Applied Mathematics E-Notes, 24(2024), 339-349 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Invariant Convergence In Asymmetric Metric Spaces*

Büşra Söylemez†, Fatih Nuray‡

Received 23 August 2023

Abstract

The main purpose of this paper is to extend the invariant convergence, statistical invariant convergence, invariant Cauchy sequence and invariant continuity in asymmetric metric spaces. Also, we investigate relations between forward and backward invariant convergent sequences. Furthermore, some results obtained are related invariant Cauhy conditions, invariant continuity and invariant compactness in asymmetric metric spaces.

1 Introduction

Raimi [14] defined the concept of invariant convergence (σ-convergence) which is generalization of almost convergence. Following his terminology, many authors studied invariant convergent sequences ([8], [12], [16]).

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean, if and only if,

1. $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n.
2. $\phi(e) = 1$, where $e = (1, 1, 1, ...)$.
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in l_\infty$.

The mappings σ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$ denotes the mth iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [11].

Definition 1 (Invariant Convergence) *A sequence (x_k) is said to be invariant convergent to L if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} = L$$

uniformly in m. In this case we write $(x_k) \longrightarrow L(V_\sigma)$ and L is called the σ-limit of x [9].

Wilson [18] introduced asymmetric metric spaces as quasi-metric spaces in 1931. Asymmetric metric spaces, sometimes called quasi-metric spaces, are defined as metric spaces that lack the axiom of symmetry in metric spaces. Without symmetry, an 'embarassing richness of material' is revealed [18]. An asymmetric metric space has two topologies which are the forward and backward topology in the same space. So some notions, such as convergence, continuity, completeness and compactness have two notions namely forward and backward.

Taxicab geometry topology including one-way streets, where we can have a path from point A to point B contains a different set of streets than a path from B to A. Therefore, asymmetric metric spaces are common

*Mathematics Subject Classifications: 40A05, 26A15.

†Faculty of Science, Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey
‡Faculty of Science, Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey

in real life [17]. Also, in the realms of applied mathematics and materials science there are many applications of asymmetric metric spaces [1]. Several authors studied the asymmetric context for many years ([1], [2], [5], [7], [15], [17], [19]).

If one crucial asymmetric assumption fails, classic statements of symmetric analysis fail. So, we investigate the question of whether invariant convergence, compactness and continuity can be generalized in the asymmetric metric spaces. Firstly, we recall some definitons in asymmetric metric spaces.

Definition 2 (Asymmetric metric space) *A function $d : X \times X \to \mathbb{R}$ is an asymmetric metric and (X, d) is an asymmetric metric space if*

1. *For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$,*
2. *For every $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$ [1].*

Henceforth, throughout this work (X, d) shall be an asymmetric metric space. The following definitions are in [1].

Definition 3 (Forward and backward topologies) *The forward topology τ_+ induced by d is the topology generated by the forward open balls for $x \in X$, $\forall \varepsilon > 0$*

$$B^+(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

Likewise, the backward topology τ_- induced by d is the topology generated by the backward open balls for $x \in X$, $\forall \varepsilon > 0$

$$B^-(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}.$$

Definition 4 (Boundedness) *A sequence $(x_k) \in X$ is forward (backward) bounded, if there exists $x \in X$ and $M > 0$ such that $d(x, x_k) \leq M$ $(d(x_k, x) \leq M)$.*

Definition 5 (Convergence) *A sequence $(x_k) \in X$ forward (backward) convergent to $x_0 \in X$ if and only if*

$$\lim_{k \to \infty} d(x, x_k) = 0 \qquad (\lim_{k \to \infty} d(x_k, x) = 0)$$

and it is written $x_k \xrightarrow{f} x_0$ $(x_k \xrightarrow{b} x_0)$.

Definition 6 (Cauchyness) *A sequence $(x_k) \in X$ is forward (backward) Cauchy if for $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $k \geq n \geq N$, $d(x_k, x_n) < \varepsilon$ $(d(x_k, x_n) < \varepsilon)$ holds.*

Definition 7 (Continuity) *A funciton $f : X \longrightarrow Y$ is forward (backward) continuous at $x \in X$ if and only if whenever $x_k \xrightarrow{f} x$ $(x_k \xrightarrow{b} x)$ in (X, d_X) we have $f(x_k) \xrightarrow{f} f(x)$ $(f(x_k) \xrightarrow{b} f(x))$ in (Y, d_Y).*

Definition 8 (Compactness) *A set $S \subset X$ is forward (backward) compact if every open cover of S in the forward (backward) topology has a finite subcover.*

2 Invariant convergence in Asymmetric Metric Spaces

In this section, we will define invariant convergence, invariant Cauchy sequence, statistical invariant convergence and statistical invariant Cauchy sequence in any asymmetric metric spaces. Then we will have some results.

