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Existence Of m-Tuple Best Proximity Points With Applications To Delay Differential Equations And Integral Equations*

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Abstract

In this paper, we first introduce the notions of m-tuple best proximity point and later establish the new results of existence and convergence theorems of m-tuple best proximity point in metric spaces. We provide some illustrations to support our claims. Furthermore, as a consequence, we obtain m-tuple fixed point results for cyclic contraction mappings. As applications, we solve some systems of delay differential equations and integral equations.

1 Introduction and preliminaries

Approximation theory is a subject with a long history and a huge importance in classical and contemporary research. It has many applications, such as problems obtaining a solution for split feasibility, variational inequalities, systems of nonlinear matrix, integral and delay differential equations, etc. Delay differential equations are used in many physical phenomena of interest in biology, medicine, chemistry, physics, engineering, economics etc. There exist several methods for clarifying distinct classes of delay differential equations problems (see [1], [6], [10] and the references therein).

The study of non-self mappings is fascinating because, in this case, the best approximation exists by K. Fan [12] technique. In this case, we find an approximate solution such that the error $d(\tilde{a}, \Im'\tilde{a})$ is minimum, that is, the point $\tilde{a}$ is in close proximity to $\Im'\tilde{a}$, where $\Im'$ is a non-self mapping. After that, many authors gave the subsequent extensions and variants of Fan's theorem in different spaces and gave the best approximation results (see [21, 26, 28]). When a subset $\mathcal{P}$ of metric space $(\mathcal{Y}, d)$ is mapped into another subset $\mathcal{Q}$ by a mapping $\Im'$, the problem extends to determining the points that estimate the distance between these two subsets. These are referred to as best proximity points. That is, if $\Im' : \mathcal{P} \to \mathcal{Q}$ is a non-self mapping then a point $\tilde{a}$ is known as the best proximity point of a non-self mapping $\Im'$, if

$$d(\tilde{a}, \Im'\tilde{a}) = d(\mathcal{P}, \mathcal{Q}) = \inf\{d(\tilde{a}, \tilde{b}) : \tilde{a} \in \mathcal{P}, \tilde{b} \in \mathcal{Q}\}.$$

When the intersection of $\mathcal{P}$ and $\mathcal{Q}$ is non-empty, then a best proximity point reduces to a fixed point. The fixed point technique was extended to a coupled fixed point by Bhaskar and Lakshimikanttham [5] in 2006. They proved some coupled fixed point results. In 2011, Berinde and Borcut [4] gave the notion of tripled fixed point and proved new results. In the same year quadruple fixed points are introduced by Karapinar [19]. For more details, see ([2, 7, 13, 14, 15, 25]). In 2013, Paknazar et al. [20] generalized the quadruple fixed point to m-tuple ($m \geq 1$) fixed point and obtained some results.

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In 2006, Eldred and Veeramani [11] proved the existence of a best proximity point for cyclic contraction mappings. There are many extensions of the Eldred and Veeramani [11] result in many directions (see also [3, 17, 18, 22, 23, 24, 27, 30]). Later, Sintunavarat and Kumam [29] introduced the concept of coupled best proximity point and property UC*. They also obtained some coupled best proximity point results. In 2013, Cho et al. [8] introduced the notion of tripled best proximity point and proved the new tripled best proximity point results for cyclic contraction mappings. Recently, Hammad et al. [16] presented the notion of quadruple best proximity point and established the existence and convergence theorems of quadruple best proximity point for cyclic contraction mappings in metric space.

Motivated by these works, we first introduce the notion of m-tuple best proximity point and later establish the new results for the existence and convergence of m-tuple best proximity point in metric spaces. Consequently, we obtain some results on the existence and convergence of m-tuple fixed point. We also provide some illustrations to back up our work. As applications of our obtained results, we find the solution for a system of delay differential equations and integral equations.

Now, we need the following definitions and notations to be used in the sequel. Throughout the paper, $\mathcal{P}$ and $\mathcal{Q}$ are two nonempty closed subsets of a metric space $(\mathcal{Y}, d)$.

Definition 1 *A normed vector space $\mathcal{Y}$ is said to be a uniformly convex Banach space (UCBS) [9] if for every $0 < \epsilon \leq 2$ there is some $\delta > 0$ such that for any two vectors with $||\tilde{a}|| = 1$ and $||\tilde{b}|| = 1$, the condition*

$$||\tilde{a} - \tilde{b}|| \geq \epsilon,$$

implies

$$\left|\left|\frac{\tilde{a} + \tilde{b}}{2}\right|\right| \leq 1 - \delta.$$

Definition 2 *Let $\Im : \mathcal{P}^m \to \mathcal{Q}$ be a non-self mapping. A point $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_m) \in \mathcal{P}^m$ is a m-tuple fixed point of $\Im$ [20] if*

$$\tilde{a}_1 = d(\tilde{a}_1, \Im(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), \tilde{a}_2 = d(\tilde{a}_2, \Im(\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_1)), \ldots, \tilde{a}_m = d(\tilde{a}_m, \Im(\tilde{a}_m, \tilde{a}_1, \ldots, \tilde{a}_{m-1})).$$

Definition 3 *A pair $(\mathcal{P}, \mathcal{Q})$ satisfies the property UC [30] if there exist $\{\tilde{a}_n\}$, $\{\tilde{a'}_n\} \subset \mathcal{P}$ and $\{\tilde{b}_n\} \subset \mathcal{Q}$ such that $d(\tilde{a}_n, \tilde{b}_n) \to d(\mathcal{P}, \mathcal{Q})$ and $d(\tilde{a'}_n, \tilde{b}_n) \to d(\mathcal{P}, \mathcal{Q})$, then $d(\tilde{a}_n, \tilde{a'}_n) \to 0$ as $n \to \infty$.*

Definition 4 *A pair $(\mathcal{P}, \mathcal{Q})$ satisfies the property UC* [29] if $(\mathcal{P}, \mathcal{Q})$ has the property UC and the following condition holds: if $\{\tilde{a}_n\}$ and $\{\tilde{a'}_n\}$ are sequences in $\mathcal{P}$ and $\{\tilde{b}_n\}$ is a sequence in $\mathcal{Q}$ satisfying:*

(i) $d(\tilde{a'}_n, \tilde{b}_n) \to d(\mathcal{P}, \mathcal{Q})$,

(ii) for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $m > n \geq N_0$, $d(\tilde{a}_m, \tilde{b}_n) \leq d(\mathcal{P}, \mathcal{Q}) + \epsilon$.

Then there exists N_1 such that for all $m > n \geq N_1$, $d(\tilde{a}_m, \tilde{a'}_n) \leq \epsilon$.

Example 1 ([29]) *Suppose that $\mathcal{P}$, $\mathcal{Q}$ are two nonempty closed subsets of a UCBS such that $\mathcal{P}$ is convex. Then the pair $(\mathcal{P}, \mathcal{Q})$ has the property UC*.*

2 Main Results

We start the section by the following definition of m-tuple $(m \geq 1)$ best proximity point.

Definition 5 *Let $\Im : \mathcal{P}^m \to \mathcal{Q}$ be a non-self mapping. A point $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_m) \in \mathcal{P}^m$ is a m-tuple best proximity point of $\Im$ if*

$$d(\tilde{a}_1, \Im(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m) = d(\tilde{a}_2, \Im(\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_1)) = \ldots = d(\tilde{a}_m, \Im(\tilde{a}_m, \tilde{a}_1, \ldots, \tilde{a}_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

If $m = 1, 2, 3 \cdots$, we get best proximity point, coupled best proximity point, triplet best proximity point so on. If $\mathcal{P} = \mathcal{Q}$ in the above definition, then a m-tuple best proximity point reduces to a m-tuple fixed point.

Definition 6 *We say that $\Im : \mathcal{P}^m \to \mathcal{Q}$ and $\sigma : \mathcal{Q}^m \to \mathcal{P}$ are cyclic contractions if there exists $k \in [0, 1)$ such that*

$$d(\Im(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), \sigma(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)) \leq \frac{k}{m}(d(\tilde{a}_1, \tilde{b}_1) + d(\tilde{a}_2, \tilde{b}_2) + \ldots + d(\tilde{a}_m, \tilde{b}_m) + (1-k)d(\mathcal{P}, \mathcal{Q}), \quad (1)$$

for all $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m) \in \mathcal{P}^m$ and $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m \in \mathcal{Q}^m$.

Note that if the pair $(\Im, \sigma)$ is a cyclic contraction, then the pair $(\sigma, \Im)$ is a cyclic contraction, too.

Example 2 *Consider $\mathcal{Y} = \mathbb{R}^m$ endowed with*

$$d((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)) = |\tilde{a}_1 - \tilde{b}_1| + |\tilde{a}_2 - \tilde{b}_2| + \ldots + |\tilde{a}_m - \tilde{b}_m|,$$

for $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m)$, $(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \in \mathcal{Y}$ and suppose that

$$\mathcal{P} = \{(\tilde{a}_1, 0, \ldots, 0) \in \mathcal{Y} : 0 \leq \tilde{a}_1 \leq 1\}, \quad \mathcal{Q} = \left\{(\tilde{b}_1, 0, \ldots, 0) \in \mathcal{Y} : 0 \leq \tilde{b}_1 \leq 2\right\}.$$

Clearly $d(\mathcal{P}, \mathcal{Q}) = 0$. Define $\Im : \mathcal{P}^m \to \mathcal{Q}$ and $\sigma : \mathcal{Q}^m \to \mathcal{P}$ by

$$\Im((\tilde{a}_1, 0, \ldots, 0), (\tilde{a}_2, 0, \ldots, 0), \ldots, (\tilde{a}_m, 0, \ldots, 0)) = \left(\frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_m}{2m}, 0, 0, \ldots, 0\right)$$

and

$$\sigma((\tilde{b}_1, 0, \ldots, 0), (\tilde{b}_2, 0, \ldots, 0), \ldots, (\tilde{b}_m, 0, \ldots, 0)) = \left(\frac{\tilde{b}_1 + \tilde{b}_2 + \ldots + \tilde{b}_m}{2m}, 0, 0, \ldots, 0\right).$$

Then we obtain

$$\begin{aligned}
&\begin{pmatrix} \Im((\tilde{a}_1, 0, \ldots, 0), (\tilde{a}_2, 0, \ldots, 0), \ldots, (\tilde{a}_m, 0, \ldots, 0)), \\ \sigma((\tilde{b}_1, 0, \ldots, 0), (\tilde{b}_2, 0, \ldots, 0), \ldots, (\tilde{b}_m, 0, \ldots, 0)) \end{pmatrix} \\
&d\left(\left(\frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_m}{2m}, 0, 0, \ldots, 0\right), \left(\frac{\tilde{b}_1 + \tilde{b}_2 + \ldots + \tilde{b}_m}{2m}, 0, 0, \ldots, 0\right)\right) \\
= \;& \left|\frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_m}{2m} - \frac{\tilde{b}_1 + \tilde{b}_2 + \ldots + \tilde{b}_m}{2m}\right| \\
= \;& \left|\frac{(\tilde{a}_1 - \tilde{b}_1) + (\tilde{a}_2 - \tilde{b}_2) + \ldots + (\tilde{a}_m - \tilde{b}_m)}{2m}\right| \\
\leq \;& \frac{|\tilde{a}_1 - \tilde{b}_1| + |\tilde{a}_2 - \tilde{b}_2| + \ldots + |\tilde{a}_m - \tilde{b}_m|}{2m} \\
\leq \;& \frac{k}{m}(d(\tilde{a}_1, \tilde{b}_1) + d(\tilde{a}_2, \tilde{b}_2) + \ldots + d(\tilde{a}_m, \tilde{b}_m)) + (1-k)d(\mathcal{P}, \mathcal{Q}); \quad k = \frac{1}{2}.
\end{aligned}$$

The pair $(\Im, \sigma)$ is a cyclic contraction with $k = \frac{1}{2}$.

The lemmas below are very important in the sequel.

