Applied Mathematics E-Notes, 24(2024), 150-156 © ISSN 1607-2510
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On Starlike And Convex Functions Associated With Booth Lemniscate Domain*

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Received 13 May 2023

Abstract

Let f be an analytic and normalized function in the unit disk $\mathbb{D} := \{z \colon |z| < 1\}$, such that the quantity zf'/f or $1 + zf''/f'$ respectively lies in a domain bounded by the Booth Lemniscate

$$\frac{4(1-\gamma)^2}{(1-\alpha)^2}(u-1)^2 + \frac{4(1-\gamma)^2}{(1+\alpha)^2}v^2 = \left((u-1)^2 + v^2\right)^2,$$

where $0 < \alpha < 1$ and $0 \leq \gamma < 1$. We present some results on the function treatment of f that are related to Booth Lemniscate.

1 Introduction

Let's define two classes of functions: $\mathcal{A}$ and $\mathcal{S}$. The class $\mathcal{A}$ consists of functions of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. The class $\mathcal{S}$ is a subclass of $\mathcal{A}$ and contains functions that are univalent in $\mathbb{D}$.

Furthermore, we have two additional subclasses of $\mathcal{S}$ called $\mathcal{ST}(\beta)$ and $\mathcal{CV}(\beta)$, which are introduced by Robertson [9]. These subclasses consist of functions in $\mathcal{S}$ that are, respectively, starlike and convex, with an order parameter $0 \leq \beta < 1$ within the unit disk $\mathbb{D}$. In other words, these functions satisfy the conditions $\Re\{zf'(z)/f(z)\} > \beta$ for starlike functions, and $1 + \Re\{zf''(z)/f'(z)\} > \beta$ for convex functions, in the domain $\mathbb{D}$. The classes $\mathcal{ST}$ and $\mathcal{CV}$ are the special cases of $\mathcal{ST}(\beta)$ and $\mathcal{CV}(\beta)$, respectively, when $\beta = 0$. Therefore, $\mathcal{ST}$ is the class of starlike functions and $\mathcal{CV}$ is the class of convex functions.

We say that an analytic function f is subordinate to another analytic function g, denoted as $f \prec g$, if there exists an analytic function w such that $|w(z)| \leq |z|$ and $w(0) = 0$ and satisfies $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Ma and Minda [5] provided a unified representation for various geometric subclasses of $\mathcal{S}$ as follows:

$$\mathcal{ST}(\varphi) = \left\{ f \in \mathcal{A} \colon \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \quad \mathcal{CV}(\varphi) = \left\{ f \in \mathcal{A} \colon 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\},$$

Here, φ is an analytic univalent function with a positive real part ($\Re\{\varphi\} > 0$), which maps $\mathbb{D}$ onto domains symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$, such that $\varphi'(0) > 1$ (see [2, 3]). The classes introduced by Ma and Minda encompass several well-known subclasses as special cases.

For example, when considering the function

$$\mathfrak{B}_{\alpha,\gamma}(z) = 1 + \frac{2(1-\gamma)z}{1-\alpha z^2} = 1 + 2(1-\gamma)\left(z + \alpha z^3 + \alpha^2 z^5 + \alpha^3 z^7 + \cdots\right), \tag{2}$$

*Mathematics Subject Classifications: 30C80, 30C45.

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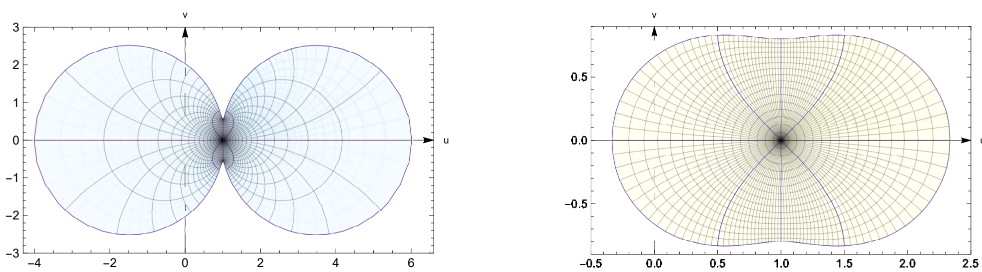
where