Definition 9 *Let (X, d) be an asymmetric metric space. A sequence (x_k) forward invariant converges to $x \in X$ if and only if*

$$\lim_{p\to\infty} \frac{1}{p} \sum_{j=1}^{p} d(x, x_{\sigma^j(m)}) = 0$$

uniformly in m. In this case, it is written $x_k \xrightarrow{f} x(V_\sigma)$. Similarly, a sequence (x_k) backward invariant converges to $x \in X$ if and only if

$$\lim_{p\to\infty} \frac{1}{p} \sum_{j=1}^{p} d(x_{\sigma^j(m)}, x) = 0$$

uniformly in m. In this case, it is written $x_k \xrightarrow{b} x(V_\sigma)$.

Theorem 1 *Let (X, d) be an asymmetric metric space. If a sequence $(x_k) \in X$ is forward convergent to $x \in X$ and backward convergent to $y \in X$, then $x = y$.*

Proof. Fix $\varepsilon > 0$. By assumption, (x_k) is forward convergent to x so there exists $\exists n_1 \in \mathbb{N}$ for all m and $p \geq n_1$ such that

$$\frac{1}{p} \sum_{j=1}^{p} d(x, x_{\sigma^j(m)}) < \frac{\varepsilon}{2}.$$

Also, (x_k) is backward convergent to y so there exists $\exists n_2 \in \mathbb{N}$ for all m and $p \geq n_2$ such that

$$\frac{1}{p} \sum_{j=1}^{p} d(x_{\sigma^j(m)}, y) < \frac{\varepsilon}{2}.$$

Then for all $p \geq n = \max\{n_1, n_2\}$,

$$d(x, y) \leq \frac{1}{p} \sum_{j=1}^{p} d(x, x_{\sigma^j(m)}) + \frac{1}{p} \sum_{j=1}^{p} d(x_{\sigma^j(m)}, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for each m. As ε is arbitrary, we deduce that $d(x, y) = 0$, which implies $x = y$. ∎

Corollary 2 *If forward invariant convergence of a sequence implies backward invariant convergence, then forward invariant limit is unique.*

Example 1 *The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined by*

$$d(x, y) = \begin{cases} y - x, & y \geq x, \\ 1, & y < x, \end{cases}$$

is an asymmetric metric wich is called Sorgenfrey asymmetric metric. Here, $B^+(x, \epsilon) = [x, x - \epsilon)$ and $B^-(x, \epsilon) = (x - \epsilon, x]$, provided $\epsilon \leq 1$. It is easy to find a sequence which does not invariant convergent in this metric but invariant convergent in a metric space. For example, let

$$x_k = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}. \end{cases}$$

This sequence is invariant convergent to $\frac{1}{2}$ in a metric space. But it isn't invariant convergent to $\frac{1}{2}$ in asymmetric metric space. Let $\sigma(m) = m + 1$. According to Sorgenfrey asymmetric metric space for (x_k)

sequence:

$$\begin{aligned}
\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(\frac{1}{2}, x_{\sigma^j(m)}) &= \lim_{p\to\infty}\frac{d(\frac{1}{2},x_{m+1})+d(\frac{1}{2},x_{m+2})+\cdots+d(\frac{1}{2},x_{m+p})}{p}\\
&= \lim_{p\to\infty}\frac{1+\frac{1}{2}+1+\frac{1}{2}+\cdots+1+\frac{1}{2}}{p}\\
&= \lim_{p\to\infty}\frac{\frac{p}{2}+\frac{p}{4}}{p}=\frac{3}{4},
\end{aligned}$$

$$\begin{aligned}
\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(x_{\sigma^j(m)}, \frac{1}{2}) &= \lim_{p\to\infty}\frac{d(x_{m+1},\frac{1}{2})+d(x_{m+2},\frac{1}{2})+\ldots d(x_{m+p},\frac{1}{2})}{p}\\
&= \lim_{p\to\infty}\frac{\frac{1}{2}+1+\frac{1}{2}+1+\cdots+\frac{1}{2}+1}{p}\\
&= \lim_{p\to\infty}\frac{\frac{p}{2}+\frac{p}{4}}{p}=\frac{3}{4}.
\end{aligned}$$

Thus, (x_k) sequence does not invariant convergent in Sorgenfrey asymmetric metric.