Lemma 1 *Suppose that* $\Im : \mathcal{P}^m \to \mathcal{Q}$ *and* $\sigma : \mathcal{Q}^m \to \mathcal{P}$ *are two cyclic contraction mappings. If* $(\tilde{a}_1^0, \tilde{a}_2^0, \ldots, \tilde{a}_m^0) \in \mathcal{P}^m$ *and the sequences* $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}, \ldots, \{\tilde{a}_m^n\}$ *are defined by*

$$\begin{cases} \tilde{a}_1^{2n+1} = \Im(\tilde{a}_1^{2n}, \tilde{a}_2^{2n}, \ldots, \tilde{a}_m^{2n}), \\ \tilde{a}_2^{2n+1} = \Im(\tilde{a}_2^{2n}, \tilde{a}_3^{2n}, \ldots, \tilde{a}_1^{2n}), \\ \vdots \\ \tilde{a}_m^{2n+1} = \Im(\tilde{a}_m^{2n}, \tilde{a}_1^{2n}, \tilde{a}_2^{2n}, \ldots, \tilde{a}_{m-1}^{2n}), \end{cases} \quad \text{and} \quad \begin{cases} \tilde{a}_1^{2n+2} = \sigma(\tilde{a}_1^{2n+1}, \tilde{a}_2^{2n+1}, \ldots, \tilde{a}_m^{2n+1}), \\ \tilde{a}_2^{2n+2} = \sigma(\tilde{a}_2^{2n+1}, \tilde{a}_3^{2n+1}, \ldots, \tilde{a}_1^{2n+1}), \\ \vdots \\ \tilde{a}_m^{2n+2} = \sigma(\tilde{a}_m^{2n+1}, \tilde{a}_1^{2n+1}, \ldots, \tilde{a}_{m-1}^{2n+1}), \end{cases} \tag{2}$$

for all $n \geq 0$, *then*

$$\begin{cases} d(\tilde{a}_1^{2n}, \tilde{a}_1^{2n+1}) \to d(\mathcal{P}, \mathcal{Q}), \\ d(\tilde{a}_2^{2n}, \tilde{a}_2^{2n+1}) \to d(\mathcal{P}, \mathcal{Q}), \\ \vdots \\ d(\tilde{a}_m^{2n}, \tilde{a}_m^{2n+1}) \to d(\mathcal{P}, \mathcal{Q}), \end{cases} \quad \text{and} \quad \begin{cases} d(\tilde{a}_1^{2n+1}, \tilde{a}_1^{2n+2}) \to d(\mathcal{P}, \mathcal{Q}), \\ d(\tilde{a}_2^{2n+1}, \tilde{a}_2^{2n+2}) \to d(\mathcal{P}, \mathcal{Q}), \\ \vdots \\ d(\tilde{a}_m^{2n+1}, \tilde{a}_m^{2n+2}) \to d(\mathcal{P}, \mathcal{Q}). \end{cases}$$

Proof. Consider

$$\begin{aligned} d(\tilde{a}_1^{2n}, \tilde{a}_1^{2n+1}) &= d(\tilde{a}_1^{2n}, \Im(\tilde{a}_1^{2n}, \tilde{a}_2^{2n}, \ldots, \tilde{a}_m^{2n})) \\ &= d\left(\begin{array}{c} \sigma(\tilde{a}_1^{2n-1}, \tilde{a}_2^{2n-1}, \ldots, \tilde{a}_m^{2n-1}) \\ \Im\left(\sigma(\tilde{a}_1^{2n-1}, \tilde{a}_2^{2n-1}, \ldots, \tilde{a}_m^{2n-1}), \sigma(\tilde{a}_2^{2n-1}, \tilde{a}_3^{2n-1}, \ldots, \tilde{a}_1^{2n-1}), \ldots, \sigma(\tilde{a}_m^{2n-1}, \tilde{a}_1^{2n-1}, \ldots, \tilde{a}_{m-1}^{2n-1})\right) \end{array}\right) \\ &\leq \frac{k}{m}\big[d(\tilde{a}_1^{2n-1}, \sigma(\tilde{a}_1^{2n-1}, \tilde{a}_2^{2n-1}, \ldots, \tilde{a}_m^{2n-1})) + d(\tilde{a}_2^{2n-1}, \sigma(\tilde{a}_2^{2n-1}, \tilde{a}_3^{2n-1}, \ldots, \tilde{a}_1^{2n-1})) \\ &\quad + \ldots + d(\tilde{a}_m^{2n-1}, \sigma(\tilde{a}_m^{2n-1}, \tilde{a}_1^{2n-1}, \ldots, \tilde{a}_{m-1}^{2n-1}))\big] + (1-k)d(\mathcal{P}, \mathcal{Q}) \\ &= \frac{k}{m}\left[d\left(\begin{array}{c} \Im(\tilde{a}_1^{2n-2}, \tilde{a}_2^{2n-2}, \ldots, \tilde{a}_m^{2n-2}) \\ \sigma\left(\Im(\tilde{a}_1^{2n-2}, \tilde{a}_2^{2n-2}, \ldots, \tilde{a}_m^{2n-2}), \Im(\tilde{a}_2^{2n-2}, \tilde{a}_3^{2n-2}, \ldots, \tilde{a}_1^{2n-2}), \ldots, \Im(\tilde{a}_m^{2n-2}, \tilde{a}_1^{2n-2}, \ldots, \tilde{a}_{m-1}^{2n-2})\right) \end{array}\right)\right. \\ &\quad + d\left(\begin{array}{c} \Im(\tilde{a}_2^{2n-2}, \tilde{a}_3^{2n-2}, \ldots, \tilde{a}_1^{2n-2}) \\ \sigma\left(\Im(\tilde{a}_2^{2n-2}, \tilde{a}_3^{2n-2}, \ldots, \tilde{a}_1^{2n-2}), \Im(\tilde{a}_3^{2n-2}, \tilde{a}_4^{2n-2}, \ldots, \tilde{a}_2^{2n-2}), \ldots, \Im(\tilde{a}_1^{2n-2}, \tilde{a}_2^{2n-2}, \ldots, \tilde{a}_m^{2n-2})\right) \end{array}\right) \\ &\quad \vdots \\ &\quad \left. + d\left(\begin{array}{c} \Im(\tilde{a}_m^{2n-2}, \tilde{a}_1^{2n-2}, \ldots, \tilde{a}_{m-1}^{2n-2}) \\ \sigma\left(\Im(\tilde{a}_m^{2n-2}, \tilde{a}_1^{2n-2}, \ldots, \tilde{a}_{m-1}^{2n-2}), \Im(\tilde{a}_1^{2n-2}, \tilde{a}_2^{2n-2}, \ldots, \tilde{a}_m^{2n-2}), \ldots, \Im(\tilde{a}_{m-1}^{2n-2}, \tilde{a}_m^{2n-2}, \ldots, \tilde{a}_{m-2}^{2n-2})\right) \end{array}\right)\right]. \end{aligned}$$

Using (1), we get

$$\begin{aligned} d(\tilde{a}_1^{2n}, \tilde{a}_1^{2n+1}) \leq{} & \frac{k}{m}\left[\frac{k}{m}(d(\tilde{a}_1^{2n-2}, \Im(\tilde{a}_1^{2n-2}, \tilde{a}_2^{2n-2}, \ldots, \tilde{a}_m^{2n-2})) + d(\tilde{a}_2^{2n-2}, \Im(\tilde{a}_2^{2n-2}, \tilde{a}_3^{2n-2}, \ldots, \tilde{a}_1^{2n-2}))\right. \\ & + \ldots + d(\tilde{a}_m^{2n-2}, \Im(\tilde{a}_m^{2n-2}, \tilde{a}_1^{2n-2}, \ldots, \tilde{a}_{m-1}^{2n-2}))) + (1-k)d(\mathcal{P}, \mathcal{Q}) \\ & + \frac{k}{m}(d(\tilde{a}_2^{2n-2}, \Im(\tilde{a}_2^{2n-2}, \tilde{a}_3^{2n-2}, \ldots, \tilde{a}_1^{2n-2})) + d(\tilde{a}_3^{2n-2}, \Im(\tilde{a}_3^{2n-2}, \tilde{a}_4^{2n-2}, \ldots, \tilde{a}_2^{2n-2})) \\ & + \ldots + d(\tilde{a}_1^{2n-2}, \Im(\tilde{a}_1^{2n-2}, \tilde{a}_2^{2n-2}, \ldots, \tilde{a}_m^{2n-2}))) + (1-k)d(\mathcal{P}, \mathcal{Q}) \end{aligned}$$

$$\begin{aligned}
&\vdots\\
&+\frac{k}{m}(d(\tilde{a}_m^{2n-2},\Im(\tilde{a}_m^{2n-2},\tilde{a}_1^{2n-2},\ldots,\tilde{a}_{m-1}^{2n-2}))+d(\tilde{a}_1^{2n-2},\Im(\tilde{a}_1^{2n-2},\tilde{a}_2^{2n-2},\ldots,\tilde{a}_m^{2n-2}))\\
&+\ldots+d(\tilde{a}_{m-1}^{2n-2},\Im(\tilde{a}_{m-1}^{2n-2},\tilde{a}_m^{2n-2},\ldots,\tilde{a}_{m-2}^{2n-2})))+(1-k)d(\mathcal{P},\mathcal{Q})\big]+(1-k)d(\mathcal{P},\mathcal{Q})\\
=\;&\frac{k^2}{m}(d(\tilde{a}_1^{2n-2},\Im(\tilde{a}_1^{2n-2},\tilde{a}_2^{2n-2},\ldots,\tilde{a}_m^{2n-2}))+d(\tilde{a}_2^{2n-2},\Im(\tilde{a}_2^{2n-2},\tilde{a}_3^{2n-2},\ldots,\tilde{a}_1^{2n-2}))\\
&+\ldots+d(\tilde{a}_m^{2n-2},\Im(\tilde{a}_m^{2n-2},\tilde{a}_1^{2n-2},\ldots,\tilde{a}_{m-1}^{2n-2})))+(1-k^2)d(\mathcal{P},\mathcal{Q}).
\end{aligned}$$

By mathematical induction, we get

$$\begin{aligned}
d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+1}) \;\le\; &\frac{k^{2n}}{m}(d(\tilde{a}_1^0,\Im(\tilde{a}_1^0,\tilde{a}_2^0,\ldots,\tilde{a}_m^0))+d(\tilde{a}_2^0,\Im(\tilde{a}_2^0,\tilde{a}_3^0,\ldots,\tilde{a}_1^0))\\
&+\ldots+d(\tilde{a}_m^0,\Im(\tilde{a}_m^0,\tilde{a}_1^0,\ldots,\tilde{a}_{m-1}^0)))+(1-k^{2n})d(\mathcal{P},\mathcal{Q}).
\end{aligned}$$

Taking $n\to\infty$, we have

$$d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+1})\to d(\mathcal{P},\mathcal{Q}).$$

Again, for each $n\in\mathbb{N}$, by induction one can write

$$\begin{aligned}
d(\tilde{a}_1^{2n+1},\tilde{a}_1^{2n+2}) \;\le\; &\frac{k^{2n}}{m}(d(\tilde{a}_1^1,\Im(\tilde{a}_1^1,\tilde{a}_2^1,\ldots,\tilde{a}_m^1))+d(\tilde{a}_2^1,\Im(\tilde{a}_2^1,\tilde{a}_3^1,\ldots,\tilde{a}_1^1))\\
&+\ldots+d(\tilde{a}_m^1,\Im(\tilde{a}_m^1,\tilde{a}_1^1,\ldots,\tilde{a}_{m-1}^1)))+(1-k^{2n})d(\mathcal{P},\mathcal{Q}).
\end{aligned}$$

After taking $n\to\infty$, we have

$$d(\tilde{a}_1^{2n+1},\tilde{a}_1^{2n+2})\to d(\mathcal{P},\mathcal{Q}).$$

Similarly, we can show that

$$d(\tilde{a}_2^{2n},\tilde{a}_2^{2n+1})\to d(\mathcal{P},\mathcal{Q}),d(\tilde{a}_2^{2n+1},\tilde{a}_2^{2n+2})\to d(\mathcal{P},\mathcal{Q}),\ldots,$$

$$d(\tilde{a}_m^{2n},\tilde{a}_m^{2n+1})\to d(\mathcal{P},\mathcal{Q}),d(\tilde{a}_m^{2n+1},\tilde{a}_m^{2n+2})\to d(\mathcal{P},\mathcal{Q}).$$

■

Lemma 2 *Let $\mathcal{P}$, $\mathcal{Q}$ be two nonempty closed subsets of a metric space $(\mathcal{Y},d)$ such that $(\mathcal{P},\mathcal{Q})$, $(\mathcal{Q},\mathcal{P})$ satisfy the peoperty UC and $\Im:\mathcal{P}^m\to\mathcal{Q}$, $\sigma:\mathcal{Q}^m\to\mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0,\tilde{a}_2^0,\ldots,\tilde{a}_m^0)\in\mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}$,..., $\{\tilde{a}_m^n\}$ are defined by (2), for all $n\in\mathbb{N}\cup\{0\}$, then for each $\epsilon>0$, there is $N_0>0$ so that*

$$\frac{1}{m}\Big(d(\tilde{a}_1^{2p},\tilde{a}_1^{2n+1})+d(\tilde{a}_2^{2p},\tilde{a}_2^{2n+1})+\ldots+d(\tilde{a}_m^{2p},\tilde{a}_m^{2n+1})\Big)<d(\mathcal{P},\mathcal{Q})+\epsilon, \tag{3}$$

for all $p>n\ge N_0$.

