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Some Hyperstability Results For Quadratic Type Functional Equations*

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Abstract

The objective of the present manuscript is to establish some stability results for quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ and Jensen type quadratic functional equation $2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y)$ using a well established approach given by Moghimi and Najati [24]. We have discussed some hyperstability results for quadratic type functional equations. Also, we have given some instances that show the necessity of the assumptions made to establish the stability results on quadratic type functional equations.

1 Introduction

Let V and W be linear spaces. A map $f : V \to W$ is said to be

1. quadratic if $f(x+y)+f(x-y)=2f(x)+2f(y)$,
2. quadratic of Jensen type if $2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y)$ for all $x, y \in V$.

In his lecture at the University of Wisconsin, Ulam [34] raised the problem regarding the stability of functional equations. Since then, it has become a great attraction for researchers exploring functional equations and their solutions. Hyers [17] provides the first conclusive response to Ulam's problem, which is referred to as Hyers-Ulam stability. Following that, numerous researchers generalised Hyers' findings in distinct ways. For more information, see [1, 2, 3, 4, 9, 14, 15, 16, 18, 19, 20, 21, 22, 23, 25, 26, 33, 35] and the references therein.

Brezdek and Pietrzyk [11] presented the stability result for a general linear functional equation $f(\alpha\varrho+\beta\nu+\gamma)=Af(\varrho)+Bf(\nu)+C$. Based on the fixed point theorem, Brzdek [12] offered a complementary result to the existing Rassias [31, 32] stability result. The result is stated as hyperstability of the Cauchy functional equation.

Theorem 1 ([12]) *Consider two normed spaces V and W and a non-empty subset $U \subset V - \{0\}$. Choose $\epsilon \geq 0$ and real numbers p and q satisfying $p+q<0$. Assume that for each $x \in U$, there exists a natural number n_x such that $nx \in U$ for all $n \geq n_x$. If a function $f : V \to W$ defined on V satisfies*

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon\|x\|^p\|y\|^q,$$

for all $x,\ y,\ x+y \in U$, then f is an additive map on U, i.e.,

$$f(x+y)=f(x)+f(y),$$

for all $x,\ y,\ x+y \in U$.

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Many researchers established hyperstability results for linear as well as quadratic-type functional equations. For more information, (see [5, 6, 7, 8, 10, 13, 27, 28, 29, 30, 36]).

Recently, Moghimi and Najati [24] provided a brief and simple justification for Brzdek's result regarding the hyperstability of the Cauchy and Jensen functional equation.

In this manuscript, we proved various hyperstability results for quadratic type functional equations, which were driven by the work of Moghimi and Najati [24]. Additionally, several examples have been given to demonstrate the significance of the conditions considered in the stability results.

2 Main Result

Theorem 2 *Consider two normed spaces V and W and a non-empty subset $U \subset V - \{0\}$. Choose $\epsilon \geq 0$ and real numbers p and q satisfying $p + q < 0$. Assume that for each $x \in U$, there exists a natural number n_x such that $nx \in U$ for all $n \geq n_x$. If a function $f : V \to W$ defined on V satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon \|x\|^p \|y\|^q, \tag{1}$$

for all $x, y, x+y, x-y \in U$, then f is quadratic on U, i.e.,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

for all $x, y, x+y, x-y \in U$.

Proof. It is given that $p + q < 0$. Therefore, either $p < 0$ or $q < 0$. Without loss of generality assume that $p < 0$. Let $x, y \in U$ with $x + y, x - y \in U$. Hence, by the given hypothesis there exists a natural number m such that $nx, ny, n(x+y)$ and $n(x-y) \in U$ for every $n \geq m$.