$$0 < \alpha < 1 \quad \textit{and} \quad \frac{1+\alpha}{2} \leq \gamma < 1.$$

These classes $\mathcal{ST}\,(\mathfrak{B}_{\alpha,\gamma})$ and $\mathcal{CV}\,(\mathfrak{B}_{\alpha,\gamma})$ reduce to the classes $\mathcal{ST}_{BL}(\alpha,\gamma)$ and $\mathcal{CV}_{BL}(\alpha,\gamma)$, respectively. These classes are of the Ma-Minda type and consist of starlike and convex functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ (see e.g., [2]) lie in the domain defined by the Booth lemniscate, given by:

$$\left\{ u + iv\colon \quad \frac{4(1-\gamma)^2}{(1-\alpha)^2}(u-1)^2 + \frac{4(1-\gamma)^2}{(1+\alpha)^2}v^2 > \left((u-1)^2 + v^2\right)^2 \right\}.$$

The figure in the document labeled as Figure 1 illustrates the image of $\mathbb{D}$ under the mapping function $\mathfrak{B}_{\alpha,\gamma}(z)$ for different values of α and γ.



(a) $\alpha = 0.8, \gamma = 0.5$ (b) $\alpha = 0.25, \gamma = 0.5$

Figure 1: The image of $\mathbb{D}$ under $\mathfrak{B}_{\alpha,\gamma}(z)$.

Considering the Alexander relation between the classes $\mathcal{ST}_{BL}(\alpha,\gamma)$ and $\mathcal{CV}_{BL}(\alpha,\gamma)$, which states that a function $f \in \mathcal{CV}_{BL}(\alpha,\gamma)$ if and only if $zf'(z) \in \mathcal{ST}_{BL}(\alpha,\gamma)$, we can obtain the properties of functions in $\mathcal{CV}_{BL}(\alpha,\gamma)$ from the corresponding results for $\mathcal{ST}_{BL}(\alpha,\gamma)$. Therefore, we can focus our attention on the class $\mathcal{ST}_{BL}(\alpha,\gamma)$.

Now, we will explore specific examples of functions in the classes $\mathcal{ST}_{BL}(\alpha,\gamma)$ and $\mathcal{CV}_{BL}(\alpha,\gamma)$ that serve as extremal functions for various problems within these subclasses. In particular, we consider the case when $\gamma = 1/2$ (see e.g., [8]).

The equivalence $g \in \mathcal{ST}_{BL}(\alpha,\gamma)$ if and only if $zg'(z)/g(z) \prec \mathcal{B}_{\alpha,\gamma}(z)$ allows us to determine the structural formula for functions in $\mathcal{ST}_{BL}(\alpha,\gamma)$. A function g belongs to the class $\mathcal{ST}_{BL}(\alpha,\gamma)$ if and only if there exists an analytic function $p \prec \mathcal{B}_{\alpha,\gamma}$ such that:

$$g(z) = z\exp\left(\int_0^z \frac{p(t)-1}{t}\,dt\right) \quad \text{for} \quad z \in \mathbb{D}.$$

The integral representation mentioned above provides numerous examples of functions belonging to the class $\mathcal{ST}_{BL}(\alpha,\gamma)$. Let's consider the function $p(z) = \mathcal{B}_{\alpha,\gamma}(z^n) \in \mathcal{ST}_{BL}(\alpha,\gamma)$ for $n = 1, 2, \ldots$. Then, for $0 < \alpha < 1$, $(1+\alpha)/2 \leq \gamma < 1$, and $n \geq 1$, the functions defined as:

$$\begin{aligned}
\Psi_{\alpha,\gamma,n}(z) &= z\exp\left(\int_0^z \frac{\mathcal{B}_{\alpha,\gamma}(t^n)-1}{t}\,dt\right) = z\exp\left(\frac{2(1-\gamma)}{n\sqrt{\alpha}}\tanh^{-1}\left(\sqrt{\alpha}\,z^n\right)\right) \\
&= z\left(\frac{1+\sqrt{\alpha}\,z^n}{1-\sqrt{\alpha}\,z^n}\right)^{\frac{1-\gamma}{n\sqrt{\alpha}}} \\
&= z + \frac{2(1-\gamma)}{n}z^{n+1} + \frac{2(1-\gamma)^2}{n^2}z^{2n+1} + \frac{2(1-\gamma)}{3n^3}\left(2(1-\gamma)^2 + \alpha n^2\right)z^{3n+1} + \cdots, \qquad (3)
\end{aligned}$$

serve as extremal functions for several problems within the class $\mathcal{ST}_{BL}(\alpha,\gamma)$.

For the special case of $n=1$, we have:

$$\begin{aligned}\Psi_{\alpha,\gamma}(z) \quad &: \quad = \Psi_{\alpha,\gamma,1}(z) = z\left(\frac{1+\sqrt{\alpha}\,z}{1-\sqrt{\alpha}\,z}\right)^{\frac{1-\gamma}{\sqrt{\alpha}}} \\ &= \quad z+2(1-\gamma)z^2+2(1-\gamma)^2z^3+\frac{2(1-\gamma)}{3}\left(2(1-\gamma)^2+\alpha\right)z^4+\cdots \quad \text{for} \quad z\in\mathbb{D}. \qquad (4)\end{aligned}$$

The Appell series F_1 is defined for $|z_1|<1$ and $|z_2|<1$ as a double series given by

$$F_1\left(a;b_1,b_2;c;z_1,z_2\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}z_1^m z_2^n,$$

where the Pochhammer symbol $(d)_n$ represents the rising factorial defined as:

$$(d)_n=d(d+1)\cdots(d+n-1)=\begin{cases}\dfrac{\Gamma(d+n)}{\Gamma(d)} & \text{for} \quad n=1,2,\ldots, \\ 1 & \text{for} \quad n=0,\end{cases}$$

with the second equality holding for all complex d except $d=0,-1,-2,\ldots$.

When $\Re\{c-a\}>0$, the Appell function can also be expressed through an Euler-type integral as follows [6, p. 413]:

$$F_1\left(a;b_1,b_2;c;z_1,z_2\right)=\frac{\Gamma(c)\Gamma(c-a)}{\Gamma(a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-z_1t)^{-b_1}(1-z_2t)^{-b_2}dt.$$

The given relationship states that a function h belongs to the function $\mathcal{CV}_{BL}(\alpha,\gamma)$ if and only if $1+zh''(z)/h'(z)$ is subordinated to the $\mathcal{B}_{\alpha,\gamma}(z)$. Using this, we can derive the structural formula for functions in $\mathcal{CV}_{BL}(\alpha,\gamma)$. Specifically, a function h is in the class $\mathcal{CV}_{BL}(\alpha,\gamma)$ if and only if there exists an analytic function p such that p is subordinated to the function $\mathcal{B}_{\alpha,\gamma}$, and the function h can be represented as follows:

$$h(z)=\int_0^z \exp\left(\int_0^w \frac{p(t)-1}{t}\,dt\right)dw \quad \text{for} \quad z\in\mathbb{D}.$$

This representation provides several examples of functions in the class $\mathcal{CV}_{BL}(\alpha,\gamma)$. In particular, when we choose $p(z)=\mathcal{B}_{\alpha,\gamma}(z^n)\in\mathcal{CV}_{BL}(\alpha,\gamma)$ for some $n\geq 1$, the functions $K_{\alpha,\gamma,n}(z)$ defined as shown in Figure 2 satisfy

