $\det(A^*MB)$ does not have the bilinearity property; for example, $\det((A_1 + A_2)^*MB)$ is not necessarily equal to $\det(A_1^*MB) + \det(A_2^*MB)$.*

Remark 2 *The problem treated here is a special case of a general problem treated by Marcus and Moore in [2, Eq. (1) on p. 111 and Theorem on p. 115]. These authors use advanced techniques to obtain the relations which must exist between the $m \times m$ matrices M_1, M_2, M_3, M_4 so that the relation*

$$\det(A^* M_1 B) \det(B^* M_2 A) \leq \det(A^* M_3 A) \det(B^* M_4 B)$$

holds for all $m \times n$ matrices A and B when $m > n$.

Because we are restricting our problem to the special case in which $M_1 = M_2 = M_3 = M_4 = M$, M being an $m \times m$ positive definite hermitian matrix, we are able to carry out our analysis by employing rather elementary techniques of linear algebra that are easily accessible.

References

[1] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[2] M. Marcus and K. Moore, A determinant formulation of the Cauchy-Schwarz inequality, Linear Algebra Appl., 36(1981), 111–127.