Proof. By Lemma 1, we get

$$d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+1})\to d(\mathcal{P},\mathcal{Q}),d(\tilde{a}_1^{2n+1},\tilde{a}_1^{2n+2})\to d(\mathcal{P},\mathcal{Q}),d(\tilde{a}_2^{2n},\tilde{a}_2^{2n+1})\to d(\mathcal{P},\mathcal{Q}),$$

$$d(\tilde{a}_2^{2n+1},\tilde{a}_2^{2n+2})\to d(\mathcal{P},\mathcal{Q}),\ldots,d(\tilde{a}_m^{2n},\tilde{a}_m^{2n+1})\to d(\mathcal{P},\mathcal{Q}),d(\tilde{a}_m^{2n+1},\tilde{a}_m^{2n+2})\to d(\mathcal{P},\mathcal{Q}).$$

Because the pair $(\mathcal{P},\mathcal{Q})$ fulfills the property UC, then we have

$$d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+2})\to 0,\;\; d(\tilde{a}_2^{2n},\tilde{a}_2^{2n+2})\to 0,\ldots,\;\; d(\tilde{a}_m^{2n},\tilde{a}_m^{2n+2})\to 0.$$

Also the pair $(\mathcal{Q}, \mathcal{P})$ fulfills the property UC, and then we have

$$d(\tilde{a}_1^{2n+1}, \tilde{a}_1^{2n+3}) \to 0,\ d(\tilde{a}_2^{2n+1}, \tilde{a}_2^{2n+3}) \to 0, \ldots,\ d(\tilde{a}_m^{2n+1}, \tilde{a}_m^{2n+3}) \to 0.$$

Suppose (3) is not true. Then for each $k' \in \mathbb{N}$ with $p_{k'} > n_{k'} \geq k'$, there is $\epsilon' > 0$ so that

$$\frac{1}{m}\left(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2n_{k'}+1}) + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2n_{k'}+1})\right) \geq d(\mathcal{P}, \mathcal{Q}) + \epsilon'.$$

This p_k' can be chosen such that it is the least integer greater than $n_{k'}$ to satisfy the above inequality. Now

$$\frac{1}{m}\left(d(\tilde{a}_1^{2p_{k'}-2}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}-2}, \tilde{a}_2^{2n_{k'}+1}) + \ldots + d(\tilde{a}_m^{2p_{k'}-2}, \tilde{a}_m^{2n_{k'}+1})\right) < d(\mathcal{P}, \mathcal{Q}) + \epsilon'.$$

Thus, we obtained

$$\begin{aligned} d(\mathcal{P}, \mathcal{Q}) + \epsilon' &\leq \frac{1}{m}\left(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2n_{k'}+1}) + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2n_{k'}+1})\right) \\ &\leq \frac{1}{m}\big(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2p_{k'}-2}) + d(\tilde{a}_1^{2p_{k'}-2}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2p_{k'}-2}) + d(\tilde{a}_2^{2p_{k'}-2}, \tilde{a}_2^{2n_{k'}+1}) \\ &\quad + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2p_{k'}-2}) + d(\tilde{a}_m^{2p_{k'}-2}, \tilde{a}_m^{2n_{k'}+1})\big) \\ &< \frac{1}{m}\big(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2p_{k'}-2}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2p_{k'}-2}) + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2p_{k'}-2})\big) + d(\mathcal{P}, \mathcal{Q}) + \epsilon'. \end{aligned}$$

As $k' \to \infty$, in above inequality we can write

$$\frac{1}{m}\left(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2n_{k'}+1}) + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2n_{k'}+1})\right) \to d(\mathcal{P}, \mathcal{Q}) + \epsilon'.$$

Using triangle inequality, we get

$$\begin{aligned} &\frac{1}{m}\left(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2n_{k'}+1}) + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2n_{k'}+1})\right) \\ \leq\ &\frac{1}{m}\big[d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2p_{k'}+2}) + d(\tilde{a}_1^{2p_{k'}+2}, \tilde{a}_1^{2n_{k'}+3}) + d(\tilde{a}_1^{2n_{k'}+3}, \tilde{a}_1^{2n_{k'}+1}) \\ &+ d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2p_{k'}+2}) + d(\tilde{a}_2^{2p_{k'}+2}, \tilde{a}_2^{2n_{k'}+3}) + d(\tilde{a}_2^{2n_{k'}+3}, \tilde{a}_2^{2n_{k'}+1}) + \ldots \\ &+ d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2p_{k'}+2}) + d(\tilde{a}_m^{2p_{k'}+2}, \tilde{a}_m^{2n_{k'}+3}) + d(\tilde{a}_m^{2n_{k'}+3}, \tilde{a}_m^{2n_{k'}+1})\big] \\ =\ &\frac{1}{m}\big[d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2p_{k'}+2}) + d(\sigma(\tilde{a}_1^{2p_{k'}+1}, \tilde{a}_2^{2p_{k'}+1}, \ldots, \tilde{a}_m^{2p_{k'}+1}), \Im(\tilde{a}_1^{2n_{k'}+2}, \tilde{a}_2^{2n_{k'}+2}, \ldots, \tilde{a}_m^{2n_{k'}+2})) \\ &+ d(\tilde{a}_1^{2n_{k'}+3}, \tilde{a}_1^{2n_{k'}+1})\big] + \frac{1}{m}\big[d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2p_{k'}+2}) + d(\sigma(\tilde{a}_2^{2p_{k'}+1}, \ldots, \tilde{a}_1^{2p_{k'}+1}), \Im(\tilde{a}_2^{2n_{k'}+2}, \ldots, \tilde{a}_1^{2n_{k'}+2})) \\ &+ d(\tilde{a}_2^{2n_{k'}+3}, \tilde{a}_2^{2n_{k'}+1})\big] + \ldots + \frac{1}{m}\big[d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2p_{k'}+2}) + d(\sigma(\tilde{a}_m^{2p_{k'}+1}, \ldots, \tilde{a}_{m-1}^{2p_{k'}+1}), \Im(\tilde{a}_m^{2n_{k'}+2}, \ldots, \tilde{a}_{m-1}^{2n_{k'}+2})) \\ &+ d(\tilde{a}_m^{2n_{k'}+3}, \tilde{a}_m^{2n_{k'}+1})\big]. \end{aligned}$$

Applying (1), we have

$$\begin{aligned} &\frac{1}{m}\left(d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2n_{k'}+1}) + d(\tilde{a}_2^{2p_{k'}}, \tilde{a}_2^{2n_{k'}+1}) + \ldots + d(\tilde{a}_m^{2p_{k'}}, \tilde{a}_m^{2n_{k'}+1})\right) \\ \leq\ &\frac{1}{m}\big[d(\tilde{a}_1^{2p_{k'}}, \tilde{a}_1^{2p_{k'}+2}) + \frac{k}{m}\big(d(\tilde{a}_1^{2p_{k'}+1}, \tilde{a}_1^{2n_{k'}+2}) + d(\tilde{a}_2^{2p_{k'}+1}, \tilde{a}_2^{2n_{k'}+2}) + \ldots + d(\tilde{a}_m^{2p_{k'}+1}, \tilde{a}_m^{2n_{k'}+2})\big) \end{aligned}$$

$$
\begin{aligned}
&+(1-k)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})+\ d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2p_{k'}+2})+\frac{k}{m}\big(d(\tilde{a}_2^{2p_{k'}+1},\tilde{a}_2^{2n_{k'}+2})\\
&+d(\tilde{a}_3^{2p_{k'}+1},\tilde{a}_3^{2n_{k'}+2})+\ldots+d(\tilde{a}_1^{2p_{k'}+1},\tilde{a}_1^{2n_{k'}+2})\big)+(1-k)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_2^{2n_{k'}+3},\tilde{a}_2^{2n_{k'}+1})+\ldots\\
&+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2p_{k'}+2})+\frac{k}{m}\big(d(\tilde{a}_m^{2p_{k'}+1},\tilde{a}_m^{2n_{k'}+2})+d(\tilde{a}_1^{2p_{k'}+1},\tilde{a}_1^{2n_{k'}+2})+\ldots+d(\tilde{a}_{m-1}^{2p_{k'}+1},\tilde{a}_{m-1}^{2n_{k'}+2})\big)\\
&+(1-k)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_m^{2n_{k'}+3},\tilde{a}_m^{2n_{k'}+1})\big]\\
=\ &\frac{1}{m}\big[d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+1},\tilde{a}_1^{2n_{k'}+3})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2p_{k'}+2})+d(\tilde{a}_2^{2n_{k'}+1},\tilde{a}_2^{2n_{k'}+3})+\ldots\\
&+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2p_{k'}+2})+d(\tilde{a}_m^{2n_{k'}+1},\tilde{a}_m^{2n_{k'}+3})\big]+\frac{k}{m}(d(\tilde{a}_1^{2p_{k'}+1},\tilde{a}_1^{2n_{k'}+2})+d(\tilde{a}_2^{2p_{k'}+1},\tilde{a}_2^{2n_{k'}+2})+\ldots\\
&+d(\tilde{a}_m^{2p_{k'}+1},\tilde{a}_m^{2n_{k'}+2}))+(1-k)d(\mathcal{P},\mathcal{Q}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{1}{m}\Big(d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2n_{k'}+1})+\ldots+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2n_{k'}+1})\Big)\\
\le\ &\frac{1}{m}\big[d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+1},\tilde{a}_1^{2n_{k'}+3})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2p_{k'}+2})+d(\tilde{a}_2^{2n_{k'}+1},\tilde{a}_2^{2n_{k'}+3})+\ldots\\
&+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2p_{k'}+2})+d(\tilde{a}_m^{2n_{k'}+1},\tilde{a}_m^{2n_{k'}+3})\big]+\frac{k}{m}\big[d(\Im(\tilde{a}_1^{2p_{k'}},\ldots,\tilde{a}_m^{2p_{k'}}),\sigma(\tilde{a}_1^{2n_{k'}+1},\ldots,\tilde{a}_m^{2n_{k'}+1}))\\
&+d(\Im(\tilde{a}_2^{2p_{k'}},\ldots,\tilde{a}_1^{2p_{k'}}),\sigma(\tilde{a}_2^{2n_{k'}+1},\ldots,\tilde{a}_1^{2n_{k'}+1}))+\ldots+d(\Im(\tilde{a}_m^{2p_{k'}},\ldots,\tilde{a}_{m-1}^{2p_{k'}}),\sigma(\tilde{a}_m^{2n_{k'}+1},\ldots,\tilde{a}_{m-1}^{2n_{k'}+1}))\big]\\
&+(1-k)d(\mathcal{P},\mathcal{Q})\\
\le\ &\frac{1}{m}\big[d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+1},\tilde{a}_1^{2n_{k'}+3})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2p_{k'}+2})+d(\tilde{a}_2^{2n_{k'}+1},\tilde{a}_2^{2n_{k'}+3})+\ldots\\
&+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2p_{k'}+2})+d(\tilde{a}_m^{2n_{k'}+1},\tilde{a}_m^{2n_{k'}+3})\big]+\frac{k^2}{m}(d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2n_{k'}+1})+\ldots\\
&+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2n_{k'}+1}))+k((1-k)d(\mathcal{P},\mathcal{Q}))+(1-k)d(\mathcal{P},\mathcal{Q}).
\end{aligned}
$$

Take $n\to\infty$, we obtain that

$$k^2\left(d(\mathcal{P},\mathcal{Q})+\epsilon'\right)+\left(1-k^2\right)d(\mathcal{P},\mathcal{Q})=d(\mathcal{P},\mathcal{Q})+k^2\epsilon',$$

which is a contradiction since $k<1$. This implies equation (3) holds. ■

Lemma 3 *Let $\mathcal{P}$, $\mathcal{Q}$ be two nonempty closed subsets of a metric space $(\mathcal{Y},d)$ such that $(\mathcal{P},\mathcal{Q})$, $(\mathcal{Q},\mathcal{P})$ satisfy the peoperty UC and $\Im:\mathcal{P}^m\to\mathcal{Q}$, $\sigma:\mathcal{Q}^m\to\mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0,\tilde{a}_2^0,\ldots,\tilde{a}_m^0)\in\mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}$,..., $\{\tilde{a}_m^n\}$ are defined by (2), for all $n\in\mathbb{N}\cup\{0\}$ then $\{\tilde{a}_1^{2n}\}$, $\{\tilde{a}_2^{2n}\}$,..., $\{\tilde{a}_m^{2n}\}$ are Cauchy sequences.*