Substituting $x = nx$ and $y = x$ in (1), we have

$$\big\|f(nx+x) + f(nx-x) - 2f(nx) - 2f(x)\big\| \leq \epsilon n^p \|x\|^{p+q}$$

or

$$\Big\|f\big((n+1)x\big) + f\big((n-1)x\big) - 2f(nx) - 2f(x)\Big\| \leq \epsilon n^p \|x\|^{p+q}.$$

Similarly, we have

$$\Big\|f\big((n+1)y\big) + f\big((n-1)y\big) - 2f(ny) - 2f(y)\Big\| \leq \epsilon n^p \|y\|^{p+q},$$

$$\Big\|f\big((n+1)(x+y)\big) + f\big((n-1)(x+y)\big) - 2f\big(n(x+y)\big) - 2f(x+y)\Big\| \leq \epsilon n^p \|x+y\|^{p+q}$$

and

$$\Big\|f\big((n+1)(x-y)\big) + f\big((n-1)(x-y)\big) - 2f\big(n(x-y)\big) - 2f(x-y)\Big\| \leq \epsilon n^p \|x-y\|^{p+q}.$$

As $p < 0$. On taking limit as $n \to \infty$ in above inequalities, we have

$$2f(x) = \lim_{n\to\infty}\Big(f\big((n+1)x\big) + f\big((n-1)x\big) - 2f(nx)\Big),$$

$$2f(y) = \lim_{n\to\infty}\Big(f\big((n+1)y\big) + f\big((n-1)y\big) - 2f(ny)\Big),$$

$$2f(x+y) = \lim_{n\to\infty}\Big(f\big((n+1)(x+y)\big) + f\big((n-1)(x+y)\big) - 2f\big(n(x+y)\big)\Big)$$

and

$$2f(x-y) = \lim_{n\to\infty}\Big(f\big((n+1)(x-y)\big) + f\big((n-1)(x-y)\big) - 2f\big(n(x-y)\big)\Big).$$

Now,

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
=\;& \lim_{n\to\infty}\left\|\frac{1}{2}\Big(f\big((n+1)(x+y)\big)+f\big((n-1)(x+y)\big)-2f(n(x+y))\Big)\right. \\
&+\frac{1}{2}\Big(f\big((n+1)(x-y)\big)+f\big((n-1)(x-y)\big)-2f\big(n(x-y)\big)\Big) \\
&-\Big(f\big((n+1)x\big)+f\big((n-1)x\big)-2f(nx)\Big) \\
&\left.-\Big(f\big((n+1)y\big)+f\big((n-1)y\big)-2f(ny)\Big)\right\| \\
\le\;& \lim_{n\to\infty}\frac{1}{2}\Big\|f\big((n+1)(x+y)\big)+f\big((n+1)(x-y)\big)-2f\big((n+1)x\big)-2f\big((n+1)y\big)\Big\| \\
&+\frac{1}{2}\Big\|f\big((n-1)(x+y)\big)+f\big((n-1)(x-y)\big)-2f\big((n-1)x\big)-2f\big((n-1)y\big)\Big\| \\
&+\Big\|f\big(n(x+y)\big)+f\big(n(x-y)\big)-2f(nx)-2f(ny)\Big\| \\
\le\;& \lim_{n\to\infty}\epsilon\Big(\frac{1}{2}(n+1)^{p+q}+\frac{1}{2}(n-1)^{p+q}+n^{p+q}\Big)\|x\|^p\|y\|^q \\
=\;& 0,
\end{aligned}$$

implies

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$

for all $x,\ y,\ x+y,\ x-y\in U$. ■

Remark 1 *The condition $(p+q<0)$ specified in Theorem 2 is necessary for the hyperstability result. In Example 1, we illustrated that if the criterion $(p+q<0)$ is not met, then the function may not be quadratic.*

Example 1 *Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=x^4$. Consider*

$$\begin{aligned}
\Big\|f(x+y)+f(x-y)-2f(x)-2f(y)\Big\| &= \Big|(x+y)^4+(x-y)^4-2x^4-2y^4\Big| \\
&= \Big|12x^2y^2\Big| \le 12|x|^2|y|^2 .
\end{aligned}$$

Then, for $\epsilon=12$ and $p=q=2$, hypothesis (1) holds but f is not quadratic.

