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Some Approximation Properties Of A Certain Summation Integral Type Operator*

Suruchi Maindola[†], Asha Ram Gairola[‡], Laxmi Rathour[¶], Lakshmi Narayan Mishra[‖]

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Abstract

We construct an operator of summation integral type and investigate its approximation properties. For smooth functions, we prove a convergence formula and find the rate of approximation in both a normed space and a weighted normed space. Relevant examples are used to support the findings.

1 Introduction

Let $B[0,1]$ and $H_n[0,1]$ denote the space of bounded functions and the space of all polynomials of degree at most n, respectively on $[0,1]$. Here, $\mathbb{N}$ is the set of positive integers and $n \in \mathbb{N}$. The Bernstein operator $B_n f : B[0,1] \longrightarrow H_n[0,1]$ is defined by

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

The Bernstein operators offer intriguing characteristics, such as the preservation of linearity, convexity, Lipschitz continuity, monotone, etc. (see [6]). The operators $B_n f(x)$, however, are not appropriate for approximating integrable functions. In order to approximate bounded and integrable functions on $[0,1]$, Durrmeyer [11] and Lupaş [25] introduced their integral modification of $B_n(f;x)$ by using the following equation:

$$D_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) \left((n+1) \int_0^1 p_{n,k}(u) f(u)\, \mathrm{d}u \right),$$

where $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ for $0 \le k \le n$. The operator $D_n(f;x)$ has been the subject of extensive research(see [1, 3, 7, 9, 10, 11, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32]). Derrienic [8] extensively studied the operator $D_n(f;x)$ for direct theorems and a number of approximation properties. For further work in this direction, we refer to [14, 15, 16, 17, 29]. It is noted that whenever $f \in C[0,1]$, the convergence of operators $D_n(f;x)$ to $f(x)$ is uniform. It turns out that no matter how smooth the function is, the order of approximation by the operators $D_n(f;x)$ is at best $O(n^{-1})$.

Baskakov [5] proposed the following set of operators for approximation on the positive axis:

$$V_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

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[†]Department of Mathematics, Doon University, Dehradun, Uttarakhand 248001, India
[‡]Department of Mathematics, Doon University, Dehradun, Uttarakhand 248001, India
[§]Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India
[¶]Corresponding author.
[‖]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India

For weighted simultaneous approximation, Moreno introduced a generalised form of the operator $V_n(f;x)$ in [24]. In [4], the q-variant of the operator $V_n(f;x)$ based on quantum calculus was presented and examined for approximation properties. In [33], Wang and Xu investigated the weighted approximation properties of $V_n(f;x)$ using Jacobi weights. L.S. Xie provided the following intriguing direct estimate of the local approximation in [34].

$$|V_n(f;x) - f(x)| = O\left(\omega^2_{\phi^\lambda}\left(f; n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)\right), \quad 0 \le \lambda \le 1.$$

For $\lambda = 1$, we get the ordinary rate of local approximation, and for $\lambda = 0$, we get the estimates of the Ditzian type (see [10]). The following Durrmeyer type modification of the Baskakov operator was defined by Gupta [19] in 1994 using the basis functions of the operator $V_n(f;x)$.

$$B_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \int_0^\infty b_{n,k}(y) f(y)\, dy, \quad x \in [0,\infty), \tag{1}$$

where $b_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^k}{(1+t)^{n+k+1}}$ and the Eüler beta function is $B(k+1,n)$. We modify the Baskakov beta operator's Durrmeyer variant (1) in an effort to achieve higher rates of convergence. Let $W_\mu[0,\infty)$ be the class of functions defined by

$$W_\mu[0,\infty) := \left\{ f \in C[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^\mu} < \infty \right\}.$$

Let $C^\infty[0,\infty)$ denote the class of all analytic functions on $[0,\infty)$. Then, for each $n \in \mathbb{N}$ the operator $\widetilde{L}_n f : W_\mu[0,\infty) \longrightarrow C^\infty[0,\infty)$

$$\widetilde{L}_n f(x) = (n+1) \sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x) \int_0^\infty \widetilde{q}_{n,k}(y) f(y)\, dy, \quad n \ge [\mu+1]. \tag{2}$$

Here $[x]$ is the integer part of x,

$$\widetilde{p}_{n,k}(x) = \binom{n+k}{k-1} \frac{x^{k-1}}{(1+x)^{n+k+1}},$$

$$\widetilde{q}_{n,k}(y) = \frac{1}{1+y}\Big(2nyP_{n-1,k-1}(y) + (1-ny)P_{n-2,k}(y) - (n-1)yP_{n,k-2}(y)\Big)$$

and $P_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$. The purpose of this study is to investigate error estimates in terms of modulus of continuity, approximation in the Lipschitz type space, weighted space, and degree of approximation. Although less exact, the notation $\widetilde{L}_n(f;x)$, is frequently used in the approximation theory for the value of the function $\widetilde{L}_n f$ at x.