Example 2 *Let $X=\mathbb{R}$ and d be an asymmetric metric on X defined by*

$$d(x,y)=\begin{cases} y-x, & x\geq y.\\ 0, & x<y.\end{cases}$$

Consider the sequence $x_k=(-1)^k$ and $\sigma(m)=m+1$. Then $x_{\sigma^j(m)}=(-1)^{m+j}$, i.e. $x_{\sigma^j(m)}\in\{-1,1\}$. Therefore,

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(-1, x_{\sigma^j(m)})=0 \quad \textit{uniformly in } m,$$

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(x_{\sigma^j(m)},1)=0 \quad \textit{uniformly in } m,$$

i.e. the sequence (x_k) forward invariant convergent to 1 and backward invariant convergent to -1.

Corollary 3 *The forward (backward) invariant limit of a sequence (x_k) in an asymmetric metric space (X,d) is not unique in general. This can be seen in the preceding example using the fact that, for all $k\in\mathbb{N}$, (x_k) is forward invariant convergent in $x\in[1,\infty)$ and (x_k) is backward invariant convergent in $x\in(-\infty,1]$.*

Remark 1 *In general, forward invariant convergence does not implies backward invariant convergence in asymmetric metric spaces or vice versa.*

Example 3 *Let $X=\mathbb{R}$ and d be the asymmetric metric on X defined by*

$$d(x,y)=\begin{cases} 0, & y\geq x,\\ 1, & y<x.\end{cases}$$

Consider the sequence (x_k) given as $(x_k)=2(1+\frac{1}{k})$ and $\sigma(m)=m+1$. This sequence is invariant convergent to 2 in metric space. But in asymmetric metric space,

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(2, x_{\sigma^j(m)})=\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(2, 2+\frac{2}{m+j})=0,$$

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p}d(x_{\sigma^j(m)},2)=\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p}d(2+\frac{2}{m+j},2)=1\neq 0.$$

So, this sequence is forward invariant convergent to 2, but it is not backward invariant convergent to 2. Consequently, it is not invariant convergent in asymmetric metric space.

The proposition below, gives the relation between forward and backward limits [1].

Proposition 4 *Let (X,d) be an asymmetric metric space. Suppose for any $x,y\in X$*

$$d(y,x)\leq c(x,y)d(x,y) \tag{1}$$

where $c:X\times X\to R$ satisfies the following constraint:

$$\forall x\in X\quad \exists\epsilon>0\quad \text{such that}\quad y\in B^+(x,\epsilon)\implies c(x,y)\leq C(x)$$

where C is some function which depends only on x. In this situation, the existence of forward limits implies the existence of backward limits and limits are unique.

Proof. It is easy to see that C which depend only on x. For example,

$$d(x,y):=\begin{cases}e^y-e^x, & y\geq x,\\ e^{-y}-e^{-x}, & y<x,\end{cases}$$

is an asymmetric metric. Assumption of Proposition 4 is satisfied with

$$C(x):=\max\{e^x(e^x+\varepsilon),e^{-x}(e^{-x}+\varepsilon)\}.$$

Suppose (x_k) forward converges to x_0. From (1), $\exists\delta>0$ such that $c(x_0,x_k)\leq C(x_0)$ for $d(x_0,x_k)<\delta$. Fix $\varepsilon>0$. Since (x_k) forward converges to x_0, $\exists N\in\mathbb{N}$ such that $\forall k\geq N$, we have $d(x_0,x_k)<\min\{\delta,\frac{\varepsilon}{C(x_0)}\}$ and thus

$$d(x_k,x_0)\leq c(x_0,x_k).d(x_0,x_k)\leq C(x_0).d(x_0,x_k)<\varepsilon.$$

By the Corollary 2, limit must be unique. ∎

Theorem 5 *Let (X,d) be an asymmetric metric space satisfying (1). Then forward invariant convergence of (x_k) sequence implies backward invariant convergence sequence and limits are unique.*