Proof. Based on Lemma 1, one can get

$$d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+1})\to d(\mathcal{P},\mathcal{Q})\ \text{ and }\ d(\tilde{a}_1^{2n+1},\tilde{a}_1^{2n+2})\to d(\mathcal{P},\mathcal{Q}).$$

As $(\mathcal{P},\mathcal{Q})$ satisfies the property UC, then $d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+2})\to 0$. Similarly, since $(\mathcal{Q},\mathcal{P})$ verifies the property UC, then $d(\tilde{a}_1^{2n+1},\tilde{a}_1^{2n+3})\to 0$. Now we show that for every $\epsilon>0$, there exists $N\in\mathbb{N}$ so that

$$d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})\le d(\mathcal{P},\mathcal{Q})+\epsilon,\ \text{for all } p>n\ge N. \tag{4}$$

Suppose (4) is not true. Then for each $k'\in\mathbb{N}$ there exists $p_{k'}>n_{k'}\ge k'$ and $\epsilon>0$ such that

$$d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})>d(\mathcal{P},\mathcal{Q})+\epsilon.$$

This p'_k can be chosen such that it is the least integer greater than $n_{k'}$ to satisfy the above inequality. Now

$$\begin{aligned} d(\mathcal{P},\mathcal{Q})+\epsilon < d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1}) &\le d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}-2})+d(\tilde{a}_1^{2p_{k'}-2},\tilde{a}_1^{2n_{k'}+1})\\ &\le d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})+d(\mathcal{P},\mathcal{Q})+\epsilon. \end{aligned}$$

Taking $k'\to\infty$, we have

$$d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})\to d(\mathcal{P},\mathcal{Q})+\epsilon.$$

By Lemma 2, we can write

$$\frac{1}{m}\Big(d(\tilde{a}_1^{2p},\tilde{a}_1^{2n+1})+d(\tilde{a}_2^{2p},\tilde{a}_2^{2n+1})+\ldots+d(\tilde{a}_m^{2p},\tilde{a}_m^{2n+1})\Big)<d(\mathcal{P},\mathcal{Q})+\epsilon,$$

for all $p_{k'}>n_{k'}\ge N_0$. Then

$$\begin{aligned} d(\mathcal{P},\mathcal{Q})+\epsilon &< d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})\\ &\le d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2p_{k'}+2},\tilde{a}_1^{2n_{k'}+3})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &= d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\sigma(\tilde{a}_1^{2p_{k'}+1},\tilde{a}_2^{2p_{k'}+1},\ldots,\tilde{a}_m^{2p_{k'}+1}),\Im(\tilde{a}_1^{2n_{k'}+2},\tilde{a}_2^{2n_{k'}+2},\ldots,\tilde{a}_m^{2n_{k'}+2}))\\ &\quad +d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &\le d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+\frac{k}{m}\big(d(\tilde{a}_1^{2p_{k'}+1},\tilde{a}_1^{2n_{k'}+2})+d(\tilde{a}_2^{2p_{k'}+1},\tilde{a}_2^{2n_{k'}+2})+\ldots\\ &\quad +d(\tilde{a}_m^{2p_{k'}+1},\tilde{a}_m^{2n_{k'}+2})\big)+(1-k)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &= \frac{k}{m}\big[d(\Im(\tilde{a}_1^{2p_{k'}},\tilde{a}_2^{2p_{k'}},\ldots,\tilde{a}_m^{2p_{k'}}),\sigma(\tilde{a}_1^{2n_{k'}+1},\tilde{a}_2^{2n_{k'}+1},\ldots,\tilde{a}_m^{2n_{k'}+1}))\\ &\quad +d(\Im(\tilde{a}_2^{2p_{k'}},\tilde{a}_3^{2p_{k'}},\ldots,\tilde{a}_1^{2p_{k'}}),\sigma(\tilde{a}_2^{2n_{k'}+1},\tilde{a}_3^{2n_{k'}+1},\ldots,\tilde{a}_1^{2n_{k'}+1}))+\ldots\\ &\quad +d(\tilde{a}_m^{2p_{k'}},\tilde{a}_1^{2p_{k'}},\ldots,\tilde{a}_{m-1}^{2p_{k'}}),\sigma(\tilde{a}_m^{2n_{k'}+1},\tilde{a}_1^{2n_{k'}+1},\ldots,\tilde{a}_{m-1}^{2n_{k'}+1})\big]\\ &\quad +(1-k)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &\le \frac{k}{m}\Big[\frac{k}{m}\big(d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2n_{k'}+1})+\ldots+d(\tilde{a}_m^{2p_{k'}+1},\tilde{a}_m^{2n_{k'}+2})\big)\\ &\quad +(1-k)d(\mathcal{P},\mathcal{Q})+\frac{k}{m}\big(d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2n_{k'}+1})+d(\tilde{a}_3^{2p_{k'}},\tilde{a}_3^{2n_{k'}+1})+\ldots+d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})\big)\\ &\quad +(1-k)d(\mathcal{P},\mathcal{Q})+\ldots+\frac{k}{m}\big(d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2n_{k'}+1})+\ldots+d(\tilde{a}_{m-1}^{2p_{k'}},\tilde{a}_{m-1}^{2n_{k'}+1})\big)\\ &\quad +(1-k)d(\mathcal{P},\mathcal{Q})\big]+(1-k)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &= \frac{k^2}{m}\big(d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2n_{k'}+1})+d(\tilde{a}_2^{2p_{k'}},\tilde{a}_2^{2n_{k'}+1})+\ldots+d(\tilde{a}_m^{2p_{k'}},\tilde{a}_m^{2n_{k'}+1})\big)\\ &\quad +(1-k^2)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &< k^2(d(\mathcal{P},\mathcal{Q})+\epsilon)+(1-k^2)d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1})\\ &= k^2\epsilon+d(\mathcal{P},\mathcal{Q})+d(\tilde{a}_1^{2p_{k'}},\tilde{a}_1^{2p_{k'}+2})+d(\tilde{a}_1^{2n_{k'}+3},\tilde{a}_1^{2n_{k'}+1}). \end{aligned}$$

Letting $n\to\infty$, we have

$$d(\mathcal{P},\mathcal{Q})+\epsilon\le d(\mathcal{P},\mathcal{Q})+k^2\epsilon,$$

which is a contradiction. By inequality (4) $d(\tilde{a}_1^{2n},\tilde{a}_1^{2n+1})\to d(\mathcal{P},\mathcal{Q})$ and from the property of UC*, we get $\{\tilde{a}_1^{2n}\}$ is a Cauchy sequence. In a similar way, we can prove that $\{\tilde{a}_2^{2n}\}$, $\{\tilde{a}_3^{2n}\}$,..., $\{\tilde{a}_m^{2n}\}$ are Cauchy sequences. ■

We shall discuss the existence and convergence of m-tuple best proximity point for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC*.

Theorem 1 *Let $\mathcal{P}$, $\mathcal{Q}$ be two nonempty closed subsets of a metric space $(\mathcal{Y}, d)$ such that $(\mathcal{P}, \mathcal{Q})$, $(\mathcal{Q}, \mathcal{P})$ satisfy the peoperty UC* and $\Im : \mathcal{P}^m \to \mathcal{Q}$, $\sigma : \mathcal{Q}^m \to \mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0, \tilde{a}_2^0, \dots, \tilde{a}_m^0) \in \mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}$,..., $\{\tilde{a}_m^n\}$ are defined by (2), for all $n \in \mathbb{N} \cup \{0\}$. Then $\Im$ has a m-tuple best proximity point $(\Gamma_1, \Gamma_2, \dots, \Gamma_m) \in \mathcal{P}^m$ and σ has a m-tuple best proximity point $(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m) \in \mathcal{Q}^m$. Moreover, we have*

$$\tilde{a}_1^{2n} \to \Gamma_1,\ \tilde{a}_2^{2n} \to \Gamma_2, \dots,\ \tilde{a}_m^{2n} \to \Gamma_m \quad \textit{and} \quad \tilde{a}_1^{2n+1} \to \Gamma'_1,\ \tilde{a}_2^{2n+1} \to \Gamma'_2, \dots,\ \tilde{a}_m^{2n+1} \to \Gamma'_m.$$

In addition, if $\Gamma_2 = \Gamma_3 \dots = \Gamma_m$ and $\Gamma'_2 = \Gamma'_3 \dots = \Gamma'_m$, then

$$d(\Gamma_1, \Gamma'_1) + d(\Gamma_2, \Gamma'_2) + \dots + d(\Gamma_m, \Gamma'_m) = md(\mathcal{P}, \mathcal{Q}). \tag{5}$$

Proof. Based on Lemma 1, we conclude that $d(\tilde{a}_1^{2n}, \tilde{a}_1^{2n+1}) \to d(\mathcal{P}, \mathcal{Q})$. From Lemma 2, we have $\{\tilde{a}_2^{2n}\}$, $\{\tilde{a}_3^{2n}\}$,..., $\{\tilde{a}_m^{2n}\}$ are Cauchy sequences. Thus there are $\Gamma_1, \Gamma_2, \dots, \Gamma_m \in \mathcal{P}^m$, so that $\tilde{a}_1^{2n} \to \Gamma_1$, $\tilde{a}_2^{2n} \to \Gamma_2$,..., $\tilde{a}_m^{2n} \to \Gamma_m$. Hence, we have

$$d(\mathcal{P}, \mathcal{Q}) \leq d(\Gamma_1, \tilde{a}_1^{2n-1}) \leq d(\Gamma_1, \tilde{a}_1^{2n}) + d(\tilde{a}_1^{2n}, \tilde{a}_1^{2n-1}). \tag{6}$$

Taking $n \to \infty$ in equation (6), we find that

$$d(\Gamma_1, \tilde{a}_1^{2n-1}) \to d(\mathcal{P}, \mathcal{Q}).$$

By similar way, we have

$$d(\Gamma_2, \tilde{a}_2^{2n-1}) \to d(\mathcal{P}, \mathcal{Q}),\ d(\Gamma_3, \tilde{a}_3^{2n-1}) \to d(\mathcal{P}, \mathcal{Q}), \dots,\ d(\Gamma_m, \tilde{a}_m^{2n-1}) \to d(\mathcal{P}, \mathcal{Q}).$$

Now consider

$$\begin{aligned} d(\tilde{a}_1^{2n}, \Im(\Gamma_1, \Gamma_2, \dots, \Gamma_m)) =& d(\sigma(\tilde{a}_1^{2n-1}, \tilde{a}_2^{2n-1}, \dots, \tilde{a}_m^{2n-1}), \Im(\Gamma_1, \Gamma_2, \dots, \Gamma_m)) \\ \leq& \frac{k}{m}(d(\tilde{a}_1^{2n-1}, \Gamma_1) + \dots + d(\tilde{a}_m^{2n-1}, \Gamma_m)) + (1-k)d(\mathcal{P}, \mathcal{Q}). \end{aligned}$$

Taking $n \to \infty$, we obtain

$$d(\Gamma_1, \Im(\Gamma_1, \Gamma_2, \dots, \Gamma_m)) = d(\mathcal{P}, \mathcal{Q}).$$

Analogously, we can obtain

$$d(\Gamma_2, \Im(\Gamma_2, \Gamma_3, \dots, \Gamma_1)) = d(\mathcal{P}, \mathcal{Q}), \dots,\ d(\Gamma_m, \Im(\Gamma_m, \Gamma_1, \dots, \Gamma_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

Therefore, $(\Gamma_1, \Gamma_2, \dots, \Gamma_m)$ is a m-tuple best proximity point of $\Im$. Similarly, we can prove that there are $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m \in \mathcal{Q}^m$ so that $\tilde{a}_1^{2n+1} \to \Gamma'_1$, $\tilde{a}_2^{2n+1} \to \Gamma'_2, \dots,$ $\tilde{a}_m^{2n+1} \to \Gamma'_m$. Moreover we get

$$d(\Gamma'_1, \sigma(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m)) = d(\mathcal{P}, \mathcal{Q}), \dots,\ d(\Gamma'_m, \sigma(\Gamma'_m, \Gamma'_1, \dots, \Gamma'_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