2 Preliminaries

Some preliminary lemmas are supplied in order to establish the main results of this work. Let e_i be the function $e_i(y) = y^i, i \in \mathbb{N} \cup \{0\}, y \in [0,\infty)$.

Lemma 1 *For the sequence $\widetilde{L}_n(f;x)$, we have*

1. $$\widetilde{L}_n(e_0,x) = 1 + \frac{12}{(2-n)(n-3)}, \quad n > 3.$$

2. $$\widetilde{L}_n(e_1,x) = x + \frac{4(4+4n+15nx)}{(n-4)(n-3)(2-n)}, \quad n > 4.$$

3.

$$\widetilde{L}_n(e_2,x) = x^2 + \frac{6(1+n)(10+n)+90(1+n)(2+n)x+60\left(4+n+3n^2\right)x^2}{(n-5)(n-4)(n-3)(2-n)}, \quad n>5.$$

Lemma 2 *For the operators $\widetilde{L}_n$ we have the following*

1. *If $n>4$ then*

$$\widetilde{L}_n((e_1-xe_0);x) = \frac{16(1+n)(3x+1)}{((n-4)(n-3)(2-n))}.$$

2. *If $n>5$ then*

$$\widetilde{L}_n((e_1-xe_0)^2;x) = -\frac{2(n+1)\left(n\left(36x^2+29x+3\right)+10\left(24x^2+17x+3\right)\right)}{(n-5)(n-4)(n-3)(n-2)}.$$

3. *If $n>7$ then*

$$\begin{aligned}\widetilde{L}_n((e_1-xe_0)^4;x) \quad &= \quad \frac{12(1+n)\Big\{n^3x^2(1+x)^2+3n^2x(1+x)(8+x(47+59x))}{(n-7)(n-6)(n-5)(n-4)(n-3)(2-n)}\\ &+\frac{28(5+3x(18+x(71+x(123+80x))))}{(n-7)(n-6)(n-5)(n-4)(n-3)(2-n)}\\ &+\frac{n(30+2x(258+x(1222+x(2225+1351x))))\Big\}}{(n-7)(n-6)(n-5)(n-4)(n-3)(2-n)}.\end{aligned}$$

Remark 1 *It follows from simple calculations that $\widetilde{L}_n((e_1-xe_0)^s;x) = O\left(n^{-2}\right)$, $s-1\in\mathbb{N}$. This order is double that of the comparable classical operator in (1). However, for small values of n, the positivity is lost. As a result, we will assume n to be large enough that $\widetilde{L}_n(f,x)\geq 0$ whenever $f(x)$ is a positive function.*

3 Convergence of $\widetilde{L}_n$

We investigate several local approximation properties of the operators $\widetilde{L}_n$ using the usual modulus of continuity, the second modulus of smoothness, and Peetre's K-functional. The K-functional is defined as follows.

$$K_{2,\phi}(f,\rho^2) := \inf_{g\in AC_{loc}[0,\infty)}\left\{\|f-g\|+\rho^2\|\phi^2 g''\|\right\},$$

where g is differentiable and g' is absolutely continuous in every closed finite subinterval interval $[c,d]$ of $\mathbb{R}_0^+$. This fact is denoted by $g'\in AC_{loc}[0,\infty)$. To prove the direct theorem, we will use the relationship between Peetre K-functional and the corresponding Ditzian-Totik modulus of smoothness defined by

$$\omega_\phi^2(f,\rho) = \sup_{0\leq h\leq\rho}\ \sup_{x\pm h\phi(x)\in[0,\infty)} |f(x+h\phi^\lambda(x))-2f(x)+f(x-h\phi^\lambda(x))|$$

where the step weight function $\phi(x)=\sqrt{x}$, and $x\in[0,\infty)$. The equivalence of $K_{2,\phi}$ and ω_ϕ^2 is well known (see pp.11, [10]) i.e. there exist $C_1, C_2>0$ such that

$$C_1\omega_\phi^2(f,\rho)\leq K_{2,\phi}(f,\rho^2)\leq C_2\omega_\phi^2(f,\rho).$$

First, we have the following convergence theorem.

Theorem 1 *Let $\widetilde{L}_n f(x)$ be the sequence (2) and $x \in [a,b]$ where $0 \le a < b < \infty$. Then*

$$\lim_{n\to\infty} \widetilde{L}_n f(x) = f(x).$$

Moreover, the convergence is uniform.
Proof. *Since $\lim_{n\to\infty} \widetilde{L}_n(e_i;x) = e_i(x)$ for $i = 0,1,2,$ the proof follows by an application of Bohman-Korovkin's theorem.* ■

The following is a Voronovskaja-type convergence result.