Theorem 3 *Consider two normed spaces V and W and a non-empty subset $U\subset V-\{0\}$. Choose $\epsilon\ge 0$ and real numbers p and q satisfying $p+q<0$. Assume that for each $x\in U$, there exists a natural number n_x such that for all $n\ge n_x$, we have $\frac{nx}{2}\in U$. If a function $f:V\to W$ defined on V satisfies*

$$\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\|\le\epsilon\|x\|^p\|y\|^q, \tag{2}$$

for all $x,\ y,\ \frac{x+y}{2},\ \frac{x-y}{2}\in U$, then f is quadratic of Jensen type on U, i.e.,

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y),$$

for all $x,\ y,\ \frac{x+y}{2},\ \frac{x-y}{2}\in U$.

Proof. It is given that $p+q<0$. Therefore, either $p<0$ or $q<0$. Without loss of generality assume that $p<0$. Let $x,y\in U$ with $\frac{x+y}{2}$, $\frac{x-y}{2}\in U$. Hence, by the given hypothesis there exists a natural number m such that $\frac{nx}{2}$, $\frac{ny}{2}$, $\frac{n(x+y)}{2}$ and $\frac{n(x-y)}{2}\in U$ for every $n\geq m$.

Substituting $x=nx$ and $y=x$ in (2), we have

$$\left\|2f\left(\frac{nx+x}{2}\right)+2f\left(\frac{nx-x}{2}\right)-f(nx)-f(x)\right\|\leq \epsilon n^p\|x\|^{p+q}.$$

Similarly, we have

$$\left\|2f\left(\frac{(n+1)y}{2}\right)+2f\left(\frac{(n-1)y}{2}\right)-f(ny)-f(y)\right\|\leq \epsilon n^p\|y\|^{p+q},$$

$$\left\|2f\left(\frac{(n+1)(x+y)}{4}\right)+2f\left(\frac{(n-1)(x+y)}{4}\right)-f\left(\frac{n(x+y)}{2}\right)-f\left(\frac{x+y}{2}\right)\right\|\leq \epsilon n^p\left\|\frac{x+y}{2}\right\|^{p+q}$$

and

$$\left\|2f\left(\frac{(n+1)(x-y)}{4}\right)+2f\left(\frac{(n-1)(x-y)}{4}\right)-f\left(\frac{n(x-y)}{2}\right)-f\left(\frac{x-y}{2}\right)\right\|\leq \epsilon n^p\left\|\frac{x-y}{2}\right\|^{p+q}.$$

As $p<0$. On taking limit as $n\to\infty$ in above inequalities, we have

$$f(x)=\lim_{n\to\infty}\left(2f\left(\frac{(n+1)x}{2}\right)+2f\left(\frac{(n-1)x}{2}\right)-f(nx)\right),$$

$$f(y)=\lim_{n\to\infty}\left(2f\left(\frac{(n+1)y}{2}\right)+2f\left(\frac{(n-1)y}{2}\right)-f(ny)\right),$$

$$f\left(\frac{x+y}{2}\right)=\lim_{n\to\infty}\left(2f\left(\frac{(n+1)(x+y)}{4}\right)+2f\left(\frac{(n-1)(x+y)}{4}\right)-f\left(\frac{n(x+y)}{2}\right)\right)$$

and

$$f\left(\frac{x-y}{2}\right)=\lim_{n\to\infty}\left(2f\left(\frac{(n+1)(x-y)}{4}\right)+2f\left(\frac{(n-1)(x-y)}{4}\right)-f\left(\frac{n(x-y)}{2}\right)\right).$$