Proof. Suppose (x_k) forward invariant converges to x_0. From (1), there exists $\exists\delta>0$ such that $c(x_0,x_{\sigma^j(m)})\leq C(x_0)$ for $d(x_0,x_{\sigma^j(m)})<\delta$ for each m. Fix $\varepsilon>0$. Since (x_k) forward converges to x_0, there exists $\exists N\in\mathbb{N}$ such that for every $\forall k\geq N$, we have $d(x_0,x_{\sigma^j(m)})<\min\{\delta,\frac{\varepsilon}{C(x_0)}\}$ for each m and thus

$$d(x_{\sigma^j(m)},x_0)\leq c(x_0,x_{\sigma^j(m)}).d(x_0,x_{\sigma^j(m)})\leq C(x_0).d(x_0,x_{\sigma^j(m)})<\varepsilon$$

for each m. By the Theorem 1, limit must be unique. ∎

Definition 10 *Let (X,d) be an asymmetric metric space and p be a positive real number. A sequence (x_k) forward (backward) p-invariant convergent to $x\in X$ if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}[d(x,x_{\sigma^j(m)})]^p=0\qquad(\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}[d(x_{\sigma^j(m)},x)]^p=0)$$

uniformly in m. In this case, it is written $(x_k)\xrightarrow{f}x(V_{p\sigma})$ $((x_k)\xrightarrow{b}x(V_{p\sigma}))$.

Statistical convergence of sequences of points was introduced by Fast [3]. Fridy [4], Nuray and Savaş [11], Nuray [13], Hudal and Gürdal [10] and many authors studied statistical convergence.

Definition 11 *Let (X,d) be an asymmetric metric space. A sequence (x_k) is forward (backward) statistical invariant convergent to $x \in X$ if and only if*

$$\lim_{p\to\infty}\frac{1}{p}\left|\{j\le p : d(x,x_{\sigma^j(m)})\ge\varepsilon\}\right| = 0 \qquad \left(\lim_{p\to\infty}\frac{1}{p}\left|\{j\le p : d(x_{\sigma^j(m)},x)\ge\varepsilon\}\right| = 0\right)$$

uniformly in m. In this case, it is written $(x_k)\xrightarrow{f}x(S_\sigma)$ $((x_k)\xrightarrow{b}x(S_\sigma))$.

Theorem 6 *Let (X,d) be an asymmetric metric space.*

1. *If the sequence $(x_k)\subset X$ is forward invariant convergent, then $(x_k)\subset X$ is forward statistical invariant convergent,*
2. *If the sequence $(x_k)\subset X$ is bounded and forward statistical invariant convergent, then $(x_k)\subset X$ is forward invariant convergent.*

Proof.

1. Let $(x_k)\xrightarrow{f}x(V_\sigma)$. Then we can write

$$\sum_{j=1}^{p} d(x,x_{\sigma^j(m)}) = \sum_{d(x,x_{\sigma^j(m)})\ge\varepsilon} d(x,x_{\sigma^j(m)}) + \sum_{d(x,x_{\sigma^j(m)})<\varepsilon} d(x,x_{\sigma^j(m)})$$

and

$$\sum_{j=1}^{p} d(x,x_{\sigma^j(m)}) \ge \sum_{d(x,x_{\sigma^j(m)})\ge\varepsilon} d(x,x_{\sigma^j(m)}) \ge \varepsilon.|\{j\le p : d(x,x_{\sigma^j(m)})\ge\varepsilon\}|.$$

Here, both sides are multiplied by positive $\frac{1}{p}$ and if the limit is passed for $p\to\infty$

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p} d(x,x_{\sigma^j(m)}) \ge \varepsilon.\lim_{p\to\infty}\frac{1}{p}|\{j\le p : d(x,x_{\sigma^j(m)})\ge\varepsilon\}|.$$

Since $(x_k)\xrightarrow{f}x(V_\sigma)$, the left side of the inequality is 0 while $p\to\infty$. So

$$\varepsilon.\lim_{p\to\infty}\frac{1}{p}|\{j\le p : d(x,x_{\sigma^j(m)})\ge\varepsilon\}| \le 0$$

and we have that $(x_k)\xrightarrow{f}x(S_\sigma)$.

2. Let (x_k) be bounded and $(x_k)\xrightarrow{f}x(S_\sigma)$. Then $\forall x\in X$ and for all j and m such that $M>0$ $d(x,x_{\sigma^j(m)})<M$.