Theorem 2 *Let $f \in W_\mu[0,\infty)$ and let $f''(x)$ exist at a certain point $x \in [a,b]$. Then*

$$\lim_{n\longrightarrow\infty} n^2 \left(\widetilde{L}_n f(x) - f(x)\right) = -16(1+3x)f'(x) - x(3+x(29+36x))f''(x).$$

Proof. By the Taylor's theorem,

$$f(y) = f(x) + (y-x)f'(x) + \frac{1}{2}(y-x)^2 f''(x) + \varepsilon(y,x)\,(y-x)^2 \tag{3}$$

where, $\lim_{y\longrightarrow x} \varepsilon(y,x) = 0$. Operating equation (3) by $\widetilde{L}_n$, we have

$$\widetilde{L}_n f(x) - f(x) = f'(x)\widetilde{L}_n((y-x);x) + \frac{1}{2}f''(x)\widetilde{L}_n((y-x)^2;x) + \widetilde{L}_n(\varepsilon(y;x)(y-x)^2;x). \tag{4}$$

Applying the Cauchy-Schwarz inequality to the last term of the equation (4), it follows that

$$\widetilde{L}_n(\varepsilon(y;x)(y-x)^2;x) \le \left(\widetilde{L}_n(\varepsilon(y;x);x)\right)^{\frac{1}{2}} \widetilde{L}_n(\varepsilon(y-x)^4;x)^{\frac{1}{2}}.$$

By Lemma 2 and method of [21], we have that

$$\lim_{n\to\infty} n^2 \widetilde{L}_n(\varepsilon(t;x)(y-x)^2;x) = 0. \tag{5}$$

Finally, by using Lemma 2 and combining (5) and (4) the proof follows. ■

Remark 2 *For a twice differentiable function f at x we have the order*

$$\widetilde{L}_n f(x) - f(x) = \mathcal{O}\left(\frac{1}{n^2}\right) \; as\, n \to \infty.$$

4 Degree of Approximation by $\widetilde{L}_n$

Theorem 3 (Local rate of approximation) *Let $\widetilde{L}_n f(x)$ be the operator (2), $x \in [a,b], 0 \le a < b < \infty$. Then for n sufficiently large*

$$\left|\widetilde{L}_n f(x) - f(x)\right| \le 2(1+r_n)\,\omega\left(f, \sqrt{\frac{\widetilde{L}_n(e_1 - xe_0)^2}{1+r_n}}\right) + r_n|f(x)|,$$

where $r_n = \frac{12}{(n-2)(n-3)}$.

Proof. We have

$$\left|\widetilde{L}_n f(x) - f(x)(1+r_n)\right| \leq (n+1)\sum_{k=0}^{\infty}\widetilde{p}_{n,k}(x)\int_0^{\infty}\widetilde{q}_{n,k}(y)|f(y)-f(x)|dy$$
$$\leq (n+1)\sum_{k=0}^{\infty}\widetilde{p}_{n,k}(x)\int_0^{\infty}\widetilde{q}_{n,k}(y)\omega(f,|y-x|)dy. \tag{6}$$

In view of the estimate,

$$\omega(f,\lambda\delta) \leq (\lambda+1)\omega(f,\delta), \qquad \lambda \geq 0,$$

the inequality (6) becomes

$$\left|\widetilde{L}_n f(x) - f(x)(1+r_n)\right| \leq \omega\left(f,\frac{1}{m}\right)\left((1+r_n) + m(n+1)\sum_{k=0}^{\infty}\widetilde{p}_{n,k}(x)\int_0^{\infty}\widetilde{q}_{n,k}(y)|y-x|dy\right). \tag{7}$$

Now using Schwarz inequality and Lemma 2 we get

$$(n+1)\sum_{k=0}^{\infty}\widetilde{p}_{n,k}(x)\int_0^{\infty}\widetilde{q}_{n,k}(y)|y-x|dy$$
$$\leq \left((n+1)\sum_{k=0}^{\infty}\widetilde{p}_{n,k}(x)\int_0^{\infty}\widetilde{q}_{n,k}(y)(y-x)^2dy\right)^{\frac{1}{2}}\left((n+1)\sum_{k=0}^{\infty}\widetilde{p}_{n,k}(x)\int_0^{\infty}\widetilde{q}_{n,k}(y)dy\right)^{\frac{1}{2}}$$
$$\leq (1+r_n)^{\frac{1}{2}}\left(\widetilde{L}_n(e_1-xe_0)^2\right)^{\frac{1}{2}}.$$

Therefore (7) implies that

$$\left|\widetilde{L}_n f(x) - f(x)\right| \leq \omega\left(f,\frac{1}{m}\right)(1+r_n)\left(1 + m\frac{\left(\widetilde{L}_n(e_1-xe_0)^2\right)^{\frac{1}{2}}}{(1+r_n)^{\frac{1}{2}}}\right).$$