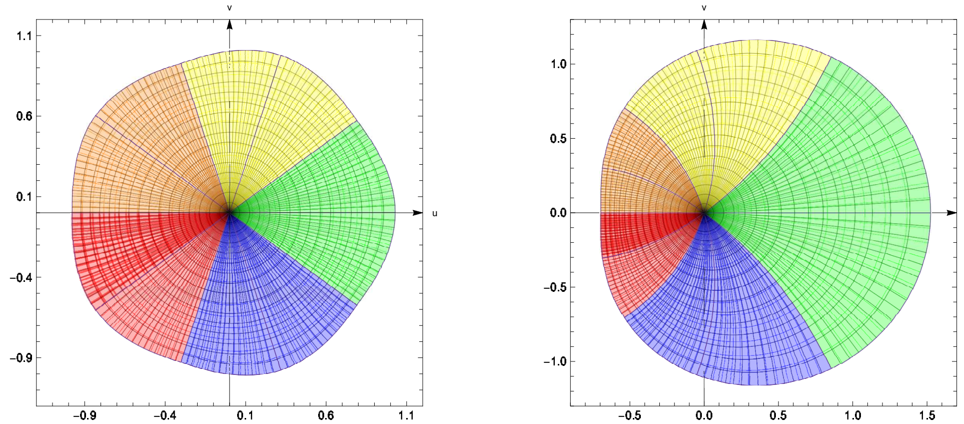
$$\begin{aligned}K_{\alpha,\gamma,n}(z) \quad &= \quad \int_0^z \exp\left(\int_0^w \frac{\mathcal{B}_{\alpha,\gamma}(t^n)-1}{t}\,dt\right)dw=\int_0^z\left(\frac{1+\sqrt{\alpha}\,t^n}{1-\sqrt{\alpha}\,t^n}\right)^{\frac{1-\gamma}{n\sqrt{\alpha}}}dt \\ &= \quad \frac{z}{n}\int_0^1 u^{\frac{1-n}{n}}\left(1+\sqrt{\alpha}\,z^n u\right)^{\frac{1-\gamma}{n\sqrt{\alpha}}}\left(1-\sqrt{\alpha}\,z^n u\right)^{-\frac{1-\gamma}{n\sqrt{\alpha}}}du \\ &= \quad z\,F_1\left(\frac{1}{n};\frac{1-\gamma}{n\sqrt{\alpha}},-\frac{1-\gamma}{n\sqrt{\alpha}};\frac{n+1}{n};\sqrt{\alpha}\,z^n,-\sqrt{\alpha}\,z^n\right) \quad \text{for} \quad z\in\mathbb{D}. \qquad (5)\end{aligned}$$

Furthermore, when $n=1$, we have the special case:

$$K_{\alpha,\gamma}(z):=K_{\alpha,\gamma,1}(z)=z\,F_1\left(1;\frac{1-\gamma}{\sqrt{\alpha}},-\frac{1-\gamma}{\sqrt{\alpha}};2;\sqrt{\alpha}\,z,-\sqrt{\alpha}\,z\right) \quad \text{for} \quad z\in\mathbb{D}. \qquad (6)$$

The bound for the Fekete Szegö inequality for the classes can be estimated as in [1, Theorem 1, p.38] and [7]. If $f\in\mathcal{ST}_{BL}(\alpha,\gamma)$ of the form (1) and λ is a real number, then

$$\left|a_3-\lambda a_2^2\right|\leq(1-\gamma)\begin{cases}-2(1-\gamma)(2\lambda-1) & \text{for} \quad \lambda\leq\dfrac{1-2\gamma}{4(1-\gamma)}, \\ 1 & \text{for} \quad \dfrac{1-2\gamma}{4(1-\gamma)}\leq\lambda\leq\frac{3-2\gamma}{4(1-\gamma)}, \\ 2(1-\gamma)(2\lambda-1) & \text{for} \quad \lambda\geq\dfrac{3-2\gamma}{4(1-\gamma)}.\end{cases}$$



(a) $n = 5$ (b) $n = 1$

Figure 2: The image of $\mathbb{D}$ under $K_{\alpha,\gamma,n}(\mathbb{D})$ for $\alpha = 0.25, \gamma = 0.625$ with $\frac{1+\alpha}{2} \leq \gamma < 1$.