Thus for $\varepsilon>0$ we have that

$$\begin{aligned}\frac{1}{p}\sum_{j=1}^{p} d(x,x_{\sigma^j(m)}) &= \frac{1}{p}\sum_{d(x,x_{\sigma^j(m)})\ge\varepsilon} d(x,x_{\sigma^j(m)}) + \frac{1}{p}\sum_{d(x,x_{\sigma^j(m)})<\varepsilon} d(x,x_{\sigma^j(m)})\\ &\leqslant \frac{M}{p}\left|\{j\le p : d(x,x_{\sigma^j(m)})\ge\varepsilon\}\right| + \varepsilon\end{aligned}$$

so

$$\frac{1}{p}\sum_{j=1}^{p} d(x,x_{\sigma^j(m)}) \le \frac{M}{p}|\{j\le p : d(x,x_{\sigma^j(m)})\ge\varepsilon\}| + \varepsilon.$$

Here, both sides are limit for $p\to\infty$, we obtained that $(x_k)\xrightarrow{f}x(V_\sigma)$.

■

Definition 12 *Let (X,d) be an asymmetric metric space. A sequence (x_k) is said to be a forward (backward) invariant Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer N such that if $n > p > N$ then for all $m \in \mathbb{N}$ we have that*

$$d(t_{pm}, t_{nm}) < \varepsilon \quad (d(t_{nm}, t_{pm}) < \varepsilon)$$

where,

$$t_{nm} = \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)}$$

Definition 13 *Let (X,d) be an asymmetric metric space. A sequence (x_k) is said to be a forward (backward) statistical invariant Cauchy sequence if for every $\varepsilon > 0$, there exists a number $N(= N(\varepsilon))$ such that*

$$\lim_{p\to\infty} \frac{1}{p} \left|\{n \le p : d(t_{Nm}, t_{nm}) \ge \varepsilon\}\right| = 0 \qquad (\lim_{p\to\infty} \frac{1}{p} |\{n \le p : d(t_{nm}, t_{Nm}) \ge \varepsilon\}| = 0)$$

uniformly in m.

Theorem 7 *Let (X,d) be an asymmetric metric space. If (x_k) is a invariant convergent sequence in (X,d), then it is a forward statistical invariant Cauchy sequence.*

Proof. Suppose that (x_k) is invariant convergent to x and $\varepsilon > 0$, then (x_k) is both forward invariant convergent to x and backward invariant convergent to x. Then, we can write

$$d(x, t_{nm}) < \frac{\varepsilon}{2}, \qquad d(t_{nm}, x) < \frac{\varepsilon}{2}$$

for all m. If N is chosen so that

$$d(t_{Nm}, x) < \frac{\varepsilon}{2}$$

for all m. Then we have

$$d(t_{Nm}, t_{nm}) < d(t_{Nm}, x) + d(x, t_{nm}) < \varepsilon$$

for all m. Hence, (x_k) is a forward statistical invariant Cauchy sequence. ■

Corollary 8 *If the sequence (x_k) is statistical invariant convergent, then sequence (x_k) is a statistical invariant Cauchy sequence.*

Theorem 9 *Let (X,d) be an asymmetric metric space. If (x_k) is a forward (backward) invariant Cauchy sequence in (X,d), then it is a forward (backward) statistical invariant Cauchy sequence.*

Proof. If (x_k) is a forward invariant Cauchy sequence then $\forall \varepsilon > 0$, there exists a positive integer N_0 such that if $n > p > N_0$ then for all m

$$d(t_{pm}, t_{nm}) < \varepsilon.$$

We choose a N

$$d(t_{Nm}, t_{nm}) < \varepsilon$$

uniformly in m. So

$$\lim_{p\to\infty} \frac{1}{p} |\{n \le p : d(t_{Nm}, t_{nm}) \ge \varepsilon\}| = 0.$$

The proof is similar for the backward case. ■

Forward convergent sequence doesn't imply forward Cauchy sequence [6]. The following example shows that in general forward (backward) invariant statistical Cauchy sequence does not imply forward (backward) invariant Cauchy sequence.

Theorem 10 *Let (x_k) be a forward invariant Cauchy sequence. If*

$$\frac{1}{p+1}\sum_{j=1}^{p} d(x_k, x_{\sigma^j(m)}) \longrightarrow 0 \quad \textit{uniformly in } m,$$

$$\frac{1}{n+1}\sum_{j=1}^{p} d(x_{\sigma^j(m)}, x_k) \longrightarrow 0 \quad \textit{uniformly in } m,$$

forward and backward conditions holds, then it is forward Cauchy and vice versa.