The desired result follows by choosing $m = \left(\frac{1+r_n}{\widetilde{L}_n(e_1-xe_0)^2}\right)^{1/2}$. ∎

Theorem 4 *Let $f \in W_\mu[0,\infty)$, $x \in [\frac{16}{15},\infty)$. Then there exist a constant $C > 0$ independent of f, n and x such that*

$$|\widetilde{L}_n f(x) - f(x)| \leq C\omega_2\left(f,\sqrt{\psi_n(x)}\right) + \omega(f,\delta) + r_n|f(x)|, \quad \text{as } n \to \infty$$

where

$$\psi_n(x) = \max\left\{\left|\widetilde{L}_n((y-x)^2;x)\right|, \left(\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right)^2\right\}.$$

Proof. Let $g \in AC_{loc}[0,\infty)$ and the operator

$$\overline{L}_n f(x) = \widetilde{L}_n f(x) + f(x) - f\left(x - \frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right). \tag{8}$$

Then, by smoothness of g

$$g(y) = g(x) + (y-x)g'(x) + \int_x^y (y-u)g''(u)\,du. \tag{9}$$

By an application of $\overline{L}_n$ on (9) it follows that

$$\begin{aligned}\overline{L}_n g(x)-g(x)&=r_n g(x)+\overline{L}_n((y-x);x)g'(x)+\overline{L}_n\left(\int_x^y (y-u)g''(u)\,du;x\right)\\&=r_n g(x)+\overline{L}_n\left(\int_x^y (y-u)g''(u)\,du;x\right)\\&=r_n g(x)+\widetilde{L}_n\left(\int_x^y (y-u)g''(u)\,du;x\right)\\&\quad-\int_x^{x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}}\left(x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}-u\right)g''(u)\,du.\end{aligned}$$

Thus

$$\begin{aligned}\left|\overline{L}_n g(x)-g(x)\right|&\le r_n|g(x)|+\left|\widetilde{L}_n\left(\int_x^y (y-u)g''(u)\,du;x\right)\right|\\&\quad+\left|\int_x^{x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}}\left(x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}-u\right)g''(u)\,du\right|.\end{aligned}$$

Since

$$\left|\widetilde{L}_n\left(\int_x^y (y-u)g''(u)\,du;x\right)\right|\le\left|\widetilde{L}_n((y-x)^2;x)\right|\|g''\|$$

and

$$\left|\int_x^{x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}}\left(x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}-u\right)g''(u)\,du\right|\le\left(\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right)^2\|g''\|,$$

we have

$$\left|\overline{L}_n g(x)-g(x)\right|\le r_n|g(x)|+\left\{\left|\widetilde{L}_n((y-x)^2;x)\right|+\left(\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right)^2\right\}\|g''\|.$$

Thus

$$\left|\overline{L}_n g(x)-g(x)\right|\le r_n|g(x)|+2\psi_n(x)\,\|g''\|. \tag{10}$$

By (8), we get

$$\begin{aligned}\left|\overline{L}_n f(x)-f(x)\right|&\le\left|\overline{L}_n(f-g;x)\right|+|(f-g)(x)|\\&\quad+\left|\overline{L}_n g(x)-g(x)\right|+\left|f\left(x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right)-f(x)\right|.\end{aligned}$$

Since $\left|\overline{L}_n f(x)\right|\le C\|f\|$,

$$\left|\overline{L}_n f(x)-f(x)\right|\le C\|f-g\|+\left|\overline{L}_n g(x)-g(x)\right|+\left|f\left(x-\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right)-f(x)\right|.$$

Again by (10) we have

$$\left|\overline{L}_n f(x)-f(x)\right|\le(C+r_n)\|f-g\|+r_n|f(x)|+\psi_n(x)\,\|g''\|+\omega\left(f,\frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}\right).$$

By taking infimum of right hand side over all $g\in AC_{loc}[0,\infty)$ we have that

$$\begin{aligned}\left|\overline{L}_n f(x)-f(x)\right|&\le Cr_n|f(x)|+\inf_g\{\|f-g\|+\psi_n(x)\,\|g''\|\}+\omega(f,\delta)\\&\le CK_2(f,\psi_n(x))+\omega(f,\delta)+r_n|f(x)|.\end{aligned}$$

Finally by (3), we get

$$\left|\overline{L}_n f(x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\psi_n(x)}\right) + \omega(f, \delta_n(x)) + r_n|f(x)|,$$

where,

$$\delta_n(x) = \frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)}.$$

Hence the proof is completed. ■

Finally, we obtain degree of approximation of a Lipschitz-type function. We consider the following Lipschitz-type space (see [12]),

$$\mathrm{Lip}^*_M(r) := \left\{ f \in W_\mu[0,\infty) : |f(y) - f(x)| \leq M \frac{|y-x|^r}{(y+x)^{\frac{r}{2}}}; x, y \in (0,\infty) \right\},$$

where M is a positive constant and $0 < r \leq 1$.