Hence $(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m)$ is a m-tuple best proximity point of σ. Ultimately, let $\Gamma_2 = \Gamma_3 = \dots = \Gamma_m$ and $\Gamma'_2 = \Gamma'_3 = \dots = \Gamma'_m$, then we claim that equation (5) holds. For each $n \in \mathbb{N}$, one can write

$$\begin{aligned} d(\tilde{a}_1^{2n}, \tilde{a}_1^{2n+1}) =& d(\sigma(\tilde{a}_1^{2n-1}, \tilde{a}_2^{2n-1}, \dots, \tilde{a}_m^{2n-1}), \Im(\tilde{a}_1^{2n}, \tilde{a}_2^{2n}, \dots, \tilde{a}_m^{2n})) \\ \leq& \frac{k}{m}(d(\tilde{a}_1^{2n-1}, \tilde{a}_1^{2n}) + d(\tilde{a}_2^{2n-1}, \tilde{a}_2^{2n}) + \dots + d(\tilde{a}_m^{2n-1}, \tilde{a}_m^{2n})) + (1-k)d(\mathcal{P}, \mathcal{Q}). \end{aligned}$$

Taking $n \to \infty$, we get

$$d(\Gamma_1, \Gamma_1') \leq \frac{k}{m}(d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m')) + (1-k)d(\mathcal{P}, \mathcal{Q}). \tag{7}$$

Also for each $n \in \mathbb{N}$, one can write

$$\begin{aligned} d(\tilde{a}_2^{2n}, \tilde{a}_2^{2n+1}) =& d(\sigma(\tilde{a}_2^{2n-1}, \tilde{a}_3^{2n-1}, \ldots, \tilde{a}_1^{2n-1}), \Im(\tilde{a}_2^{2n}, \tilde{a}_3^{2n}, \ldots, \tilde{a}_1^{2n})) \\ \leq& \frac{k}{m}(d(\tilde{a}_2^{2n-1}, \tilde{a}_2^{2n}) + d(\tilde{a}_3^{2n-1}, \tilde{3}_2^{2n}) + \ldots + d(\tilde{a}_1^{2n-1}, \tilde{a}_1^{2n})) + (1-k)d(\mathcal{P}, \mathcal{Q}). \end{aligned}$$

Letting $n \to \infty$, we get

$$d(\Gamma_2, \Gamma_2') \leq \frac{k}{m}(d(\Gamma_2, \Gamma_2') + d(\Gamma_3, \Gamma_3') + \ldots + d(\Gamma_1, \Gamma_1')) + (1-k)d(\mathcal{P}, \mathcal{Q}). \tag{8}$$

Similarly we obtain

$$d(\Gamma_m, \Gamma_m') \leq \frac{k}{m}(d(\Gamma_m, \Gamma_m') + d(\Gamma_1, \Gamma_1') + \ldots + d(\Gamma_{m-1}, \Gamma_{m-1}')) + (1-k)d(\mathcal{P}, \mathcal{Q}). \tag{9}$$

By equations (7), (8) and (9), we get

$$\begin{aligned} d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m') \leq& k(d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m')) \\ &+ m(1-k)d(\mathcal{P}, \mathcal{Q}), \end{aligned}$$

this implies

$$d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m') \leq md(\mathcal{P}, \mathcal{Q}). \tag{10}$$

Since $d(\mathcal{P}, \mathcal{Q}) \leq d(\Gamma_1, \Gamma_1')$, $d(\mathcal{P}, \mathcal{Q}) \leq d(\Gamma_2, \Gamma_2')$,..., $d(\mathcal{P}, \mathcal{Q}) \leq d(\Gamma_m, \Gamma_m')$, we have

$$d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m') \geq md(\mathcal{P}, \mathcal{Q}). \tag{11}$$

By equations (10) and (11), we have

$$d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m') = md(\mathcal{P}, \mathcal{Q}).$$

■

Note that every pair of nonempty closed subsets $\mathcal{P}$, $\mathcal{Q}$ of a uniformly convex Banach space $\mathcal{Y}$, such that $\mathcal{P}$, is convex satisfies the property UC*. Therefore, we obtain the following result.

Corollary 1 *Let $\mathcal{Y}$ be a UCBS, $\mathcal{P}$, $\mathcal{Q}$ be two nonempty closed convex subsets of $\mathcal{Y}$ and $\Im : \mathcal{P}^m \to \mathcal{Q}$ and $\sigma : \mathcal{Q}^m \to \mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0, \tilde{a}_2^0, \ldots, \tilde{a}_m^0) \in \mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}$,..., $\{\tilde{a}_m^n\}$ are defined by (2), for all $n \in \mathbb{N} \cup \{0\}$. Then $\Im$ has a m-tuple best proximity point $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) \in \mathcal{P}^m$ and σ has a m-tuple best proximity point $(\Gamma_1', \Gamma_2', \ldots, \Gamma_m') \in \mathcal{Q}^m$. Moreover, we have*

$$\tilde{a}_1^{2n} \to \Gamma_1,\ \tilde{a}_2^{2n} \to \Gamma_2, \ldots,\ \tilde{a}_m^{2n} \to \Gamma_m \quad \text{and} \quad \tilde{a}_1^{2n+1} \to \Gamma_1',\ \tilde{a}_2^{2n+1} \to \Gamma_2', \ldots,\ \tilde{a}_m^{2n+1} \to \Gamma_m'.$$

In addition, if $\Gamma_2 = \Gamma_3 \ldots = \Gamma_m$ and $\Gamma_2' = \Gamma_3' \ldots = \Gamma_m'$, then

$$d(\Gamma_1, \Gamma_1') + d(\Gamma_2, \Gamma_2') + \ldots + d(\Gamma_m, \Gamma_m') = md(\mathcal{P}, \mathcal{Q}).$$

Theorem 2 *Let $\mathcal{P}$, $\mathcal{Q}$ two nonempty compact subsets of a metric space $(\mathcal{Y}, d)$ and $\Im : \mathcal{P}^m \to \mathcal{Q}$, $\sigma : \mathcal{Q}^m \to \mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0, \tilde{a}_2^0, \dots, \tilde{a}_m^0) \in \mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}, \dots, \{\tilde{a}_m^n\}$ are defined in Lemma 1, for all $n \in \mathbb{N} \cup \{0\}$. Then $\Im$ has a m-tuple best proximity point $(\Gamma_1, \Gamma_2, \dots, \Gamma_m) \in \mathcal{P}^m$ and σ has a m-tuple best proximity point $(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m) \in \mathcal{Q}^m$. Moreover, we have*

$$\tilde{a}_1^{2n} \to \Gamma_1,\ \tilde{a}_2^{2n} \to \Gamma_2, \dots,\ \tilde{a}_m^{2n} \to \Gamma_m \quad and \quad \tilde{a}_1^{2n+1} \to \Gamma'_1,\ \tilde{a}_2^{2n+1} \to \Gamma'_2, \dots,\ \tilde{a}_m^{2n+1} \to \Gamma'_m.$$

In addition, if $\Gamma_2 = \Gamma_3 \dots = \Gamma_m$ and $\Gamma'_2 = \Gamma'_3 \dots = \Gamma'_m$ then

$$d(\Gamma_1, \Gamma'_1) + d(\Gamma_2, \Gamma'_2) + \dots + d(\Gamma_m, \Gamma'_m) = md(\mathcal{P}, \mathcal{Q}).$$

Proof. Since $(\tilde{a}_1^0, \tilde{a}_2^0, \dots, \tilde{a}_m^0) \in \mathcal{P}$ and (2) holds for each $n \in \mathbb{N} \cup \{0\}$, we get

$$(\tilde{a}_1^{2n}, \tilde{a}_2^{2n}, \dots, \tilde{a}_m^{2n}) \in \mathcal{P} \quad \text{and} \quad (\tilde{a}_1^{2n+1}, \tilde{a}_2^{2n+1}, \dots, \tilde{a}_m^{2n+1}) \in \mathcal{Q}.$$

The compactness of $\mathcal{P}$ implies that sequence $(\tilde{a}_1^{2n}, \tilde{a}_2^{2n}, \dots, \tilde{a}_m^{2n})$ have convergent sequence $(\tilde{a}_1^{2n_{k'}}, \dots, \tilde{a}_m^{2n_{k'}})$, respectively, so that $\tilde{a}_1^{2n_{k'}} \to \Gamma_1$, $\tilde{a}_2^{2n_{k'}} \to \Gamma_2, \dots,$ $\tilde{a}_m^{2n_{k'}} \to \Gamma_m$ By same approach used in proof of Theorem 1, we have

$$d(\Gamma_1, \Im(\Gamma_1, \Gamma_2, \dots, \Gamma_m)) = d(\mathcal{P}, \mathcal{Q}), \dots,\ d(\Gamma_m, \Im(\Gamma_m, \Gamma_1, \dots, \Gamma_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

Since $\mathcal{Q}$ is compact, we can also show that

$$d(\Gamma'_1, \sigma(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m)) = d(\mathcal{P}, \mathcal{Q}), \dots,\ d(\Gamma'_m, \sigma(\Gamma'_m, \Gamma'_1, \dots, \Gamma'_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

Hence $(\Gamma_1, \Gamma_2, \dots, \Gamma_m)$ is a m-tuple best proximity point of $\Im$ in $\mathcal{P}^m$, $(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m)$ is a m-tuple best proximity point of σ in $\mathcal{Q}^m$ and

$$d(\Gamma_1, \Gamma'_1) + d(\Gamma_2, \Gamma'_2) + \dots + d(\Gamma_m, \Gamma'_m) = md(\mathcal{P}, \mathcal{Q}).$$

■

3 Consequences

In this section, we give some consequences of our main results.

Theorem 3 *Let $\mathcal{P}$, $\mathcal{Q}$ be two nonempty closed subsets of a complete metric space $(\mathcal{Y}, d)$ and $\Im : \mathcal{P}^m \to \mathcal{Q}$, $\sigma : \mathcal{Q}^m \to \mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0, \tilde{a}_2^0, \dots, \tilde{a}_m^0) \in \mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}, \dots, \{\tilde{a}_m^n\}$ are defined by (2), for all $n \in \mathbb{N} \cup \{0\}$. If $d(\mathcal{P}, \mathcal{Q}) = 0$, then $\Im$ has a m-tuple fixed point $(\Gamma_1, \Gamma_2, \dots, \Gamma_m) \in \mathcal{P}^m$ and σ has a m-tuple fixed point $(\Gamma'_1, \Gamma'_2, \dots, \Gamma'_m) \in \mathcal{Q}^m$. Moreover, we have*

$$\tilde{a}_1^{2n} \to \Gamma_1,\ \tilde{a}_2^{2n} \to \Gamma_2, \dots,\ \tilde{a}_m^{2n} \to \Gamma_m \quad and \quad \tilde{a}_1^{2n+1} \to \Gamma'_1,\ \tilde{a}_2^{2n+1} \to \Gamma'_2, \dots,\ \tilde{a}_m^{2n+1} \to \Gamma'_m.$$

In addition, if $\Gamma_2 = \Gamma_3 \dots = \Gamma_m$ and $\Gamma'_2 = \Gamma'_3 \dots = \Gamma'_m$, then $\Im$ and σ has common m-tuple fixed point in $(\mathcal{P} \cap \mathcal{Q})^m$.

