Applied Mathematics E-Notes, 24(2024), 350-361 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Generalized Hermite Hadamard Type Inequalities For Twice Differentiable (r,s)-Convex Functions*

Muhammad Bilal†, Asif Raza Khan‡

Received 12 September 2023

Abstract

In this research investigation, we present comprehensive findings pertaining to the estimated upper bounds of the Hermite Hadamard dual inequality for functions characterized as (r,s)-convex. These findings are derived through distinct methodologies, including the utilization of Hölder's and Power Mean integral inequalities. Consequently, our study encompasses a wide spectrum of both well-established and novel outcomes as specific instances. Furthermore, we delve into the application of these results to specialized means.

1 Introduction

In the realm of mathematics, the field of inequalities continues to captivate researchers, driven by its myriad practical applications. Within this domain, particular emphasis has been placed on the theory of convex functions over the past few decades, as it encompasses a diverse range of essential mathematical concepts. Convexity finds wide-ranging utility in everyday life, extending its influence into disciplines such as the arts, architecture, industry, management science, economics, and more. Among the various applications of convexity, the Hermite-Hadamard type inequality stands out prominently, thanks to its numerous practical implications, particularly in relation to (r,s)-convex functions. For an in-depth exploration of this topic, please refer to [2, 3, 4, 6] and [9]–[11], along with the references cited therein.

Before we proceed any further, it is important to acknowledge that this article adheres to specific notation:

1. E denotes any real interval,
2. E° is the interior of E,
3. and $\beta(a,b)=\int_0^1 v^{a-1}(1-v)^{b-1}dv, \quad a,b>0$ is the famous Euler Beta function.

Additionally, it is worth noting that in the course of this article, we consistently employ the convention that $0^0=1$.

We will commence by introducing noteworthy concepts and established findings:

Theorem 1 ([8]) *Let $f:E\to\mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(u)du\le\frac{f(a)+f(b)}{2}. \tag{1}$$

The aforementioned outcome is renowned in mathematical circles as the Hermite Hadamard dual inequality.

*Mathematics Subject Classifications: 26A46, 26A51, 26D07, 26D99.

†Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan
‡Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan

Remark 1 *It is important to note that if f is a concave function, then both inequalities will be reversed in their order. Furthermore, Equation (1) can be considered as a special case of refinement of Jensen's inequality.*

Here, we are going to recall the definitions of convex function, s-convex functions of 1^{st} and 2^{nd} kinds.

Definition 1 ([1]) *A function $f : E \to \mathbb{R}$ is said to be convex, if f satisfies the below mentioned inequality*

$$f(vx + (1-v)y) \leq vf(x) + (1-v)f(y),$$

$\forall\ x, y \in E$ *and* $v \in [0,1]$.

Definition 2 ([6]) *A function $f : E \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the 1^{st} kind, if f satisfies the below mentioned inequality*

$$f(vx + (1-v)y) \leq v^s f(x) + (1-v^s)f(y),$$

$\forall\ x, y \in E$, $v \in [0,1]$ *and* $s \in [0,1]$.

Remark 2 *If we replace $s = 1$ and $s = 0$ in Definition 2, then we acquire Definition 1 and refinement of quasi convex function [6], respectively.*

Definition 3 ([6]) *A function $f : E \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the 2^{nd} kind, if f satisfies the below mentioned inequality*

$$f(vx + (1-v)y) \leq v^s f(x) + (1-v)^s f(y),$$

$\forall\ x, y \in E, v \in [0,1]$ *and* $s \in [0,1]$.

Remark 3 *If we replace $s = 1$ and $s = 0$ in Definition 3, then we acquire Definition 1 and definition of $P-$convex function [7], respectively.*

The generalization of the definitions 2 and 3 is known as (r,s)-convex functions which we recall here as:

Definition 4 ([9]) *A function $f : E \subset (0,\infty) \to [0,\infty)$ is said to be the (r,s)-convex function, if f satisfies the below mentioned inequality*

$$f(vx + (1-v)y) \leq v^{rs} f(x) + (1-v^r)^s f(y), \tag{2}$$

$\forall\ x, y \in E$, $v \in [0,1]$ *and* $(s,r) \in [0,1]^2$.