Proof. Let (x_k) be a forward invariant Cauchy sequence and the above conditions hold. We have for $k > l$

$$\begin{aligned} d(x_l, x_k) \quad &\leq \quad d\left(x_l, \frac{1}{p+1}\sum_{j=0}^{p} x_{\sigma^j(m)}\right) \\ &\quad + d\left(\frac{1}{p+1}\sum_{j=0}^{p} x_{\sigma^j(m)}, \frac{1}{n+1}\sum_{j=0}^{n} x_{\sigma^j(m)}\right) + d\left(\frac{1}{n+1}\sum_{j=0}^{n} x_{\sigma^j(m)}, x_k\right). \end{aligned}$$

Let (x_k) be forward Cauchy sequence and conditions hold. We have for $k > l$

$$\begin{aligned} & d\left(\frac{1}{p+1}\sum_{j=0}^{p} x_{\sigma^j(m)}, \frac{1}{n+1}\sum_{j=0}^{n} x_{\sigma^j(m)}\right) \\ \leq \quad & d\left(\frac{1}{p+1}\sum_{j=0}^{p} x_{\sigma^j(m)}, x_l\right) + d(x_l, x_k) + d\left(x_k, \frac{1}{n+1}\sum_{j=0}^{n} x_{\sigma^j(m)}\right) \end{aligned}$$

so (x_k) is a forward invariant Cauchy sequence. The proof is completed. ■

3 Invariant Continuity and Invariant Compactness in Asymmetric Metric Spaces

Now, we will investigate invariant continuity and invariant compactness in asymmetric metric spaces.

Definition 14 *Let X and Y be asymmetric metric spaces. Let $f : X \longrightarrow Y$ be a mapping. f is said to be invariant continuous at a point $x \in X$ if*

$$(x_k) \longrightarrow x(V_\sigma) \quad \textit{implies} \quad f(x_k) \longrightarrow f(x)(V_\sigma).$$

We can introduce four types of forward continuity in asymmetric metric spaces:

$$(x_k) \xrightarrow{f} x \quad \text{implies} \quad f(x_k) \xrightarrow{f} f(x), \tag{2}$$

$$(x_k) \xrightarrow{f} x \quad \text{implies} \quad f(x_k) \xrightarrow{f} f(x)(V_\sigma), \tag{3}$$

$$(x_k) \xrightarrow{f} x(V_\sigma) \quad \text{implies} \quad f(x_k) \xrightarrow{f} f(x), \tag{4}$$

$$(x_k) \xrightarrow{f} x(V_\sigma) \quad \text{implies} \quad f(x_k) \xrightarrow{f} f(x)(V_\sigma). \tag{5}$$

(2) is forward continuity and (5) is forward invariant continuity. We have

$$(4) \Rightarrow (2) \Rightarrow (3), \qquad (4) \Rightarrow (5) \Rightarrow (3).$$

We can introduce four types of backward continuity in asymmetric metric spaces:

$$(x_k) \xrightarrow{b} x \quad \text{implies} \quad f(x_k) \xrightarrow{b} f(x), \tag{6}$$

$$(x_k) \xrightarrow{b} x \quad \text{implies} \quad f(x_k) \xrightarrow{b} f(x)(V_\sigma), \tag{7}$$

$$(x_k) \xrightarrow{b} x(V_\sigma) \quad \text{implies} \quad f(x_k) \xrightarrow{b} f(x), \tag{8}$$

$$(x_k) \xrightarrow{b} x(V_\sigma) \quad \text{implies} \quad f(x_k) \xrightarrow{b} f(x)(V_\sigma). \tag{9}$$

(6) is backward continuity and (9) is backward invariant continuity. We have

$$(8) \Rightarrow (6) \Rightarrow (7), \qquad (8) \Rightarrow (9) \Rightarrow (7).$$

Definition 15 *Let X be an asymmetric metric space, and S be a subset of X.*

1. *S is forward (backward) invariant compact if every sequence has a forward (backward) invariant convergent subsequence with limit in S.*
2. *S is forward (backward) invariant complete if every forward (backward) invariant Cauchy sequence is forward (backward) invariant convergent.*