Theorem 5 *Let $f \in Lip^*_M(r)$. Then for all $x \in (0,\infty)$*

$$|L_n f(x) - f(x)| \leq M\lambda^{\frac{r}{2}}(n,x), \quad n \to \infty$$

where $\lambda(n,x) = \frac{\left(\widetilde{L}_n(|y-x|;x)\right)^2}{x}$.

Proof. First we assume that $r = 1$. Then for $f \in \mathrm{Lip}^*_M(1)$,

$$\begin{aligned} |L_n f(x) - f(x)| &\leq (n+1)\sum_{n=0}^{\infty} \widetilde{p}_{n,k}(x) \int_0^\infty \widetilde{q}_{n,k}(y)|f(y) - f(x)|dy \\ &\leq (n+1)\sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x) \int_0^\infty \widetilde{q}_{n,k}(y) M \frac{|y-x|}{(y+x)^{1/2}}\, dy. \end{aligned}$$

Since, $\frac{1}{\sqrt{y+x}} \leq \frac{1}{\sqrt{x}}$, we can write

$$\begin{aligned} |L_n f(x) - f(x)| &\leq \frac{M(n+1)}{\sqrt{x}} \sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x) \int_0^\infty \widetilde{q}_{n,k}(y) M|y-x|\, dy \\ &= \frac{M}{\sqrt{x}} \widetilde{L}_n(|y-x|;x) \leq M\left(\sqrt{\frac{\widetilde{L}^2_n(|y-x|;x)}{x}}\right). \end{aligned}$$

The result is true for $r = 1$. Now suppose that $r \in (0,1)$. Considering $p = \frac{1}{r}, q = \frac{1}{1-r}$ and then applying Hölder's inequality twice we get

$$\begin{aligned} |L_n f(x) - f(x)| &\leq (n+1)\sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x) \int_0^\infty \widetilde{q}_{n,k}(y)|f(y) - f(x)|\, dy \\ &\leq \left\{ \sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x) \left((n+1)\int_0^\infty \widetilde{q}_{n,k}(y)|f(y) - f(x)|dy \right)^{\frac{1}{r}} \right\}^r \\ &\leq \left\{ \sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x)(n+1) \int_0^\infty \widetilde{q}_{n,k}(y)|f(y) - f(x)|^{\frac{1}{r}}\, dy \right\}^r. \end{aligned}$$

As $f \in Lip^*_M(r)$, this leads to

$$|L_n f(x) - f(x)| \leq M\left\{ \sum_{k=0}^{\infty} \widetilde{p}_{n,k}(x)(n+1) \int_0^\infty \widetilde{q}_{n,k}(y) \frac{|y-x|}{(y+x)^{1/2}}\, dy \right\}^r \leq \frac{M}{x^{\frac{r}{2}}} \left(\widetilde{L}_n(|y-x|;x)\right)^r.$$

This completes the proof. ■

5 Weighted Approximation

Let $D_\rho[0,\infty)$ be the class of real valued functions f on $[0,\infty)$ such that $|f(x)| \leq M_f \rho(x)$, where $\rho(x)$ is a weight function. Let $C_\rho[0,\infty)$ be the sub space of $D_\rho[0,\infty)$ consisting of of all continuous functions f such that $\lim_{x\longrightarrow\infty} \frac{|f(x)|}{\rho(x)}$ is finite. The defining norm of the normed space $C_\rho[0,\infty)$ is $\|f\|_\rho := \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$. The usual modulus of continuity of f on $[0,h]$ is defined as

$$\omega_h(f,\delta) = \sup_{|y-x|\leq\delta} \sup_{x,y\in[0,h]} |f(y)-f(x)|.$$

Theorem 6 *If $f \in C_\rho[0,\infty)$, then*

$$|L_n f(x) - f(x)| \leq 4M_f(1+h^2)\psi_n(x) + 2\omega_h(f, \sqrt{\psi_n(x)}),$$

where $\psi_n(x) = \widetilde{L}_n((y-x)^2; x)$.