Remark 4 *By altering the values of both r and s in various combinations, we can encompass the following widely recognized definitions:*

1. *If we replace $s = 1$ in (2), then we acquire the Definition 2.*
2. *If we replace $r = 1$ in (2), then we acquire the Definition 3.*
3. *If we replace $r = s = 1$ in (2), then we acquire the Definition 1.*
4. *If we replace $r = 0$ in (2), then we acquire refinement of quasi convex function [6].*
5. *If we replace $s = 0$ in (2), then we acquire P-convex function [7].*

The general form of renowned classical Hölder's integral inequality is given as follows [10]:

Theorem 2 *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p$ and $\phi \in L_q$, then $f\phi \in L_1$ and*

$$\int |f(u)\phi(u)|du \leq \|f\|_p \|\phi\|_q, \tag{3}$$

where $f \in L_p$ *if* $\|f\|_p = \left(\int |f(u)|^p du\right)^{\frac{1}{p}} < \infty$.

Remark 5 *It is to be noted that (3) becomes the Cauchy Schwarz inequality if we replace $p = q = 2$. Further, if we replace $q = 1$ forcing $p \to \infty$, then we acquire,*

$$\int |f(u)\phi(u)|du \leq \|f\|_\infty \|\phi\|_1$$

where $||f||_\infty$ denotes the essential supremum of $|f|$, i.e.,

$$\|f\|_\infty = ess \sup_{\forall u} |f(u)|.$$

Another representation of (3) is known in literature as power mean inequality (see [12]), stated as:

Proposition 1 *Let f and ϕ be real valued functions defined on E and if $|f|$ and $|f||\phi|^q$ are integrable on E, then for $q \geq 1$ we have:*

$$\int_a^b |f(u)||\phi(u)|du \leq \left(\int_a^b |f(u)|du\right)^{1-\frac{1}{q}} \left(\int_a^b |f(u)||\phi(u)|^q du\right)^{\frac{1}{q}}. \quad (4)$$

Now, we are going to recall the below stated identity extracting from [2] which will be used to derive our main results of this article.

Lemma 1 ([2]) *Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on E° and with $f'' \in L[a, b]$. Then the below mentioned identity holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du = \frac{(b-a)^2}{2}\int_0^1 v(1-v)f''(va + (1-v)b)dv.$$

The structure of this article is as follows: The initial section is dedicated to presenting preliminary results. In the subsequent section, we focus on estimating the upper bound of the well-known Hermite Hadamard inequality for the category of twice differentiable (r, s)-convex functions. These outcomes encompass various findings outlined in [2] and [11] as specific instances. The third section delves into applications related to specialized means. The fourth section is entirely devoted to concluding remarks, while the final section provides additional insights and future prospects for interested readers and researchers.

2 Various Estimations of Right bound of Hermite Hadamard Inequality for Twice Differentiable (r, s)-Convex Functions

Right now, we have been able to state and prove three generalized results related to Hermite Hadamard type inequalities for twice differentiable (r, s)-convex functions using Definition 4, Theorem 2, Proposition 1 and Lemma 1.

Theorem 3 *Let $f : E \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on E° such that $f'' \in L[a, b]$, where $a, b \in E$ and $a < b$. If $|f''|$ is (r, s)-convex on E for some fixed $s \in [0, 1]$, then below stated inequality holds:*

1. *For $r \in (0, 1]$*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{2}\times$$
$$\left[\frac{|f''(a)|}{(rs+2)(rs+3)} + \frac{|f''(b)|}{r}\left\{\beta\left(\frac{2}{r}, s+1\right) - \beta\left(\frac{3}{r}, s+1\right)\right\}\right]. \quad (5)$$

2. *For* $r = 0$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{(b-a)^2}{4}|f''(a)|. \tag{6}$$

Proof. Using Lemma 1 and the definition of absolute value, we have,

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{(b-a)^2}{2}\int_0^1 |v(1-v)||f''(va+(1-v)b)|dv. \tag{7}$$