Remark 2 *Compactness and sequentially compactness is equivalent in a metric space. Similarly, forward(backward) invariant compactness and forward (backward) invariant sequentially compactness is equivalent in asymmetric metric space (X, d).*

Theorem 11 *Let $d : X \times X \to \mathbb{R}_0^+$ be an asymmetric metric. If (X, d) is forward invariant compact and $x_k \xrightarrow{b} x(V_\sigma)$, then $x_k \xrightarrow{f} x(V_\sigma)$.*

Proof. Consider a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $(x_k) \xrightarrow{b} x(V_\sigma)$ for some $x \in X$. By invariant compactness, every subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_k\}$ has a forward invariant convergent subsequence. Say $x_{k_{n_j}} \xrightarrow{f} y(V_\sigma) \in X$ as $j \to \infty$. Then $x = y$ by Theorem 1.

Suppose that $x_k \overset{f}{\nrightarrow} x(V_\sigma)$. Then there exists $\varepsilon_0 > 0$ and a sequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ with $\frac{1}{p}\sum_{l=1}^{p} d(x, x_{\sigma^{j_l}(m)}) \geq \varepsilon_0$ for all m. But this subsequence has a subsequence $\{x_{k_{n_j}}\}_{j\in\mathbb{N}}$ forward converging to x , so there exists $J \in \mathbb{N}$ such that for $j \geq J$, one has $\frac{1}{p}\sum_{l=1}^{p} d(x, x_{\sigma^{j_l}(m)}) < \varepsilon_0$. This is obviously a contradiction, so $x_k \xrightarrow{f} x(V_\sigma)$. ∎

Remark 3 *Whenever we have invariant compactness, we have unique limits of sequences and backward invariant limits imply forward invariant limits.*

Theorem 12 *An asymmetric metric space (X, d) is forward complete if every forward invariant Cauchy sequence has a forward invariant convergent subsequence.*

Proof. Let $\{x_k\}_{k\in\mathbb{N}}$ be a forward invariant Cauchy sequence in (X, d) with a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_k\}$ forward invariant converging to $x \in X$. Fix $\varepsilon > 0$. First choose $n \in \mathbb{N}$ so that $d(t_{pm}, t_{nm}) \leq \frac{\varepsilon}{2}$ for every $n \geq p \geq N$. Then choose $r \in \mathbb{N}$ so that $k_r \geq N$, $d(x, x_{\sigma^{k_r}(n)}) < \frac{\varepsilon}{2}$. Then for $k \geq k_r \geq N$

$$d(x, t_{km}) \leq d(x, t_{k_r m}) + d(t_{k_r m}, t_{km}) < \varepsilon.$$

So $x_k \xrightarrow{f} x(V_\sigma)$, hence (X, d) is forward complete. ∎

Theorem 13 *Let X and Y be asymmetric metric spaces and $f : X \longrightarrow Y$. f is forward continuous and if*

$$d\left(f(x_k), \frac{1}{p}\sum_{j=1}^{p} f(x_{\sigma^j(m)})\right) \longrightarrow 0$$

uniformly in m, then it is forward invariant continuous.

Proof.

$$d\left(f(x), \frac{1}{p}\sum_{j=1}^{p} f(x_{\sigma^j(m)})\right) \leq d\left(f(x), f(x_k)\right) + d\left(f(x_k), \frac{1}{p}\sum_{j=1}^{p} f(x_{\sigma^j(m)})\right).$$

■

Theorem 14 *Let X and Y be asymmetric metric spaces and $f : X \longrightarrow Y$. f is forward invariant continuous and if*

$$d\left(\frac{1}{p}\sum_{j=1}^{p} f(x_{\sigma^j(m)}), f(x_k)\right) \longrightarrow 0$$

uniformly in m, then it is forward continuous.