Proof. Let $x \in [0,h]$ and $y > h+1$. Since $y - x > 1$, we have

$$|f(y) - f(x)| \leq M_f(2 + y^2 + x^2) \leq M_f(y-x)^2(1+x^2) \leq 4M_f(y-x)^2(1+h^2). \tag{11}$$

If $x \in [0,h]$ and $y \in [0,h+1]$, then for any $\delta > 0$

$$|f(y) - f(x)| \leq \omega_{h+1}(f; |y-x|) \leq \left(1 + \frac{|y-x|}{\delta}\right)\omega_{h+1}(f;\delta). \tag{12}$$

Now by the estimates (11) and (12), for all $y \geq 0$ and $x \in [0,h]$,

$$|f(y) - f(x)| \leq 4M_f(y-x)^2(1+h^2) + \left(1 + \frac{|y-x|}{\delta}\right)\omega_{h+1}(f;\delta). \tag{13}$$

An application of the Cauchy–Schwarz inequality in (13) leads to

$$|\widetilde{L}_n f(x) - f(x)| \leq 4M_f(1+h^2)\widetilde{L}_n((y-x)^2;x) + \left(1 + \omega_{h+1}(f;\delta)\frac{1}{\delta}\widetilde{L}_n((y-x)^2);x\right)^{\frac{1}{2}}.$$

Finally, by choosing $\delta = \psi_n(x)$ we get our result. ■

Theorem 7 *Let $f \in C_\rho[0,\infty)$.Then we have*

$$\lim_{n\longrightarrow\infty} \|\widetilde{L}_n f - f\|_\rho = 0. \tag{14}$$

Proof. By Korovkin type theorem (see [13]), it is sufficient to verify that $\lim_{n\longrightarrow\infty} \|\widetilde{L}_n e_i - e_i\|_\rho = 0$ for $i = 0,1,2$. Since $\widetilde{L}_n(e_0;x) \leq 1$, for $n \geq 5$, (14) holds for $i = 0$. Now for $i = 1$ we make use of the Lemma 1. Thus

$$\|\widetilde{L}_n e_1 - e_1\|_\rho = \sup_{x\in[0,\infty)} \left| x - \frac{16(n+1)(1+3x)}{(n-4)(n-3)(n-2)} - x \right| \frac{1}{1+x^2} \leq \frac{4(4+19)}{(n-4)(n-3)(n-2)}.$$

Thus

$$\lim_{n\longrightarrow\infty} \|\widetilde{L}_n e_1 - e_1\|_\rho = 0.$$

Similarly we prove for $i = 2$. By using Lemma 1 we get

$$\|\widetilde{L}_n e_2 - e_2\|_\rho = \sup_{x\in[0,\infty)} \left| \frac{6(1+n)(10+n) + 90(1+n)(2+n)x + 60\left(4+n+3n^2\right)x^2}{(n-5)(n-4)(n-3)(n-2)} \right| \frac{1}{1+x^2}$$
$$\leq \frac{6(1+n)(10+n) + 90(1+n)(2+n)x + 60\left(4+n+3n^2\right)}{(n-5)(n-4)(n-3)(n-2)}.$$

Therefore,

$$\lim_{n\longrightarrow\infty}\|\widetilde{L}_n e_2 - e_2\|_\rho = 0.$$

This completes the proof. ■

Let $f \in C_\rho[0,\infty)$. Then weighted modulus of continuity was defined by Yuksel and Ispir in [35]

$$\Omega(f;\delta) = \sup_{x\geq 0, 0\leq h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2},$$

and satisfies the following properties.

Lemma 3 *Let $f \in C_\rho[0,\infty)$. Then*

(i) $\Omega(f;\delta)$ *is monotone increasing function of* δ,

(ii) $\lim_{\delta\longrightarrow 0^+}\Omega(f;\delta) = 0$,

(iii) *for each* $p \in \mathbb{N}, \Omega(f;p\delta) \leq p\Omega(f;\delta)$,

(iv) *for each* $\lambda \in \mathbb{R}^+, \Omega(f;\lambda\delta) \leq (\lambda+1)\Omega(f;\lambda\delta)$.

Theorem 8 *Let $f \in C_\rho[0,\infty)$. Then there exists a positive constant R such that*

$$\sup_{x\in[0,\infty)}\frac{\left|\widetilde{L}_n f(x) - f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leq R\omega\left(f,\frac{1}{n}\right).$$

Proof. By the definition of $\Omega(f;\lambda\delta)$, Lemma 3, (see [2]) for $y>0$, $x\in[0,\infty)$ and $\delta>0$,

$$\begin{aligned}|f(y)-f(x)| &\leq (1+(x+|y-x|)^2)\omega\left(f,|y-x|\right)\\ &\leq 2(1+x^2)(1+(y-x)^2)\left(1+\frac{|y-x|}{\delta}\right)\omega\left(f,\delta\right).\end{aligned}$$

Therefore,

$$\left|\widetilde{L}_n f(x) - f(x)\right| \leq 2(1+x^2)\omega\left(f,\delta\right)\left\{1+\widetilde{L}_n((y-x)^2;x)+\widetilde{L}_n\left((1+(y-x)^2)\frac{|y-x|}{\delta};x\right)\right\}. \tag{15}$$

Now using Lemma 2 to the equation 15, we have

$$\widetilde{L}_n((y-x)^2;x) \leq R_1\frac{1+x^2}{n^2}$$

where R_1 is positive constant.