Now,

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\|\\
=&\lim_{n\to\infty}\left\|2\left(2f\left(\frac{(n+1)(x+y)}{4}\right)+2f\left(\frac{(n-1)(x+y)}{4}\right)-f\left(\frac{n(x+y)}{2}\right)\right)\right.\\
&+2\left(2f\left(\frac{(n+1)(x-y)}{4}\right)+2f\left(\frac{(n-1)(x-y)}{4}\right)-f\left(\frac{n(x-y)}{2}\right)\right)\\
&-\left(2f\left(\frac{(n+1)x}{2}\right)+2f\left(\frac{(n-1)x}{2}\right)-f(nx)\right)\\
&\left.-\left(2f\left(\frac{(n+1)y}{2}\right)+2f\left(\frac{(n-1)y}{2}\right)-f(ny)\right)\right\|\\
\leq&\lim_{n\to\infty}2\left\|2f\left(\frac{(n+1)(x+y)}{4}\right)+2f\left(\frac{(n+1)(x-y)}{4}\right)-f\left(\frac{(n+1)x}{2}\right)-f\left(\frac{(n+1)y}{2}\right)\right\|\\
&+2\left\|2f\left(\frac{(n-1)(x+y)}{4}\right)+2f\left(\frac{(n-1)(x-y)}{4}\right)-f\left(\frac{(n-1)x}{2}\right)-f\left(\frac{(n-1)y}{2}\right)\right\|
\end{aligned}$$

$$\begin{aligned}
&+\left\|2f\left(\frac{n(x+y)}{2}\right)+f\left(\frac{n(x-y)}{2}\right)-f(nx)-f(ny)\right\| \\
\leq\ & \lim_{n\to\infty}\epsilon\left(2\left(\frac{n+1}{2}\right)^{p+q}+2\left(\frac{n-1}{2}\right)^{p+q}+n^{p+q}\right)\|x\|^p\|y\|^q \\
=\ & 0,
\end{aligned}$$

implies

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y),$$

for all x, y, $\frac{x+y}{2}$, $\frac{x-y}{2}\in U$. ■

Remark 2 *The condition $(p+q<0)$ specified in Theorem 3 is necessary for the hyperstability result. In Example 2, we illustrated that if the criterion $(p+q<0)$ is not met, then the function may not be quadratic of Jensen type.*

Example 2 *Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=x^4$ and let $U=[1,\infty)$. Consider*

$$\begin{aligned}
\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| &= \left|2\left(\frac{x+y}{2}\right)^4+2\left(\frac{x-y}{2}\right)^4-x^4-y^4\right| \\
&= \frac{3}{4}\left|x^2-y^2\right|^2\leq\frac{3}{2}\left|x^2\right|^2\left|y^2\right|^2 \\
&= \frac{3}{2}|x|^4|y|^4,
\end{aligned}$$

for all $x,y\in U$. Then, for $\epsilon=\frac{3}{2}$, and $p=q=4$ hypothesis (2) holds, but f is not quadratic of Jensen type.

Theorem 4 *Consider two normed spaces V and W and a non-empty subset $U\subset V-\{0\}$. Choose $\epsilon\geq 0$ and real numbers p, q and r satisfying $p+q+r<0$. Assume that for each $x\in U$, there exists a natural number n_x such that for all $n\geq n_x$, we have $nx\in U$. If a function $f:V\to W$ defined on V satisfies*

$$\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(z+x)\|\leq\epsilon\|x\|^p\|y\|^q\|z\|^r, \tag{3}$$

for all x, y, $z\in U$, $x+y$, $y+z$, $x+z$, $x+y+z\in U$, then f is a quadratic type functional equation satisfying

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x),$$

for all x, y, $x+y$, $y+z$, $x+z$, $x+y+z\in U$.