Proof.

$$d\left(f(x), f(x_k)\right) \leq d\left(f(x), \frac{1}{p}\sum_{j=1}^{p} f(x_{\sigma^j(m)})\right) + d\left(\frac{1}{p}\sum_{j=1}^{p} f(x_{\sigma^j(m)}), f(x_k)\right).$$

■

Theorem 15 *Forward (backward) invariant continuous image of an forward (backward) invariant compact space is forward (backward) invariant compact.*

Proof. Let X and Y be asymmetric metric spaces, Z be a forward invariant compact subspace of X and let $f : X \longrightarrow Y$ be forward invariant continuous. We have to show that $f(Z) = \{f(x) : x \in Z\}$ is also forward invariant compact. Let $\{f(x_k)\}$ be a sequence in $f(Z)$. Then (x_k) is a sequence in Z. Since Z is forward invariant compact, there is a subsequence (x_{k_n}) which is forward invariant convergent to $x \in X$. Observe that $\{f(x_{k_n})\}$ is a subsequence of $f(x_k)$. Since f is forward invariant continuous,

$$x_{k_n} \xrightarrow{f} x(V_\sigma) \quad \text{implies} \quad f(x_{k_n}) \xrightarrow{f} f(x)(V_\sigma).$$

Thus, $f(Z)$ is forward invariant compact. The proof is parallel to backward situation. ■

4 Conclusion

Asymmetric metric spaces do not have symmetry property. Thus, unlike metric spaces, definitions are given in pairs, namely forward and backward. Therefore, the theories are also proved differently. For this reason, in this study, it is shown that a sequence that is invariant convergent in a metric space may not be invariant convergent in an asymmetric metric space. And it is shown that invariant convergent sequence must be forward and backward invariant convergent. The mutual necessity of forward and backward convergence is proved using a property. The relations between invariant convergence and invariant Cauchy sequence are given and some theorems are obtained. In addition, the concepts of invariant continuity and invariant compactness are defined and theorems giving their relations are proved. The study of invariant convergence types, which have been studied by many mathematicians in recent years, but in asymmetric metric spaces it will be a source for future studies. Nevertheless, many questions about invariant convergence in asymmetric metric spaces remain as open problems.

Acknowledgment. The authors are thankful to the referees for their valuable comments.

References

[1] J. Collins and J. Zimmer, An asymmetric Arzela-Ascoli theorem, Topol. Appl., 154(2007), 2312–2322.

[2] P. Das, S. Ghosal and S. Pal, Extending asymmetric convergence and Cauchy condition using ideals, Math. Slovaca, 63(2013), 545–562.

[3] H. Fast, Sur la convergence statistique, Colloq. Math., 2(1951), 241–244.

[4] J. A. Fridy, On statistical convergence, Analysis, 5(1985), 301–313.

[5] M. Ilkhan and E. E. Kara, On statistical convergence in quasi-metric spaces, Demons. Math., 52(2019), 225–236.

[6] J. C. Kelly, Bitopological space, Proc. London Math. Soc., 13(1963), 71–89.

[7] A. Mennuci, On Asymmetric Distances, Technical report, Scuola Normale Superiore, pisa, 2004.

[8] M. Mursaleen, Invariant means and some matrix transformations, Tamkang J. Math., 10(1979), 183–188.

[9] M. Mursaleen, On finite matrices and invariant means, Indian J. Pure Appl. Math., 10(1979), 457–460.

[10] M. B. Huban and M. Gürdal, Wijsman lacunary invariant statistical convergence for triple sequences via Orlicz function, J. Classical Anal., 17(2021), 119–128.

[11] F. Nuray and E. Savaş, Invariant statistical convergence and A-invariant statistical convergence, Indian J. Pure Appl. Math., 10(1994), 267–274.

[12] F. Nuray and E. Savaş, On *sigma*-statistically convergence and lacunary *sigma*-statistically convergence, Math. Slovaca, 43(1993), 309–315.

[13] F. Nuray, Statistical convergence in partial metric spaces, Korean J. Math., 30(2022), 155–160.

[14] R. A. Raimi, On Banach's generalized limits, Duke. Math. J., 26(1959), 17–28.

[15] I. L. Reilly, P. V. Subrahmanyam and M. K. Vamanamurthy, Cauchy sequences in quasipseudometric spaces, Monatsh. Math., 93(1982), 127–140.

[16] P. Schaefer, Infinite matrices and invariant means, Proc. Amer. Math. Soc., 36(1972), 104–110.

[17] Z. H. Toyganozu and S. Pehlivan, Some results on exhaustiveness in asymmetric metric spaces, Filomat, 29(2015), 183–192.

[18] W. A. Wilson, On quasi-metric spaces, Amer. J. Math., 53(1991), 675–684.

[19] T. Yaying, B. Hazarika and A. S. Mohiuddine, Some new types of continuity in asymmetric metric spaces, Facta Univ. Ser. Math. Inform., 35(2020), 485–493.