The inequalities are sharp for the functions

$$f(z) = \begin{cases} \overline{\mu}\Psi_{\alpha,\gamma,2}(\mu z) & \text{for} \quad \dfrac{1-2\gamma}{4(1-\gamma)} < \lambda < \frac{3-2\gamma}{4(1-\gamma)}, \\ \overline{\mu}\Psi_{\alpha,\gamma}(\mu z) & \text{for} \quad \lambda \in \left(-\infty, \dfrac{1-2\gamma}{4(1-\gamma)}\right) \cup \left(\dfrac{3-2\gamma}{4(1-\gamma)}, \infty\right), \\ \overline{\mu} f_x(\mu z) & \text{for} \quad \lambda = \dfrac{1-2\gamma}{4(1-\gamma)}, \\ \overline{\mu} g_x(\mu z) & \text{for} \quad \lambda = \dfrac{3-2\gamma}{4(1-\gamma)}, \end{cases}$$

where $\Psi_{\alpha,\gamma,2}$ and $\Psi_{\alpha,\gamma}$ are given by (3) and (4), μ is an unimodular constant, and f_x and g_x $(0 \leq x \leq 1)$ are given by

$$\frac{zf_x'(z)}{f_x(z)} = \mathfrak{B}_{\alpha,\gamma}\left(\frac{z(z+x)}{1+xz}\right) \quad \text{and} \quad \frac{zg_x'(z)}{g_x(z)} = \mathfrak{B}_{\alpha,\gamma}\left(-\frac{z(z+x)}{1+xz}\right).$$

Also, If $f \in \mathcal{CV}_{BL}(\alpha, \gamma)$ of the form (1) and λ is a real number, then

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{1-\gamma}{3} \begin{cases} -(1-\gamma)(3\lambda - 2) & \text{for} \quad \lambda \leq \dfrac{1-2\gamma}{3(1-\gamma)}, \\ 1 & \text{for} \quad \dfrac{1-2\gamma}{3(1-\gamma)} \leq \lambda \leq \dfrac{3-2\gamma}{3(1-\gamma)}, \\ (1-\gamma)(3\lambda - 2) & \text{for} \quad \lambda \geq \dfrac{3-2\gamma}{3(1-\gamma)}. \end{cases}$$

The inequalities are sharp for the functions

$$f(z) = \begin{cases} \overline{\mu}K_{\alpha,\gamma,2}(\mu z) & \text{for} \quad \dfrac{-\gamma}{3(1-\gamma)} < \lambda < \frac{2-\gamma}{3(1-\gamma)}, \\ \overline{\mu}K_{\alpha,\gamma}(\mu z) & \text{for} \quad \lambda \in \left(-\infty, \dfrac{1-2\gamma}{3(1-\gamma)}\right) \cup \left(\dfrac{3-2\gamma}{3(1-\gamma)}, \infty\right), \\ \overline{\mu} F_x(\mu z) & \text{for} \quad \lambda = \dfrac{1-2\gamma}{3(1-\gamma)}, \\ \overline{\mu} G_x(\mu z) & \text{for} \quad \lambda = \dfrac{3-2\gamma}{3(1-\gamma)}, \end{cases}$$

where $K_{\alpha,\gamma,2}$ and $K_{\alpha,\gamma}$ are given by (5) and (6), μ is an unimodular constant, and F_x and G_x $(0 \le x \le 1)$ are given by

$$1 + \frac{zF_x''(z)}{F_x'(z)} = \mathfrak{B}_{\alpha,\gamma}\left(\frac{z(z+x)}{1+xz}\right) \quad \text{and} \quad 1 + \frac{zG_x''(z)}{G_x'(z)} = \mathfrak{B}_{\alpha,\gamma}\left(-\frac{z(z+x)}{1+xz}\right).$$