Proof. Because $d(\mathcal{P}, \mathcal{Q}) = 0$, we find that the pairs $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{Q}, \mathcal{P})$ satisfy the property UC*. Using Theorem 1, we see that

$$d(\Gamma_1, \Im(\Gamma_1, \Gamma_2, \dots, \Gamma_m)) = d(\mathcal{P}, \mathcal{Q}), \dots,\ d(\Gamma_m, \Im(\Gamma_m, \Gamma_1, \dots, \Gamma_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

and

$$d(\Gamma'_1, \sigma(\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_m)) = d(\mathcal{P}, \mathcal{Q}), \ldots, \ d(\Gamma'_m, \sigma(\Gamma'_m, \Gamma'_1, \ldots, \Gamma'_{m-1})) = d(\mathcal{P}, \mathcal{Q}).$$

Since $d(\mathcal{P}, \mathcal{Q}) = 0$, we get

$$\Gamma_1 = \Im(\Gamma_1, \Gamma_2, \ldots, \Gamma_m), \Gamma_2 = \Im(\Gamma_2, \Gamma_3, \ldots, \Gamma_1)), \ldots, \ \Gamma_m = \Im(\Gamma_m, \Gamma_1, \ldots, \Gamma_{m-1}),$$

this means that $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$ is a m-tuple fixed point of $\Im$ in $\mathcal{P}^m$. Similarly

$$\Gamma'_1 = \sigma(\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_m), \Gamma'_2 = \sigma(\Gamma'_2, \Gamma'_3, \ldots, \Gamma'_1)), \ldots, \ \Gamma'_m = \sigma(\Gamma'_m, \Gamma'_1, \ldots, \Gamma'_{m-1}),$$

this means that $(\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_m)$ is a m-tuple fixed point of σ in $\mathcal{Q}^m$. Let $\Gamma_2 = \Gamma_3 \ldots = \Gamma_m$ and $\Gamma'_2 = \Gamma'_3 \ldots = \Gamma'_m$ then by Theorem 1, we can write

$$d(\Gamma_1, \Gamma'_1) + d(\Gamma_2, \Gamma'_2) + \ldots + d(\Gamma_m, \Gamma'_m) = md(\mathcal{P}, \mathcal{Q}).$$

Since $(\mathcal{P}, \mathcal{Q}) = 0$, we have

$$d(\Gamma_1, \Gamma'_1) + d(\Gamma_2, \Gamma'_2) + \ldots + d(\Gamma_m, \Gamma'_m) = 0,$$

it follows that $\Gamma_1 = \Gamma'_1$, $\Gamma_2 = \Gamma'_2, \ldots,$ $\Gamma_m = \Gamma'_m$. Hence $\Im$ and σ has common m-tuple fixed point $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) \in (\mathcal{P} \cap \mathcal{Q})^m$. ■

If $\mathcal{P} = \mathcal{Q}$ in Theorem 3, we have the result below:

Corollary 2 *Let $\mathcal{P}$ be a nonempty closed subsets of a complete metric space $(\mathcal{Y}, d)$ and $\Im : \mathcal{P}^m \to \mathcal{P}$, $\sigma : \mathcal{P}^m \to \mathcal{P}$ be two cyclic contraction mappings. If $(\tilde{a}_1^0, \tilde{a}_2^0, \ldots, \tilde{a}_m^0) \in \mathcal{P}^m$ and the sequence $\{\tilde{a}_1^n\}$, $\{\tilde{a}_2^n\}, \ldots,$ $\{\tilde{a}_m^n\}$ are defined by (2), for all $n \in \mathbb{N} \cup \{0\}$. Then $\Im$ has a m-tuple fixed point $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) \in \mathcal{P}^m$ and σ has a m-tuple fixed point $(\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_m) \in \mathcal{P}^m$. Moreover, we have*

$$\tilde{a}_1^{2n} \to \Gamma_1, \ \tilde{a}_2^{2n} \to \Gamma_2, \ldots, \ \tilde{a}_m^{2n} \to \Gamma_m \ \text{ and } \ \tilde{a}_1^{2n+1} \to \Gamma'_1, \ \tilde{a}_2^{2n+1} \to \Gamma'_2, \ldots, \ \tilde{a}_m^{2n+1} \to \Gamma'_m.$$

In addition, if $\Gamma_2 = \Gamma_3 \ldots = \Gamma_m$ and $\Gamma'_2 = \Gamma'_3 \ldots = \Gamma'_m$ then $\Im$ and σ has common m-tuple fixed point in $\mathcal{P}^m$.

Corollary 3 *Let $\mathcal{P}$ be a nonempty closed subsets of a complete metric space $(\mathcal{Y}, d)$ and $\Im : \mathcal{P}^m \to \mathcal{P}$ be a mapping*

$$d(\Im(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), \Im(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)) \leq \frac{k}{m}(d(\tilde{a}_1, \tilde{b}_1) + d(\tilde{a}_2, \tilde{b}_2) + \ldots + d(\tilde{a}_m, \tilde{b}_m),$$

for all $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m)$, $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m \in \mathcal{P}^m$ and $k \in (0,1)$. Then $\Im$ has a m-tuple fixed point $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) \in \mathcal{P}^m$.

4 Examples

In this section, we provide some illustrations to support our claims.

Example 3 *Consider $\mathcal{Y} = \mathbb{R}$ endowed with*

$$d(\tilde{a}_1, \tilde{a}_2) = |\tilde{a}_1 - \tilde{a}_2|.$$

Suppose that

$$\mathcal{P} = [1, 7] \text{ and } \mathcal{Q} = [-7, -1].$$

Then $d(\mathcal{P},\mathcal{Q}) = 2$. *Define* $\Im : \mathcal{P}^m \to \mathcal{Q}$ *and* $\sigma : \mathcal{Q}^m \to \mathcal{P}$ *by*

$$\Im((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m)) = \left(\frac{-\tilde{a}_1 - \tilde{a}_2 - \ldots - \tilde{a}_m - 3m}{4m}\right)$$

and

$$\sigma((\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) = \left(\frac{-\tilde{b}_1 - \tilde{b}_2 - \ldots - \tilde{b}_m + 3m}{4m}\right)$$

for all $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m) \in \mathcal{P}^m$, $(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \in \mathcal{Q}^m$ *and* $k = \frac{1}{4}$. *Then we obtain*

$$\begin{aligned}
& d(\Im(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), \sigma((\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)) \\
& d\left(\left(\frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_m}{2m}, 0, 0, \ldots, 0\right), \left(\frac{\tilde{b}_1 + \tilde{b}_2 + \ldots + \tilde{b}_m}{2m}, 0, 0, \ldots, 0\right)\right) \\
=& \left|\frac{-\tilde{a}_1 - \tilde{a}_2 - \ldots - \tilde{a}_m - 3m}{4m} - \frac{-\tilde{b}_1 - \tilde{b}_2 - \ldots - \tilde{b}_m + 3m}{4m}\right| \\
=& \frac{|\tilde{a}_1 - \tilde{b}_1| + |\tilde{a}_2 - \tilde{b}_2| + \ldots + |\tilde{a}_m - \tilde{b}_m|}{4m} + \frac{3}{2} \\
\leq& \frac{k}{m}(d(\tilde{a}_1, \tilde{b}_1) + d(\tilde{a}_2, \tilde{b}_2) + ... + d(\tilde{a}_m, \tilde{b}_m)) + (1-k)d(\mathcal{P},\mathcal{Q}).
\end{aligned}$$

This is a cyclic contraction with $k = \frac{1}{4}$. *Since* $\mathcal{P}$ *and* $\mathcal{Q}$ *are convex and closed subsets of a UCBS, the pairs* $(\mathcal{P},\mathcal{Q})$ *and* $(\mathcal{Q},\mathcal{P})$ *fulfils the property UC*. Therefore, all hypotheses of Corollary 1 are true. Hence,* $\Im$ *and* σ *have a* m*-tuple best proximity point say* $(1, 1, \ldots, 1) \in \mathcal{P}^m$ *and* $(-1, -1, \ldots, -1) \in \mathcal{Q}^m$ *respectively.*

Example 4 *Consider* $\mathcal{Y} = \mathbb{R}^m$ *endowed with*

$$d((\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)) = \max\left\{|\tilde{a}_1 - \tilde{b}_1|, |\tilde{a}_2 - \tilde{b}_2|, \ldots, |\tilde{a}_m - \tilde{b}_m|\right\},$$

$(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m), (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \in \mathcal{Y}$ *and suppose that*

$$\mathcal{P} = \{(\tilde{a}_1, 2, \ldots, 2) \in \mathcal{Y} : 0 \leq \tilde{a}_1 \leq 2\}, \ \mathcal{Q} = \left\{(\tilde{b}_1, 0, \ldots, 0) \in \mathcal{Y} : 0 \leq \tilde{b}_1 \leq 2\right\}.$$

Clearly $d(\mathcal{P},\mathcal{Q}) = 2$. *Define* $\Im : \mathcal{P}^m \to \mathcal{Q}$ *and* $\sigma : \mathcal{Q}^m \to \mathcal{P}$ *by*

$$\Im((\tilde{a}_1, 2, \ldots, 2), (\tilde{a}_2, 2, \ldots, 2), \ldots, (\tilde{a}_m, 2, \ldots, 2)) = \left(\frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_m}{m}, 0, 0, \ldots, 0\right)$$

and

$$\sigma((\tilde{b}_1, 0, \ldots, 0), (\tilde{b}_2, 0, \ldots, 0), \ldots, (\tilde{b}_m, 0, \ldots, 0)) = \left(\frac{\tilde{b}_1 + \tilde{b}_2 + \ldots + \tilde{b}_m}{m}, 2, 2, \ldots, 2\right).$$

Then we obtain

$$\begin{aligned}
& \begin{pmatrix} \Im((\tilde{a}_1, 2, \ldots, 2), (\tilde{a}_2, 2, \ldots, 2), \ldots, (\tilde{a}_m, 2, \ldots, 2)), \\ \sigma((\tilde{b}_1, 0, \ldots, 0), (\tilde{b}_2, 0, \ldots, 0), \ldots, (\tilde{b}_m, 0, \ldots, 0)) \end{pmatrix} \\
=& d\left(\left(\frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_m}{m}, 0, 0, \ldots, 0\right), \left(\frac{\tilde{b}_1 + \tilde{b}_2 + \ldots + \tilde{b}_m}{m}, 2, 2, \ldots, 2\right)\right)
\end{aligned}$$

$$= \max\left\{\left|\frac{\tilde{a}_1+\tilde{a}_2+\ldots+\tilde{a}_m}{2m}-\frac{\tilde{b}_1+\tilde{b}_2+\ldots+\tilde{b}_m}{2m}\right|, |2|, \ldots, |2|\right\} = 2.$$

Also

$$\begin{aligned}
&\frac{k}{m}\big(d((\tilde{a}_1,2,\ldots,2),(\tilde{b}_1,0,\ldots,0)) + d((\tilde{a}_2,2,\ldots,2),(\tilde{b}_2,0,\ldots,0)) + \ldots + \\
&d((\tilde{a}_m,2,\ldots,2),(\tilde{b}_m,0,\ldots,0))\big) + (1-k)d(\mathcal{P},\mathcal{Q}) \\
=&\frac{k}{m}\Bigg(\max\Big\{|\tilde{a}_1-\tilde{b}_1|,|2|,\ldots,|2|\Big\} + \max\Big\{|\tilde{a}_2-\tilde{b}_2|,|2|,\ldots,|2|\Big\} + \ldots + \\
&\max\Big\{|\tilde{a}_m-\tilde{b}_m|,|2|,\ldots,|2|\Big\}\Bigg) \\
=&\frac{k}{m}\times 2m + (1-k)2 = 2,
\end{aligned}$$

for any $k<1$. This show that

$$d(\Im(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m),\sigma((\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_m)) \le \frac{k}{m}(d(\tilde{a}_1,\tilde{b}_1)+...+d(\tilde{a}_m,\tilde{b}_m)) + (1-k)d(\mathcal{P},\mathcal{Q}).$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are convex and closed subsets of a UCBS, the pairs $(\mathcal{P},\mathcal{Q})$ and $(\mathcal{Q},\mathcal{P})$ fulfils the property UC. Therefore, all hypotheses of Corollary 1 are true. Hence, $\Im$ and σ have a m-tuple best proximity point say $(2,2,\ldots,2)\in\mathcal{P}^m$ and $(0,0,\ldots,0)\in\mathcal{Q}^m$ respectively.*

5 Applications

In this section, we obtain the solution for systems of delay differential and integral equations.

5.1 Application to Delay Differential Equations

In this paper, we consider the following system of delay differential equations:

$$\begin{cases}
\tilde{p}_1'(t) = f(t,\tilde{p}_1'(t),\tilde{p}_2'(t),\ldots,\tilde{p}_m'(t),\tilde{p}_1'(t-\tau),\tilde{p}_2'(t-\tau),\ldots,\tilde{p}_m'(t-\tau)),\ t\in[t_0,b],\\
\tilde{p}_2'(t) = f(t,\tilde{p}_2'(t),\tilde{p}_3'(t),\ldots,\tilde{p}_1'(t),\tilde{p}_2'(t-\tau),\tilde{p}_3'(t-\tau),\ldots,\tilde{p}_1'(t-\tau)),\ t\in[t_0,b],\\
\vdots\\
\tilde{p}_m'(t) = f(t,\tilde{p}_m'(t),\tilde{p}_1'(t),\ldots,\tilde{p}_{m-1}'(t),\tilde{p}_m'(t-\tau),\tilde{p}_1'(t-\tau),\ldots,\tilde{p}_{m-1}'(t-\tau)),\ t\in[t_0,b],
\end{cases} \tag{DFE}$$

with initial condition $\tilde{p}_1(t)=\psi_1(t),\ \tilde{p}_2(t)=\psi_2(t),\cdots,\ \tilde{p}_m(t)=\psi_m(t),\ t_0\in[t_0-\tau,t_0]$. It is well known that $\mathcal{Y}=(C([a,b]),||.||_\infty)$ is a Banach space where $C([a,b])$ denotes the space of all continuous real valued functions on a closed interval $[a,b]$ and $||.||_\infty$ is a Chebyshev norm.

$$||x-y||_\infty = \max_{t\in[a,b]}|x(t)-y(t)|.$$

Theorem 4 *Assume that the following conditions are satisfied:*

(M_1) $t_0,b\in\mathbb{R},\ \tau>0$.