Proof. *It is given that $p+q+r<0$. Therefore, without loss of generality let $q+r<0$. Let $x,y\in U$ with $x+y$, $y+z$, $z+x\in U$. Hence, by the given hypothesis there exists a natural number m such that $nx,ny,nz,n(x+y),n(y+z),n(x+z)$ and $n(x+y+z)\in U$ for every $n\geq m$.*

Substituting $x=x$, $y=nx$ and $z=nx$ in (3), we have

$$\left\|f\big((2n+1)x\big)+f(x)+2f(nx)-2f\big((n+1)x\big)-f(2nx)\right\|\leq\epsilon n^{q+r}\|x\|^{p+q+r}.$$

Similarly, we have

$$\left\|f\big((2n+1)z\big)+f(z)+2f(nz)-2f\big((n+1)z\big)-f(2nz)\right\|\leq\epsilon n^{q+r}\|z\|^{p+q+r},$$

$$\left\|f\big((2n+1)y\big)+f(y)+2f(ny)-2f\big((n+1)y\big)-f(2ny)\right\|\leq\epsilon n^{q+r}\|y\|^{p+q+r},$$

$$\Big\|f\big((2n+1)(x+y)\big)+f(x+y)+2f\big(n(x+y)\big)-2f\big((n+1)(x+y)\big)-f\big(2n(x+y)\big)\Big\|\le \epsilon n^{q+r}\|x+y\|^{p+q+r},$$

$$\Big\|f\big((2n+1)(y+z)\big)+f(y+z)+2f\big(n(y+z)\big)-2f\big((n+1)(y+z)\big)-f\big(2n(y+z)\big)\Big\|\le \epsilon n^{q+r}\|y+z\|^{p+q+r},$$

$$\Big\|f\big((2n+1)(x+z)\big)+f(x+z)+2f\big(n(x+z)\big)-2f\big((n+1)(x+z)\big)-f\big(2n(x+z)\big)\Big\|\le \epsilon n^{q+r}\|x+z\|^{p+q+r}$$

and

$$\begin{aligned}&\Big\|f\big((2n+1)(x+y+z)\big)+f(x+y+z)+2f\big(n(x+y+z)\big)-2f\big((n+1)(x+y+z)\big)-f\big(2n(x+y+z)\big)\Big\|\\ \le\ & \epsilon n^{q+r}\|x+y+z\|^{p+q+r}.\end{aligned}$$

As $q+r<0$. *On taking* $\lim_{n\to\infty}$ *in above inequalities, we have*

$$f(x)=\lim_{n\to\infty}\Big(2f\big((n+1)(x)\big)+f(2nx)-f\big((2n+1)x\big)-2f(nx)\Big),$$

$$f(y)=\lim_{n\to\infty}\Big(2f\big((n+1)(y)\big)+f(2ny)-f\big((2n+1)y\big)-2f(ny)\Big),$$

$$f(z)=\lim_{n\to\infty}\Big(2f\big((n+1)(z)\big)+f(2nz)-f\big((2n+1)z\big)-2f(nz)\Big),$$

$$f(x+y)=\lim_{n\to\infty}\Big(2f\big((n+1)(x+y)\big)+f\big(2n(x+y)\big)-f\big((2n+1)(x+y)\big)-2f\big(n(x+y)\big)\Big),$$

$$f(y+z)=\lim_{n\to\infty}\Big(2f\big((n+1)(y+z)\big)+f\big(2n(y+z)\big)-f\big((2n+1)(y+z)\big)-2f\big(n(y+z)\big)\Big),$$

$$f(x+z)=\lim_{n\to\infty}\Big(2f\big((n+1)(x+z)\big)+f\big(2n(x+z)\big)-f\big((2n+1)(x+z)\big)-2f\big(n(x+z)\big)\Big)$$

and

$$f(x+y+z)=\lim_{n\to\infty}\Big(2f\big((n+1)(x+y+z)\big)+f\big(2n(x+y+z)\big)-f\big((2n+1)(x+y+z)\big)-2f\big(n(x+y+z)\big)\Big).$$