The Fekete Szegö inequality yield the sharp first three coefficient bounds for both the classes $\mathcal{ST}_{BL}(\alpha,\gamma)$ and $\mathcal{CV}_{BL}(\alpha,\gamma)$, which are as follows: If $f \in \mathcal{ST}_{BL}(\alpha,\gamma)$ given by (1), then

$$|a_2| \le 2(1-\gamma), \quad |a_3| \le \begin{cases} 2(1-\gamma)^2 & \text{for} \quad \gamma \le \dfrac{1}{2}, \\ 1-\gamma & \text{for} \quad \gamma \ge \dfrac{1}{2} \end{cases}$$

and

$$\left|a_3 - a_2^2\right| \le \begin{cases} 2(1-\gamma)^2 & \textit{for} \quad \gamma \le \dfrac{1}{2}, \\ 1-\gamma & \textit{for} \quad \gamma \ge \dfrac{1}{2}. \end{cases}$$

Also, if $f \in \mathcal{CV}_{BL}(\alpha,\gamma)$ given by (1), then

$$|a_2| \le 1-\gamma, \quad |a_3| \le \begin{cases} \dfrac{2}{3}(1-\gamma)^2 & \text{for} \quad \gamma \le \dfrac{1}{2}, \\ \frac{1-\gamma}{3} & \text{for} \quad \gamma \ge \dfrac{1}{2} \end{cases}, \quad \left|a_3 - a_2^2\right| \le \frac{1-\gamma}{3}.$$

2 Radius Estimates

The first theorem in this section establishes the sharp radii for univalence in the class $\mathcal{ST}_{BL}(\alpha,\gamma)$ within the class $\mathcal{ST}$ for $0 \le \gamma < (1+\alpha)/2$. The largest disk centered at the origin, within which the function f is univalent, is referred to as the univalence radius of f

The radius of starlikeness for a class M is the maximum value $R_{\mathcal{ST}}(M)$ such that any function $f \in M$ is starlike within the disk $|z| < R_{\mathcal{ST}}(M)$. An example is the class $\mathcal{S}$, where the radius of starlikeness is given by $\tanh(\pi/4)$, as determined by Grunsky.

Theorem 1 *Let $0 \le \gamma < (1+\alpha)/2$, $0 < \alpha < 1$ and $0 \le \beta < 1$. If $f \in \mathcal{ST}_{BL}(\alpha,\gamma)$, then f is starlike of order β in the disc*

$$|z| < \rho := \frac{\sqrt{(1-\gamma)^2 + \alpha(1-\beta)^2} - (1-\gamma)}{\alpha(1-\beta)}.$$

The result is sharp.

Proof. From [2], for

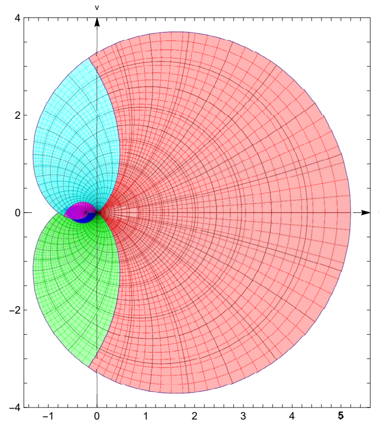
$$0 < \alpha < 1 \quad \text{and} \quad 0 \le \gamma < \frac{1+\alpha}{2},$$

we conclude that function $f \in \mathcal{ST}_{BL}(\alpha,\gamma)$ may not be univalent in $\mathbb{D}$ (see Figure 3). Thus for analytic function w with $|w(z)| \le |z|$ and $w(0) = 0$ we get

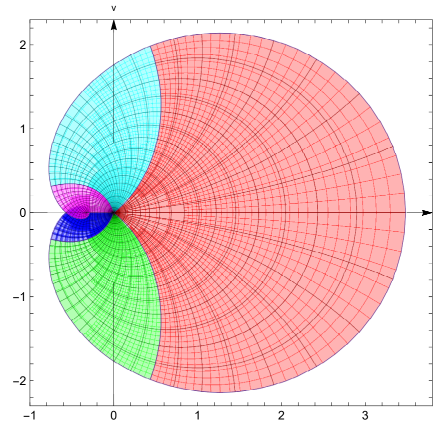
$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = 1 + 2(1-\gamma)\,\Re\left\{\frac{w(z)}{1-\alpha w(z)^2}\right\} \quad \text{for} \quad z \in \mathbb{D}. \tag{7}$$