(M_2) $f\in C([t_0,b]\times(\mathbb{R}^n)^m\times(\mathbb{R}^n)^m,\mathbb{R}^n)$.

(M_3) $\psi_1, \psi_2, \cdots, \psi_m \in C([t_0-\tau, b], \mathbb{R}^n)$.

(M_4) *There exists* $L_f > 0$ *such that*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i|, \text{ for all } u_i, v_i \in \mathbb{R},\ t \in [t_0, b].$$

(M_5) $2L_f(b-t_0) \leq \frac{1}{m^2}$.

Then the problem (DFE) has a unique solution on $C([t_0-\tau, b], \mathbb{R}^n) \cap C^1([t_0, b], \mathbb{R}^n)$.

Proof. Define the mapping $\mathcal{B} : \mathcal{Y}^m \to \mathcal{Y}$ by

$$\mathcal{B}(\tilde{p}'_1, \ldots, \tilde{p}'_m)(t) = \begin{cases} \psi_1(t), & t \in [t_0-\tau, t_0] \\ \psi_1(t_0) + \int_{t_0}^{t} f(s, \tilde{p}'_1(s), \ldots, \tilde{p}'_m(s), \tilde{p}'_1(s-\tau), \ldots, \tilde{p}'_m(s-\tau))ds, & t \in [t_0, b]. \end{cases}$$

Assume that $\tilde{p}'_1, \tilde{p}'_2, \ldots, \tilde{p}'_m, \tilde{q}'_1, \tilde{q}'_2, \ldots, \tilde{q}'_m \in \mathbb{R}^n$, we get

$$\begin{aligned}
&d(\mathcal{B}(\tilde{p}'_1, \ldots, \tilde{p}'_m)(t), \mathcal{B}(\tilde{q}'_1, \ldots, \tilde{q}'_m)(t)) \\
&= \max_{t\in[t_0-\tau,b]} |\mathcal{B}(\tilde{p}'_1, \ldots, \tilde{p}'_m)(t) - \mathcal{B}(\tilde{q}'_1, \ldots, \tilde{q}'_m)(t)| \\
&= \max_{t\in[t_0-\tau,b]} |\psi_1(t_0) + \int_{t_0}^{t} f(s, \tilde{p}'_1(s), \ldots, \tilde{p}'_m(s), \tilde{p}'_1(s-\tau), \ldots, \tilde{p}'_m(s-\tau))ds \\
&-\psi_1(t_0) - \int_{t_0}^{t} f(s, \tilde{q}'_1(s), \ldots, \tilde{q}'_m(s), \tilde{q}'_1(s-\tau), \ldots, \tilde{q}'_m(s-\tau))ds| \\
&\leq \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, \tilde{p}'_1(s), \ldots, \tilde{p}'_m(s), \tilde{p}'_1(s-\tau), \ldots, \tilde{p}'_m(s-\tau))ds \\
&-f(s, \tilde{q}'_1(s), \ldots, \tilde{q}'_m(s), \tilde{q}'_1(s-\tau), \ldots, \tilde{q}'_m(s-\tau))|ds \\
&\leq \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|\tilde{p}'_1 - \tilde{q}'_1| + \cdots + |\tilde{p}'_m - \tilde{q}'_m| + |\tilde{p}'_1(s-\tau) - \tilde{q}'_1(s-\tau)| + \cdots \\
&\quad + |\tilde{p}'_m(s-\tau) - \tilde{q}'_m(s-\tau))|ds \\
&\quad \leq \int_{t_0}^{t} L_f\big(\max_{t\in[t_0-\tau,b]} |\tilde{p}'_1 - \tilde{q}'_1| + \cdots + \max_{t\in[t_0-\tau,b]} |\tilde{p}'_m - \tilde{q}'_m| + \max_{t\in[t_0-\tau,b]} |\tilde{p}'_1(s-\tau) - \tilde{q}'_1(s-\tau)| + \cdots \\
&\quad + \max_{t\in[t_0-\tau,b]} |\tilde{p}'_m(s-\tau) - \tilde{q}'_m(s-\tau)|\big)ds \\
&\quad \leq \int_{t_0}^{t} L_f\big(||\tilde{p}'_1 - \tilde{q}'_1||_\infty + \cdots + ||\tilde{p}'_m - \tilde{q}'_m||_\infty + ||\tilde{p}'_1 - \tilde{q}'_1||_\infty + \cdots \\
&\quad + ||\tilde{p}'_m - \tilde{q}'_m||_\infty\big)ds \\
&\quad \leq 2L_f(b-t_0)(||\tilde{p}'_1 - \tilde{q}'_1||_\infty + \cdots + ||\tilde{p}'_m - \tilde{q}'_m||_\infty) \\
&\quad \leq \frac{1}{m^2}(d(\tilde{p}'_1, \tilde{q}'_1) + \cdots + d(\tilde{p}'_m, \tilde{q}'_m)) = \frac{k}{m}(d(\tilde{p}'_1, \tilde{q}'_1) + \cdots + d(\tilde{p}'_m, \tilde{q}'_m).
\end{aligned}$$

Hence Corollary 3 holds with $k = \frac{1}{m}$. Then the problem (DFE) has a unique solution on $C([t_0 - \tau, b], \mathbb{R}^n) \cap C^1([t_0, b], \mathbb{R}^n)$. ∎

Example 5 *Consider the system of 3 first order delay differential equation:*

$$\begin{cases} \hat{p}_1'(t) = \frac{1}{81}(\hat{p}_1'(t) - \hat{p}_1'(t-2) + \cdots + \hat{p}_3'(t) - \hat{p}_3'(t-2)), \\ \hat{p}_2'(t) = \frac{1}{81}(\hat{p}_2'(t) - \hat{p}_2'(t-2) + \cdots + \hat{p}_1'(t) - \hat{p}_1'(t-2)), \\ \hat{p}_3'(t) = \frac{1}{81}(\hat{p}_3'(t) - \hat{p}_3'(t-2) + \cdots + \hat{p}_2'(t) - \hat{p}_2'(t-2)), \ t \in [0,3], \end{cases} \tag{E}$$

with initial conditions

$$\hat{p}_1(t) = \hat{p}_2(t) = \psi_1(t) = \psi_2(t) = t, \ \hat{p}_3(t) = \psi_3(t) = e^t, \ t \in [-2,0]. \tag{12}$$

Next we prove that all the conditions $(M_1 - M_5)$ *are verified:*

(M_1) $t_0 = 0, \ b = 3, \ \tau = 2.$

(M_2) $f : C([0,3] \times (\mathbb{R})^3 \times (\mathbb{R})^3, \mathbb{R}).$

(M_3) $\psi \in C[-2,3] \to \mathbb{R}$ *is continuous.*

(M_4) *Define* $f : C([0,3] \times (\mathbb{R})^3 \times (\mathbb{R})^3 \to \mathbb{R}^3)$

$$\begin{aligned} f(t, \hat{p}_1'(t), \cdots, \hat{p}_3')(t), \hat{p}_1'(t-2), \cdots, \hat{p}_3'(t-2) &= \frac{1}{81}(\hat{p}_1'(t) - \hat{p}_1'(t-2) + \cdots + \hat{p}_3'(t) - \hat{p}_3'(t-2)) \\ f(t, \hat{p}_2'(t), \cdots, \hat{p}_1')(t), \hat{p}_1'(t-2), \cdots, \hat{p}_3'(t-2) &= \frac{1}{81}(\hat{p}_2'(t) - \hat{p}_2'(t-2) + \cdots + \hat{p}_1'(t) - \hat{p}_1'(t-2)) \\ f(t, \hat{p}_3'(t), \cdots, \hat{p}_2')(t), \hat{p}_1'(t-2), \cdots, \hat{p}_3'(t-2) &= \frac{1}{81}(\hat{p}_3'(t) - \hat{p}_3'(t-2) + \cdots + \hat{p}_2'(t) - \hat{p}_2'(t-2)). \end{aligned}$$

Then for any $\hat{p}_1'(t), \cdots, \hat{p}_3'(t), \ \hat{q}_1'(t), \cdots, \hat{q}_3'(t) \in \mathbb{R}$ *and* $t \in [0,3]$, *we have*

$$\begin{aligned} &|f(t, \hat{p}_1'(t), \cdots, \hat{p}_3'(t), \hat{p}_1'(t-2), \cdots, \hat{p}_3'(t-2)) - f(t, \hat{q}_1'(t), \cdots, \hat{q}_3'(t), \hat{q}_1'(t-2), \cdots, \hat{q}_3'(t-2))| \\ \leq &\frac{1}{81}|\hat{p}_1'(t) - \hat{q}_1'(t)| + |\hat{p}_1'(t-2) - \hat{q}_1'(t-2)| + \cdots + |\hat{p}_m'(t) - \hat{q}_m'(t)| + |\hat{p}_m'(t-2) - \hat{q}_m'(t-2)|. \end{aligned}$$

(M_5) $2L_f(b - t_0) = 2.\frac{1}{81}.(3-0) < \frac{1}{9}.$

All the hypotheses of Theorem 4 are satisfied with $m = 3$, *then the problem (DFE) has a solution on* $C([t_0 - \tau, b], \mathbb{R}^n) \cap C^1([t_0, b], \mathbb{R}^n)$. *The problem (E) and (12) can be reformulated in the following integral equation:*

$$\tilde{p}_1(t) = \begin{cases} t, \quad t \in [-2,0] \\ \psi_1(0) + \int_0^t \frac{1}{81}(\hat{p}_1'(t) - \hat{p}_1'(t-2) + \cdots + \hat{p}_3'(t) - \hat{p}_3'(t-2))ds, \quad t \in [0,5]. \end{cases}$$

The exact solution of the problem (E) and (12) is the function

$$\tilde{p}_1(t) = \begin{cases} t, \quad t \in [-2,0] \\ \frac{1}{81}(t^2 + (t-2)^2 + e^t - e^{t-2} + e^{-2} + 3), \quad t \in [0,5]. \end{cases}$$

5.2 Application to integral equations

Consider the following problem

$$\begin{aligned}\tilde{a}_1(\Bbbk) &= \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\tilde{a}_2(\varsigma),\ldots,\tilde{a}_m(\varsigma))d\varsigma \\ \tilde{a}_2(\Bbbk) &= \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)\eth(\Bbbk,\varsigma,\tilde{a}_2(\varsigma),\tilde{a}_3(\varsigma),\ldots,\tilde{a}_1(\varsigma))d\varsigma \\ &\vdots \\ \tilde{a}_m(\Bbbk) &= \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)\eth(\Bbbk,\varsigma,\tilde{a}_m(\varsigma),\tilde{a}_1(\varsigma),\ldots,\tilde{a}_{m-1}(\varsigma))d\varsigma, \end{aligned} \tag{13}$$

where $\Bbbk,\varsigma \in [0,\vartheta]$ with $\vartheta > 0$. Suppose that $\mathit{\Gamma} = C([0,\vartheta],\mathbb{R}^n)$ is endowed with

$$||\tilde{a}_1|| = \sup_{\Bbbk\in[0,\vartheta]} |\tilde{a}_1(\Bbbk)|, \text{ for all } \tilde{a}_1 \in \mathit{\Gamma}.$$

Moreover, define a distance on $\mathit{\Gamma}$ in the form of

$$\varrho(u,v) = \sup_{\Bbbk\in[0,\vartheta]} |\tilde{u}(\Bbbk) - \tilde{v}(\Bbbk)|$$

for all $\tilde{u},\tilde{v} \in \mathit{\Gamma}$. Hence $(\mathit{\Gamma},\varrho)$ is a complete metric space.