As we have $|f''|$ is a (r,s)-convex function, so we can take,

$$|f''(va+(1-v)b)| \leq v^{rs}|f''(a)| + (1-v^r)^s|f''(b)|.$$

Utilizing the above result (7) becomes,

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right|$$
$$\leq \frac{(b-a)^2}{2}\left[|f''(a)|\int_0^1 v^{rs+1}(1-v)dv + |f''(b)|\int_0^1 v(1-v)(1-v^r)^s dv \right].$$

After using the following facts, the result of Theorem 3 is accomplished.

$$\int_0^1 v^{rs+1}(1-v)dv = \frac{1}{(rs+2)(rs+3)}$$

and

$$\int_0^1 v(1-v)(1-v^r)^s dv = \frac{1}{r}\left\{ \beta\left(\frac{2}{r}, s+1\right) - \beta\left(\frac{3}{r}, s+1\right) \right\}.$$

■

Remark 6 *If someone replaces* $r = s = 1$ *in Theorem 3, then one has Remark 1 of [11].*

Corollary 1 *In Theorem 3, one can acquire the following results:*

1. *If someone replaces* $s = 1$*, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable* s*-convex functions in* 1*st kind:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{(b-a)^2}{12}\left[\frac{6|f''(a)| + s(s+5)|f''(b)|}{(s+2)(s+3)} \right].$$

2. *If someone replaces* $r = 1$*, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable* s*-convex functions in* 2*nd kind:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{(b-a)^2}{2}\left[\frac{|f''(a)| + |f''(b)|}{(s+2)(s+3)} \right].$$

3. *If someone replaces $s = 0$, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable P-convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{12}\left[|f''(a)| + |f''(b)|\right].$$

Theorem 4 *Let $f : E \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on E° such that $f'' \in L[a,b]$, where $a, b \in E$ and $a < b$. If $|f''|^q$ is (r,s)-convex on E for some fixed $s \in [0,1]$ and $q \geq 1$ then the below stated inequality holds:*

1. *For $r \in (0,1]$*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(b-a)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|f''(a)|^q}{(rs+1)} + \frac{|f''(b)|^q}{r}\beta\left(\frac{1}{r}, s+1\right)\right]^{\frac{1}{q}}. \tag{8}$$

2. *For $r = 0$*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}|f''(a)|.$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 1 and the definition of absolute value, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{2}\int_0^1 |v(1-v)||f''(va+(1-v)b)|dv. \tag{9}$$

Applying (3) to $\int_0^1 |v(1-v)||f''(va+(1-v)b)|dv$ implies

$$\int_0^1 |v(1-v)||f''(va+(1-v)b)|dv \leq \left(\int_0^1 |v(1-v)|^p dv\right)^{\frac{1}{p}}\left(\int_0^1 |f''(va+(1-v)b)|^q dv\right)^{\frac{1}{q}}.$$

As we have $|f''|^q$ is a (r,s)-convex function, so we can take,

$$|f''(va+(1-v)b)|^q \leq v^{rs}|f''(a)|^q + (1-v^r)^s|f''(b)|^q.$$

Utilizing the above two facts (9) becomes

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 v^p(1-v)^p dv\right)^{\frac{1}{p}}\left(|f''(a)|^q\int_0^1 v^{rs}dv + |f''(b)|^q\int_0^1 (1-v^r)^s dv\right)^{\frac{1}{q}}.$$

After using the following facts, the result of Theorem 4 is accomplished.

$$\int_0^1 v^p(1-v)^p dv = \beta(p+1, p+1),$$

$$\int_0^1 v^{rs} dv = \frac{1}{(rs+1)}$$

and

$$\int_0^1 (1-v^r)^s dv = \frac{1}{r}\beta\left(\frac{1}{r}, s+1\right).$$

∎

Remark 7 *In Theorem 4, one can capture the following results:*

1. *If someone replaces $r = s = 1$, then one has Corollary 2 of [11].*
2. *If someone replaces $r = 1$, then one has Theorem 10 of [2].*

Corollary 2 *In Theorem 4, one can acquire the following results:*

1. *If someone replaces $s = 1$, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable s-convex functions in 1st kind:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|f''(a)|^q + s|f''(b)|^q}{s+1}\right]^{\frac{1}{q}}.$$