Now,

$$\begin{aligned}&\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(x+z)\|\\ =\ &\lim_{n\to\infty}\Big\|\Big(2f\big((n+1)(x+y+z)\big)+f\big(2n(x+y+z)\big)-f\big((2n+1)(x+y+z)\big)-2f\big(n(x+y+z)\big)\big)\Big)\\ &+\Big(2f\big((n+1)(x)\big)+f(2nx)-f\big((2n+1)x\big)-2f(nx)\Big)\\ &+\Big(2f\big((n+1)(y)\big)+f(2ny)-f\big((2n+1)y\big)-2f(ny)\Big)\\ &+\Big(2f\big((n+1)(z)\big)+f(2nz)-f\big((2n+1)z\big)-2f(nz)\Big)\\ &-\Big(2f\big((n+1)(x+y)\big)+f\big(2n(x+y)\big)-f\big((2n+1)(x+y)\big)-2f\big(n(x+y)\big)\Big)\\ &-\Big(2f\big((n+1)(y+z)\big)+f\big(2n(y+z)\big)-f\big((2n+1)(y+z)\big)-2f\big(n(y+z)\big)\Big)\end{aligned}$$

$$
\begin{aligned}
& -\Big(2f\big((n+1)(x+z)\big)+f\big(2n(x+z)\big)-f\big((2n+1)(x+z)\big)-2f\big(n(x+z)\big)\Big)\Bigg\| \\
\leq\ & \lim_{n\to\infty} 2\Bigg\|f\big((n+1)(x+y+z)\big)+f\big((n+1)x\big)+f\big((n+1)y\big)+f\big((n+1)z\big) \\
& -f\big((n+1)(x+y)\big)-f\big((n+1)(y+z)\big)-f\big((n+1)(x+z)\big)\Bigg\| \\
& +\Bigg\|f\big(2n(x+y+z)\big)+f(2nx)+f(ny)+f(2nz)-f\big(2n(x+y)\big)-f(2n(y+z))-f\big(2n(x+z)\big)\Bigg\| \\
& +\Bigg\|f\big((2n+1)(x+y+z)\big)+f\big((2n+1)x\big)+f\big((2n+1)y\big)+f\big((2n+1)z\big)-f\big((2n+1)(x+y)\big) \\
& -f\big((2n+1)(y+z)\big)-f\big((2n+1)(x+z)\big)\Bigg\|+2\Bigg\|f\big(n(x+y+z)\big)+f(nx)+f(ny)+f(nz) \\
& -f\big(n(x+y)\big)-f\big(n(y+z)\big)-f\big(n(x+z)\big)\Bigg\| \\
\leq\ & \lim_{n\to\infty}\epsilon\Big(2(n+1)^{p+q+r}+(2n)^{p+q+r}+(2n+1)^{p+q+r}+2n^{p+q+r}\Big)\|x\|^p\|y\|^q\|z\|^r=0,
\end{aligned}
$$

implies

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)f(x+z),$$

for all $x,\ y,\ x+y,\ y+z,\ x+z,\ x+y+z\in U$. ∎

Remark 3 *The condition* $(p+q+r<0)$ *specified in Theorem 4 is necessary for the hyperstability result. In Example 3, we illustrated that if the criterion* $(p+q+r<0)$ *is not met, then the function may not be of quadratic type.*

Example 3 *Let* $f:\mathbb{R}\to\mathbb{R}$ *be defined as* $f(x)=x^3$. *Then we have*

$$
\begin{aligned}
& \Bigg\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(z+x)\Bigg\| \\
=\ & \Big|(x+y+z)^3+x^3+y^3+z^3-(x+y)^3-(y+z)^3-(z+x)^3\Big| \\
=\ & |6xyz|\leq 6|x||y||z|\,.
\end{aligned}
$$

Then, for $p=1,\ q=1,\ r=1,\ \epsilon=6$, *hypothesis (3) holds but* f *is not a quadratic function.*

Theorem 5 *Consider two normed spaces* V *and* W *and a non-empty subset* $U\subset V-\{0\}$. *Choose* $\epsilon,\theta\geq 0$ *and real numbers* p, q *and* r *satisfying* $p+q+r<0$ *and* $p+q+2r<0$. *Assume that for each* $x\in U$, *there exists a natural number* n_x *such that for all* $n\geq n_x$, *we have* $nx\in U$. *If a function* $f:V\to W$ *defined on* V *satisfies*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\leq\|x\|^p\|y\|^q(\epsilon\|x+y\|^r+\theta\|x-y\|^r), \tag{4}$$

for all $x,\ y,\ x+y,\ x-y\in U$, *then* f *is quadratic on* U, *i.e.,*

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$

for all $x,\ y,\ x+y,\ x-y\in U$.