Now from the relationships $-|z| \le \Re z$, we conclude that

$$\Re\left\{\frac{w(z)}{|1-\alpha w(z)^2|}\right\} \ge -\frac{|w(z)|}{|1-\alpha w(z)^2|} \ge -\frac{r}{1-\alpha r^2} \quad \text{for} \quad |z| = r. \tag{8}$$



(a) $\alpha = \gamma = 0.25$



(b) $\alpha = \gamma = 0.5$

Figure 3: The image of $\mathbb{D}$ under $\Psi_{\alpha,\gamma}(z)$ with $0 \le \gamma < \frac{1+\alpha}{2}$.

The function defined by $m(r) = 1 - \frac{2(1-\gamma)r}{1-\alpha r^2}$, $0 \le r < 1$ is a decreasing function. Let ρ is the root of the equation $m(r) = \beta$. Taking into account the relation (7) and (8), it follows that

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} \ge 1 - \frac{2(1-\gamma)r}{1-\alpha r^2} = m(r) \ge m(\rho) = \beta \quad \text{for} \quad 0 \le |z| = r \le \rho. \tag{9}$$

This shows that if $r \le \rho$, then results follows. Therefore f is starlike of order β in $|z| < \rho$. Also at the point $z = -\rho$, we see that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2(1-\gamma)z}{1-\alpha z^2} = 1 - \frac{2(1-\gamma)\rho}{1-\alpha\rho^2} = m(\rho) = \beta.$$

This proves the sharpness of the result. By way of explanation the function $\Psi_{\alpha,\gamma}$ given by (4) proves that the estimation is sharp.

From (9), for $|z| < 1$, we deduce that if

$$\frac{2r - 1 + \alpha r^2}{2r} < \gamma < \frac{1+\alpha}{2},$$

then each function $f \in \mathcal{ST}_{BL}(\alpha, \gamma)$ maps the disk $|z| < r$ onto a starlike domain. ∎

For $0 < \alpha < 1$ and $0 \le \gamma < (1+\alpha)/2$ with putting $\beta = 0$ in Theorem 1, the function $\Psi_{\alpha,\gamma}$ given by (4) is extremal function for problem in the class

$$\mathcal{ST}_{BL}(\alpha,\gamma) = \left\{ f \in \mathcal{A}\colon\ \frac{zf'(z)}{f(z)} \prec 1 + \frac{2(1-\gamma)z}{1-\alpha z^2}, \quad |z| < r \quad \text{for} \quad 0 < r \le \frac{-1+\gamma+\sqrt{(1-\gamma)^2+\alpha}}{\alpha} \right\}.$$

Putting $\gamma = 1/2$ and $\beta = 0$ in Theorem 1, we get the following result. The reader is urged to compare our next corollary with the [4, Corollary 1.1].

Corollary 1 *For $0 < \alpha < 1$, the function $\Psi_{\alpha,1/2}$ given by (4) is extremal function for problem in the class*

$$\mathcal{ST}_{BL}\left(\alpha, \frac{1}{2}\right) = \left\{ f \in \mathcal{A}\colon\ \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{1-\alpha z^2}, \quad |z| < r \quad \textit{for} \quad 0 < r \le \frac{-1+\sqrt{1+4\alpha}}{2\alpha} \right\}.$$

3 Conclusions

The paper provides detailed descriptions of the Booth Leminiscate curve, considering different parameters and offering precise specifications. It explores families of starlike and convex functions associated with the Booth Leminiscate, where common function properties are found within the regions bounded by the Booth Leminiscate curve. The paper also presents examples and characteristics of extremal functions within these defined families. Additionally, it derives extremal functions for the bounded domains of convex functions using Euler-type integrals.

Acknowledgments. The authors thank the constructive comments and suggestions by the editor and anonymous referees, which have contributed to the improvement of the presentation of this paper.

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