Next using the Cauchy-Schwarz inequality, we have

$$\widetilde{L}_n\left((1+(y-x)^2)\frac{|y-x|}{\delta};x\right) \leq \frac{1}{\delta}\sqrt{\widetilde{L}_n((y-x)^2;x)} + \frac{1}{\delta}\sqrt{\widetilde{L}_n((y-x)^4;x)}\sqrt{\widetilde{L}_n((y-x)^2;x)}. \tag{16}$$

By the Lemma 2 for some positive constant R_2 there holds

$$\sqrt{\widetilde{L}_n((y-x)^4;x)} \leq R_2\frac{1+x^2}{n^2}.$$

Combining all the above estimates we have that $R = 2(1+R_1+\sqrt{R_1}+R_2\sqrt{R_1})$. Finally, by choosing $\delta = \frac{1}{n}$, we obtain the desired result. ■

6 Numerical Verification

We discuss numerical examples in this section to analyze the theoretical findings that were presented in the preceding sections.

First, the function $f(x) = \frac{x}{(1+x)^2}$, $x \in [0, \infty)$ is compared with $\widetilde{L}_n(f;x)$ at $n = 5$, 10, and 20. This example explains the convergence of the sequence $\widetilde{L}_n(f;x)$, to the function $f(x)$ with respect to the value of n. The table 1 provides the values of $f(x)$ and $\widetilde{L}_n(f;x)$ at different nodes with step size 0.1. And, the corresponding errors $|f(x) - \widetilde{L}_n f(x)|$ is illustrated in Figure 2.

Table 1: Comparison of $\widetilde{L}_n(f;x)$ with respect to the function $f(x) = \frac{x}{(1+x)^2}$. for $n = 5, 10, 20$ and the absolute errors $|f(x) - \widetilde{L}_n f(x)|$.

		$\widetilde{L}_n f(x)$			$\lvert f(x) - \widetilde{L}_n f(x)\rvert$		
x	$f(x)$	$n=5$	$n=10$	$n=20$	$n=5$	$n=10$	$n=20$
0	0.	-0.2	-0.0444444	-0.0105263	0.2	0.0444444	0.0105263
0.1	0.0826446	-0.0676183	0.0492529	0.0747361	0.150263	0.0333918	0.00790858
0.2	0.138889	0.023148	0.113169	0.132797	0.115741	0.0257202	0.00609162
0.3	0.177515	0.0864816	0.157285	0.172724	0.0910332	0.0202296	0.00479122
0.4	0.204082	0.131195	0.187885	0.200246	0.0728863	0.016197	0.00383612
0.5	0.222222	0.162963	0.209053	0.21910	0.0592593	0.0131687	0.00311891
0.6	0.234375	0.185547	0.223524	0.231805	0.0488281	0.0108507	0.0025699
0.7	0.242215	0.201506	0.233168	0.240072	0.0407083	0.00904629	0.00214254
0.8	0.246914	0.21262	0.239293	0.245109	0.0342936	0.00762079	0.00180492
0.9	0.249307	0.220149	0.242828	0.247773	0.0291588	0.00647973	0.00153467
1	0.25	0.225	0.244444	0.248684	0.025	0.00555556	0.00131579

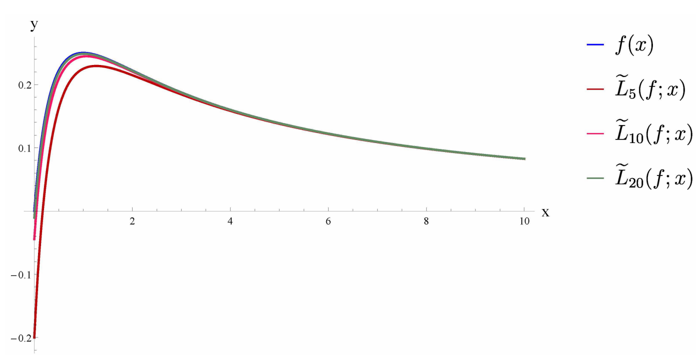


Figure 1: Comparison of the behavior of $f(x)$ and $\widetilde{L}_n(f;x)$ for $n = 5, 10, 20$.

Our next example is the function $x^2 \exp(-x)$, $0 \leq x < \infty$. Here, again we take $n = 5$, 10, 20 and 20. The convergence is shown in Figure 6, while the absolute errors are shown in Figure 6. And, the behavior of $f(x) = x^2 \exp(-x)$ and the operator $\widetilde{L}_n(f;x)$ for $n = 5$, 10 and 20 is depicted in Table 2.