2. *If someone replaces $s = 0$, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable P-convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[|f''(a)|^q + |f''(b)|^q\right]^{\frac{1}{q}}.$$

Theorem 5 *Let $f : E \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on E° such that $f'' \in L[a,b]$, where $a, b \in E^\circ$ and $a < b$. If $|f''|^q$, $q \geq 1$ is (r,s)-convex on E for some fixed $s \in [0,1]$ then the below stated inequality holds:*

1. *For $r \in (0,1]$*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{2\times(6)^{1-\frac{1}{q}}}\times$$
$$\left[\frac{|f''(a)|^q}{(rs+2)(rs+3)} + \frac{|f''(b)|^q}{r}\left\{\beta\left(\frac{2}{r}, s+1\right) - \beta\left(\frac{3}{r}, s+1\right)\right\}\right]^{\frac{1}{q}}. \tag{10}$$

2. *For $r = 0$*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{4\times(3)^{1-\frac{1}{q}}}|f''(a)|. \tag{11}$$

Proof. Using Lemma 1 and the definition of absolute value, we have,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(u)du\right| \leq \frac{(b-a)^2}{2}\int\limits_0^1 |v(1-v)||f''(va+(1-v)b)|dv. \tag{12}$$

Applying (4) to $\int\limits_0^1 |v(1-v)||f''(va+(1-v)b)|dv$ implies

$$\begin{aligned}&\int\limits_0^1 |v(1-v)||f''(va+(1-v)b)|dv \\ &\leq \left(\int\limits_0^1 |v(1-v)|dv\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 |v(1-v)||f''(va+(1-v)b)|^q dv\right)^{\frac{1}{q}}.\end{aligned}$$

As we have $|f''|^q$ is a (r,s)-convex function, so we can take,

$$|f''(va+(1-v)b)|^q \leq v^{rs}|f''(a)|^q+(1-v^r)^s|f''(b)|^q.$$

Utilizing the above result (12) becomes

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(u)du\right| \leq\ & \frac{(b-a)^2}{2}\left(\int\limits_0^1 |v(1-v)|dv\right)^{1-\frac{1}{q}} \\ & \times\left[|f''(a)|^q\int\limits_0^1 v^{rs+1}(1-v)dv+|f''(b)|^q\int\limits_0^1 v(1-v)(1-v^r)^s dv\right]^{\frac{1}{q}}.\end{aligned}$$

After using the following facts, the result of Theorem 5 is accomplished.

$$\int\limits_0^1 v(1-v)dv=\frac{1}{6},$$

$$\int\limits_0^1 v^{rs+1}(1-v)dv=\frac{1}{(rs+2)(rs+3)}$$

and

$$\int\limits_0^1 v(1-v)(1-v^r)^s dv=\frac{1}{r}\left\{\beta\left(\frac{2}{r},s+1\right)-\beta\left(\frac{3}{r},s+1\right)\right\}.$$

■

Remark 8 *If someone replaces $r=1$ in Theorem 5, then one has Theorem 8 of [2].*

Corollary 3 *In Theorem 5, one can acquire the following results:*

1. *If someone replaces $s=1$, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable s-convex functions in 1st kind:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(u)du\right| \leq \frac{(b-a)^2}{12}\left[\frac{6|f''(a)|^q+s(s+5)|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}.$$

2. *If someone replaces $r = s = 1$, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{12}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{\frac{1}{q}}.$$

3. *If someone replaces $s = 0$, then one has obtained the below mentioned result related to Hermite Hadamard type inequality for twice differentiable P-convex functions:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)^2}{12}\left[|f''(a)|^q + |f''(b)|^q\right]^{\frac{1}{q}}.$$

3 Application to Special Means

In the subsequent section, we will elucidate the relationships between various established means and the outcomes delineated in the preceding section. To accomplish this objective, we commence with the definitions of certain specialized means, carefully extracted from [5]:

1. The Geometric mean:

$$G = G(a,b) = \sqrt{ab};\ a, b \geq 0.$$

2. The Harmonic mean:

$$H = H(a,b) = \frac{2ab}{a+b};\quad a, b \in (0,\infty).$$

3. The Logarithmic mean:

$$L = L(a,b) = \frac{b-a}{\ln b - \ln a};\ a \neq b, a, b \in (0,\infty).$$

Example 1 *If f be defined by $f(u) = \frac{1}{u}$ and $0 < a < b$ then we have:*

$$\frac{1}{b-a}\int_a^b f(u)du = L^{-1}(a,b) = L^{-1},$$

$$\frac{f(a)+f(b)}{2} = H^{-1}(a,b) = H^{-1},$$

$$\frac{|f''(a)| + |f''(b)|}{2} = 2H^{-1}\left(a^3, b^3\right)$$

and

$$\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}} = 2\left[H^{-1}\left(a^{3q}, b^{3q}\right)\right]^{\frac{1}{q}}.$$

1. *Then (5) becomes,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{G^2\left(a^3,b^3\right)}\left[\frac{b^3}{(rs+2)(rs+3)}+\frac{a^3}{r}\left\{\beta\left(\frac{2}{r},s+1\right)-\beta\left(\frac{3}{r},s+1\right)\right\}\right]. \quad (13)$$

   (a) *If we put $s=0$ in (13), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{3}H^{-1}\left(a^3,b^3\right).$$

   (b) *If we put $r=1$, in (13), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{2(b-a)^2}{(s+2)(s+3)}H^{-1}\left(a^3,b^3\right).$$

   (c) *If we put $s=1$, in (13), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{6G^2\left(a^3,b^3\right)}\left[\frac{6b^3+s(s+5)a^3}{(s+2)(s+3)}\right].$$

   (d) *If we put $r=s=1$, in (13), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{6}H^{-1}\left(a^3,b^3\right).$$

2. *Then (8) becomes*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{G^2\left(a^3,b^3\right)}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{b^{3q}}{(rs+1)}+\frac{a^{3q}}{r}\beta\left(\frac{1}{r},s+1\right)\right]^{\frac{1}{q}}. \quad (14)$$

   (a) *If we put $s=0$ in (14), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{4}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[2H^{-1}\left(a^3,b^3\right)\right]^{\frac{1}{q}}.$$

   (b) *If we put $r=1$, in (14), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{4H^{-1}\left(a^3,b^3\right)}{(s+1)}\right]^{\frac{1}{q}}.$$

   (c) *If we put $s=1$ in (14), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{G^2\left(a^3,b^3\right)}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{b^{3q}+sa^{3q}}{(s+1)}\right]^{\frac{1}{q}}.$$

   (d) *If we put $r=s=1$, in (14), we acquire,*

$$\left|H^{-1}-L^{-1}\right| \leq (b-a)^2\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[H^{-1}\left(a^3,b^3\right)\right]^{\frac{1}{q}}.$$

3. Then (10) becomes,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{(6)^{1-\frac{1}{q}} G^2\left(a^3, b^3\right)}\left[\frac{b^{3q}}{(rs+2)(rs+3)}+\frac{a^{3q}}{r}\left\{\beta\left(\frac{2}{r}, s+1\right)-\beta\left(\frac{3}{r}, s+1\right)\right\}\right]^{\frac{1}{q}}. \quad (15)$$

(a) If we put $s=0$ in (15), we acquire,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{6}\left[2H^{-1}\left(a^{3q}, b^{3q}\right)\right]^{\frac{1}{q}}.$$

(b) If we put $r=1$, in (15), we acquire,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{6^{1-\frac{1}{q}}}\left[\frac{2H^{-1}\left(a^{3q}, b^{3q}\right)}{(s+1)(s+2)}\right]^{\frac{1}{q}}.$$

(c) If we put $s=1$ in (15), we acquire

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{6G^2\left(a^3, b^3\right)}\left[\frac{6b^{3q}+s(s+5)a^{3q}}{(s+2)(s+3)}\right]^{\frac{1}{q}}.$$

(d) If we put $r=s=1$, in (15), we acquire,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(b-a)^2}{12}\left[H^{-1}\left(a^{3q}, b^{3q}\right)\right]^{\frac{1}{q}}.$$