Theorem 5 *Suppose that the following hypotheses hold:*

($\mathcal{T}_1$) *the function* $\eth : [0,\vartheta]\times[0,\vartheta]\times(\mathbb{R}^n)^m \to \mathbb{R}^n$ *and* $\hbar : [0,\vartheta]\times[0,\vartheta] \to \mathbb{R}^n$ *are continuous so that*

$$\int_0^{\vartheta} \hbar(\Bbbk,\varsigma)d\varsigma \le \frac{\vartheta}{m^2}, \text{ for all } \Bbbk,\varsigma \in [0,\vartheta];$$

($\mathcal{T}_2$) *for all* $\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m,\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_m \in \mathbb{R}^n$, *we get*

$$\begin{aligned}&|\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\tilde{a}_2(\varsigma),\ldots,\tilde{a}_m(\varsigma)) - \eth(\Bbbk,\varsigma,\tilde{b}_1(\varsigma),\tilde{b}_2(\varsigma),\ldots,\tilde{b}_m(\varsigma))| \\ \le\ & \frac{1}{\vartheta}\left(|\tilde{a}_1-\tilde{b}_1| + |\tilde{a}_2-\tilde{b}_2| + \ldots + |\tilde{a}_m-\tilde{b}_m|\right), \text{ for all } \Bbbk,\varsigma \in [0,\vartheta].\end{aligned}$$

Then the problem (13) has a unique solution on $C([0,\vartheta],\mathbb{R}^n)$.

Proof. Define the mapping $\mathcal{B} : \mathit{\Gamma}^m \to \mathit{\Gamma}$ by

$$\mathcal{B}(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m)(\Bbbk) = \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\tilde{a}_2(\varsigma),\ldots,\tilde{a}_m(\varsigma))d\varsigma.$$

The existence solution of (13) is corresponding to find a m-tuple fixed point of $\mathcal{B}$. Assume that $\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m,\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_m \in \mathbb{R}^n$, we get

$$\begin{aligned}&\varrho(\mathcal{B}(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m),\mathcal{B}(\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_m)) \\ =&\sup_{\Bbbk\in[0,\vartheta]} |\mathcal{B}(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m)(\Bbbk) - \mathcal{B}(\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_m)(\Bbbk)|\end{aligned}$$

$$= \sup_{\Bbbk\in[0,\vartheta]} \left| \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\ldots,\tilde{a}_m(\varsigma))d\varsigma - \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)\eth(\Bbbk,\varsigma,\tilde{b}_1(\varsigma),\ldots,\tilde{b}_m(\varsigma))d\varsigma \right|$$
$$\leq \sup_{\Bbbk\in[0,\vartheta]} \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)|\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\ldots,\tilde{a}_m(\varsigma))d\varsigma - \eth(\Bbbk,\varsigma,\tilde{b}_1(\varsigma),\ldots,\tilde{b}_m(\varsigma))|d\varsigma$$
$$\leq \sup_{\Bbbk\in[0,\vartheta]} \int_0^{\vartheta} \hbar(\Bbbk,\varsigma)d\varsigma \left(\frac{1}{\vartheta}\left(|\tilde{a}_1-\tilde{b}_1|+|\tilde{a}_2-\tilde{b}_2|+\ldots+|\tilde{a}_m-\tilde{b}_m|\right)\right)$$
$$\leq \frac{1}{m^2}\sup_{\Bbbk\in[0,\vartheta]}\left(|\tilde{a}_1-\tilde{b}_1|+|\tilde{a}_2-\tilde{b}_2|+\ldots+|\tilde{a}_m-\tilde{b}_m|\right)$$
$$= \frac{k}{m}\left(\varrho(\tilde{a}_1,\tilde{b}_1)+\varrho(\tilde{a}_2,\tilde{b}_2)+\ldots+\varrho(\tilde{a}_m,\tilde{b}_m)\right).$$

Hence Corollary 3 holds with $k = \frac{1}{m}$. Then the problem (13) has a unique solution on $C([0,\vartheta],\mathbb{R}^n)$. ■

Example 6 *Consider a system of 5 integral equations below:*

$$\begin{cases}
\tilde{a}_1(\Bbbk) = \int_0^{25}\left(\Bbbk\varsigma^{-2} + \frac{1}{25}\frac{|\tilde{a}_1(\varsigma)|}{1+|\tilde{a}_1(\varsigma)|} + \frac{1}{75}\frac{|\tilde{a}_2(\varsigma)|}{3+|\tilde{a}_2(\varsigma)|} + \frac{1}{25}\frac{\cos|\tilde{a}_3(\varsigma)|}{1+\cos|\tilde{a}_3(\varsigma)|} + \frac{1}{25}\frac{\sin|\tilde{a}_4(\varsigma)|}{1+(\sin|\tilde{a}_4(\varsigma)|)} + \frac{1}{25}|\tilde{a}_5(\varsigma)|\right)d\varsigma,\\
\tilde{a}_2(\Bbbk) = \int_0^{25}\left(\Bbbk\varsigma^{-2} + \frac{1}{25}\frac{|\tilde{a}_2(\varsigma)|}{1+|\tilde{a}_2(\varsigma)|} + \frac{1}{75}\frac{|\tilde{a}_3(\varsigma)|}{3+|\tilde{a}_3(\varsigma)|} + \frac{1}{25}\frac{\cos|\tilde{a}_4(\varsigma)|}{1+\cos|\tilde{a}_4(\varsigma)|} + \frac{1}{25}\frac{\sin|\tilde{a}_5(\varsigma)|}{1+(\sin|\tilde{a}_5(\varsigma)|)} + \frac{1}{25}|\tilde{a}_1(\varsigma)|\right)d\varsigma,\\
\tilde{a}_3(\Bbbk) = \int_0^{25}\left(\Bbbk\varsigma^{-2} + \frac{1}{25}\frac{|\tilde{a}_3(\varsigma)|}{1+|\tilde{a}_3(\varsigma)|} + \frac{1}{75}\frac{|\tilde{a}_4(\varsigma)|}{3+|\tilde{a}_4(\varsigma)|} + \frac{1}{25}\frac{\cos|\tilde{a}_5(\varsigma)|}{1+\cos|\tilde{a}_5(\varsigma)|} + \frac{1}{25}\frac{\sin|\tilde{a}_1(\varsigma)|}{1+(\sin|\tilde{a}_1(\varsigma)|)} + \frac{1}{25}|\tilde{a}_2(\varsigma)|\right)d\varsigma,\\
\tilde{a}_4(\Bbbk) = \int_0^{25}\left(\Bbbk\varsigma^{-2} + \frac{1}{25}\frac{|\tilde{a}_4(\varsigma)|}{1+|\tilde{a}_4(\varsigma)|} + \frac{1}{75}\frac{|\tilde{a}_5(\varsigma)|}{3+|\tilde{a}_5(\varsigma)|} + \frac{1}{25}\frac{\cos|\tilde{a}_1(\varsigma)|}{1+\cos|\tilde{a}_1(\varsigma)|} + \frac{1}{25}\frac{\sin|\tilde{a}_2(\varsigma)|}{1+(\sin|\tilde{a}_2(\varsigma)|)} + \frac{1}{25}|\tilde{a}_3(\varsigma)|\right)d\varsigma,\\
\tilde{a}_5(\Bbbk) = \int_0^{25}\left(\Bbbk\varsigma^{-2} + \frac{1}{25}\frac{|\tilde{a}_5(\varsigma)|}{1+|\tilde{a}_5(\varsigma)|} + \frac{1}{75}\frac{|\tilde{a}_1(\varsigma)|}{3+|\tilde{a}_1(\varsigma)|} + \frac{1}{25}\frac{\cos|\tilde{a}_2(\varsigma)|}{1+\cos|\tilde{a}_2(\varsigma)|} + \frac{1}{25}\frac{\sin|\tilde{a}_3(\varsigma)|}{1+(\sin|\tilde{a}_3(\varsigma)|)} + \frac{1}{25}|\tilde{a}_4(\varsigma)|\right)d\varsigma,
\end{cases}$$

for all $\Bbbk \in [0,25]$ *and*

$$\begin{aligned}
&\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\tilde{a}_2(\varsigma),\ldots,\tilde{a}_5(\varsigma))\\
= &\int_0^{25}\left(\Bbbk\varsigma^{-2} + \frac{1}{25}\frac{|\tilde{a}_1(\varsigma)|}{1+|\tilde{a}_1(\varsigma)|} + \frac{1}{75}\frac{|\tilde{a}_2(\varsigma)|}{3+|\tilde{a}_2(\varsigma)|} + \frac{1}{25}\frac{\cos|\tilde{a}_3(\varsigma)|}{1+\cos|\tilde{a}_3(\varsigma)|} + \frac{1}{25}\frac{\sin|\tilde{a}_4(\varsigma)|}{1+(\sin|\tilde{a}_4(\varsigma)|)}\right.\\
&\left.+\frac{1}{25}|\tilde{a}_5(\varsigma)|\right)d\varsigma.
\end{aligned}$$

For the hypothesis $\mathcal{T}_2$*, we can write*

$$\begin{aligned}
&|\eth(\Bbbk,\varsigma,\tilde{a}_1(\varsigma),\tilde{a}_2(\varsigma),\ldots,\tilde{a}_5(\varsigma)) - \eth(\Bbbk,\varsigma,\tilde{b}_1(\varsigma),\tilde{b}_2(\varsigma),\ldots,\tilde{b}_5(\varsigma))|\\
\leq\ &\frac{1}{25}\left|\frac{|\tilde{a}_1(\varsigma)|}{1+|\tilde{a}_1(\varsigma)|} - \frac{|\tilde{b}_1(\varsigma)|}{1+|\tilde{b}_1(\varsigma)|}\right| + \frac{1}{75}\left|\frac{|\tilde{a}_2(\varsigma)|}{3+|\tilde{a}_2(\varsigma)|} - \frac{|\tilde{b}_2(\varsigma)|}{3+|\tilde{b}_2(\varsigma)|}\right|\\
&+\frac{1}{25}\left|\frac{\cos|\tilde{a}_3(\varsigma)|}{1+\cos|\tilde{a}_3(\varsigma)|} - \frac{\cos|\tilde{b}_3(\varsigma)|}{1+\cos|\tilde{b}_3(\varsigma)|}\right| + \frac{1}{25}\left|\frac{\sin|\tilde{a}_4(\varsigma)|}{1+(\sin|\tilde{a}_4(\varsigma)|)} - \frac{\sin|\tilde{b}_4(\varsigma)|}{1+(\sin|\tilde{b}_4(\varsigma)|)}\right|\\
&+\frac{1}{25}\left||\tilde{a}_5(\varsigma)| - |\tilde{b}_5(\varsigma)|\right|\\
\leq\ &\frac{1}{25}\Big(\left||\tilde{a}_1(\varsigma)| - |\tilde{b}_1(\varsigma)|\right| + \left||\tilde{a}_2(\varsigma)| - |\tilde{b}_2(\varsigma)|\right| + \left||\tilde{a}_3(\varsigma)| - |\tilde{b}_3(\varsigma)|\right| + \left||\tilde{a}_4(\varsigma)| - |\tilde{b}_4(\varsigma)|\right|
\end{aligned}$$

$$+ \left| |\tilde{a}_5(\varsigma)| - |\tilde{b}_5(\varsigma)| \right| \Big)$$

$$\leq \frac{1}{25} \left(|\tilde{a}_1 - \tilde{b}_1| + |\tilde{a}_2 - \tilde{b}_2| + |\tilde{a}_3 - \tilde{b}_3| + |\tilde{a}_4 - \tilde{b}_4| + |\tilde{a}_5 - \tilde{b}_5| \right).$$

All the hypotheses of Theorem 5 is satisfied with $\vartheta = 25$, $m = 5$ and $\hbar(\Bbbk, \varsigma) = \Bbbk\varsigma^{-2}$. Therefore, given problem has a unique solution.

6 Conclusion

The best proximity point has many applications such as finding the solution of integral equations, split feasibility and fixed point problems etc. The fixed point approach, is considered one of the different method for finding these solutions under the contraction mappings, due to its smoothness and clarity. So, in this paper, we introduce the notions of m-tuple best proximity point and establish the existence and convergence theorems of m-tuple best proximity point in metric spaces. Moreover, we apply these results in uniformly convex Banach space. We also provide some illustrations to back up our work. We also study some results on the existence and convergence of m-tuple fixed point. As an application of our obtained results, we find solutions for systems of delay differential equations and integral equations.

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