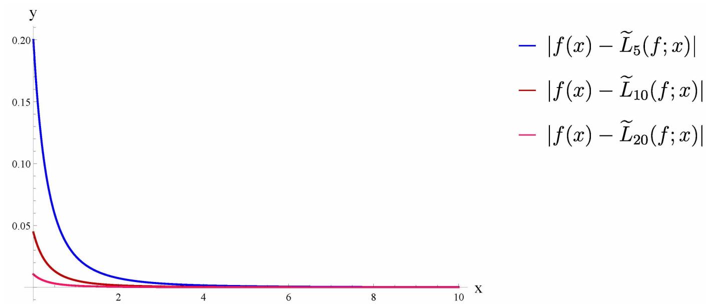
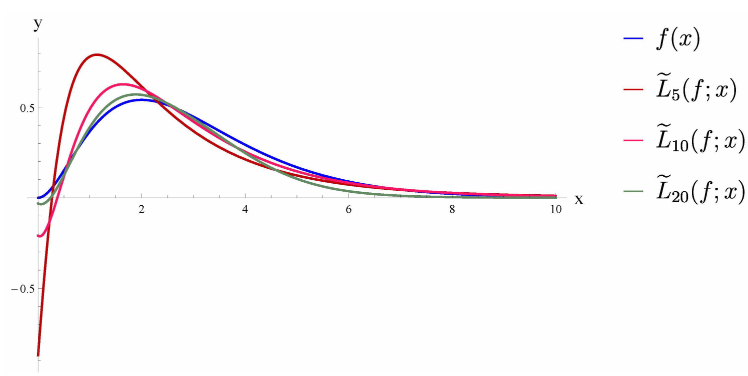


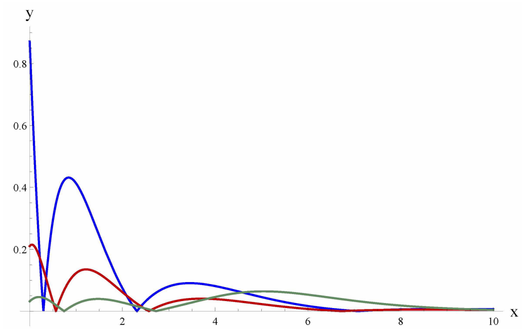
Figure 2: The absolute error function $|f(x) - \widetilde{L}_n(f;x)|$ for $n = 5, 10, 20$.

Table 2: Comparison of $\widetilde{L}_n(f;x)$ with respect to the function $f(x) = x^2e^{-x}$, for $n = 5, 10$ and 20 and the absolute errors $|f(x) - \widetilde{L}_nf(x)|$.

		$\widetilde{L}_nf(x)$			$\lvert f(x) - \widetilde{L}_nf(x)\rvert$		
3-8 x	$f(x)$	$n = 5$	$n = 10$	$n = 20$	$n = 5$	$n = 10$	$n = 20$
0	0	-0.870657	-0.210006	-0.0316959	0.870657	0.210006	0.0316959
0.1	0.367879	0.780847	0.490564	0.39126	0.412967	0.122685	0.0233805
0.2	0.541341	0.615186	0.602418	0.569141	0.0738453	0.061077	0.0278004
0.3	0.448084	0.368182	0.421251	0.438267	0.0799018	0.0268323	0.00981637
0.4	0.29305	0.212071	0.25417	0.247235	0.0809793	0.0388727	0.0458157
0.5	0.168449	0.123047	0.145799	0.104545	0.045402	0.0226497	0.0639034
0.6	0.089235	0.0727465	0.0822435	0.0360542	0.0164885	0.00699153	0.0531809
0.7	0.0446822	0.0439179	0.0462862	0.0111644	0.00076435	0.00160401	0.0335178
0.8	0.0214696	0.0270519	0.0261801	0.00331609	0.00558225	0.00471046	0.0181535
0.9	0.00999619	0.0169711	0.014945	0.000982414	0.00697495	0.00494882	0.00901378
1	0.00453999	0.0108222	0.0086332	0.000296706	0.00628222	0.0040932	0.00424329



Comparison of the behaviour of $f(x) = x^2 \exp(-x)$ and $\widetilde{L}_n(f;x)$ for $n = 5, 10, 20$.



The absolute error function $|f(x) - \widetilde{L}_n(f;x)|$ for $n = 5, 10, 20$.

7 Conclusion

When a sufficiently smooth function $f(x)$ is approximated by the sequence $\widetilde{L}_n(f;x)$, the absolute error for $|f(x) - \widetilde{L}_n(f;x)|$ is found to be of order $\mathcal{O}(n^{-2})$ whereas it is $\mathcal{O}(n^{-1})$ when the operators $B_n f(x)$ or $D_n f(x)$ are used to approximate the same function. To attain a higher degree of approximation, it is advisable to use a suitable modification $\widetilde{L}_n(f;x)$ rather than raising the degree n of the operator. The error bound $|f(x) - \widetilde{L}_n(f;x)|$ for the function decrease as n increases.

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