Remark 9 *Similarly, as demonstrated earlier, we can deduce novel relationships pertaining to specialized means by introducing another category of convex functions.*

4 Conclusion

The Hermite Hadamard inequality stands as one of the most renowned inequalities in mathematics. Throughout the literature, numerous generalizations and variations of this inequality have been developed. In this work, we present a generalization of this inequality by introducing the novel concept of (r, s)-convex functions. In Section 2, we elucidate three distinct results concerning the upper bounds estimated in an absolute sense for the Hermite Hadamard dual inequality, focusing on the category of twice-differentiable (r, s)-convex functions. We employ various techniques, including Hölder's and power mean integral inequalities, to derive these outcomes, which encompass several findings outlined in the articles [2] and [11]. In Section 3, we establish connections between our derived results and select, well-known special means. Lastly, the concluding section offers remarks and future research prospects, aimed at inspiring further investigations by researchers in the field.

Here, we will provide some insights and offer directions for future research in light of our presented findings.

5 Remarks and Future Ideas

1. All the inequalities given in this article can be stated in reverse direction for concave function using relation f is concave if and only if $-f$ is convex.

2. From [13, p. 140] for convex function f we have that

$$\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du. \tag{16}$$

   In all our results stated in Section 2, we have established bounds only for

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right|.$$

   By using relation given in (16), we automatically acquire bounds for

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du\right|.$$

3. One may also work for Fejér Hermite Hadamard type inequality by introducing symmetric weights to our obtained results.
4. One may also work on Weighted Hermite Hadamard type inequality by introducing non symmetric weights to our obtained results.
5. One may do similar work by using various distinct classes of convex functions.
6. One may try to extend all established results for fractional calculus.
7. One may try to find refined bounds of all established results of this article.
8. One can extend this work to time scale domain or Quantum Calculus.
9. One can try to attain this work for Fuzzy set theory.

Acknowledgment. This research work is supported by the Higher Education Commission of Pakistan under the Indigenous Ph.D. Fellowship for 5000 Scholars, HEC (Phase-II) (PIN: 518-108309-2PS5-088 (50043481)).

References

[1] A. Arshad and A. R. Khan, Hermite Hadamard-Fejer type inequalities for s-p-convex functions of several senses, TJMM, 11(2019), 25–40.

[2] M. I. Bhatti, M. Iqbal and M. Hussain, Hadamard-type inequalities for s-convex functions I, Punjab. Univ. J. Math., 41(2009), 51–60.

[3] M. Bilal and A. R. Khan, New Generalized Hermite Hadamard Inequalities for p-convex functions in the mixed kind, EJPAM, 14(2021), 863–880.

[4] M. Bilal and A. R. Khan, On some new Hermite Hadamard dual inequalities, JMCMS, 16(2021), 93–98.

[5] M. Bilal, N. Irshad and A. R. Khan, New version of generalized Ostrowski-Grüss type inequality, Stud. Univ. Babeş.-Bolyai Math., 66(2021), 441–455.

[6] M. Bilal and A. R. Khan, Refinement of Hermite Hadamard type inequalities for convex functions with applications, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 31(2024), 33–48.

[7] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type, Soochow J. Math., 21(1995), 335–341.

[8] Ch. Hermite, Sur deux limites d'une int'egrale d'efinie, Mathesis, 3(1883), 82.

[9] A. R. Khan, I. U. Khan and S. Muhammad, Hermite-Hadamard type fractional inequalities for s-convex functions of mixed kind, TMCS, 1(2021), 25–37.

[10] D. S. Mitrinović, J. E. Pećarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht (1993).

[11] M. E. Özdemír, M. Avci and H. Kavurmaci, Hermite Hadamard-type inequalities via (α, m) convexity, Comp. Math. App., 61(2011), 2614–2620.

[12] S. Özcan, and I. Işcan, Some new Hermite Hadamard type inequalities for s-convex functions and their applications, J. Ineq. App., (2019), Article 201.

[13] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, Partial Orderings and Statistical Applications, Academic Press, New York, 1992.