Proof. It is given that $p+q+r<0$ and $p+q+2r<0$, without loss of generality assume that $p+r<0$. Let $x,y\in U$ with $x+y,\ x-y\in U$. Hence, by the given hypothesis there exists natural number m such that nx, ny, $n(x+y)$ and $n(x-y)\in U$ for every $n\geq m$. Let

$$\phi(x,y)=\|x\|^p\|y\|^q(\epsilon\|x+y\|^r+\theta\|x-y\|^r).$$

On similar line of Theorem 2, we have

$$2f(x) = \lim_{n\to\infty}\Big(f\big((n+1)x\big) + f\big((n-1)x\big) - 2f(nx) - 2f(x)\Big),$$

$$2f(y) = \lim_{n\to\infty}\Big(f\big((n+1)y\big) + f\big((n-1)y\big) - 2f(ny)\Big),$$

$$2f(x+y) = \lim_{n\to\infty}\Big(f\big((n+1)(x+y)\big) + f\big((n-1)(x+y)\big) - 2f\big(n(x+y)\big)\Big)$$

and

$$2f(x-y) = \lim_{n\to\infty}\Big(f\big((n+1)(x-y)\big) + f\big((n-1)(x-y)\big) - 2f\big(n(x-y)\big)\Big).$$

Now, consider

$$\begin{aligned}
& \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\
= & \lim_{n\to\infty}\Bigg\|\frac{1}{2}\Big(f\big((n+1)(x+y)\big) + f\big((n-1)(x+y)\big) - 2f\big(n(x+y)\big)\Big) \\
& +\frac{1}{2}\Big(f\Big((n+1)(x-y)\big) + f\big((n-1)(x-y)\big) - 2f\big(n(x-y)\big)\Big) \\
& -\Big(f\big((n+1)x\big) + f\big((n-1)x\big) - 2f(nx)\Big) \\
& -\Big(f\big((n+1)y\big) + f\big((n-1)y\big) - 2f(ny)\Big)\Bigg\| \\
\leq & \lim_{n\to\infty}\frac{1}{2}\Big\|f\big((n+1)(x+y)\big) + f\big((n+1)(x-y)\big) - 2f\big((n+1)x\big) - 2f\big((n+1)y\big)\Big\| \\
& +\frac{1}{2}\Big\|f\big((n-1)(x+y)\big) + f\big((n-1)(x-y)\big) - 2f\big((n-1)x\big) - 2f\big((n-1)y\big)\Big\| \\
& +\Big\|f\big(n(x+y)\big) + f\big(n(x-y)\big) - 2f(nx) - 2f(ny)\Big\| \\
\leq & \lim_{n\to\infty}\left(\frac{1}{2}(n+1)^{p+q+r} + \frac{1}{2}(n-1)^{p+q+r} + n^{p+q+r}\right)\phi(x,y) \\
= & \; 0,
\end{aligned}$$

implies

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

for all x, y, $x+y$, $x-y \in U$. ■

Remark 4 *The conditions ($p+q+r<0$ and $p+q+2r<0$) specified in Theorem 5 are necessary for the hyperstability result. In Example 4, we illustrated that if the criteria ($p+q+r<0$ and $p+q+2r<0$) is not met, then the function may not be quadratic.*

Example 4 *Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=x^3$ and let $U=[1,\infty)$. Then we have*

$$\begin{aligned}
\Big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\Big\| &= \Big|(x+y)^3 + (x-y)^3 - 2x^3 - 2y^3\Big| \\
&= \Big|6xy^2 - 2y^3\Big| \\
&= 2|y|^2|3x-y|
\end{aligned}$$

$$
\begin{aligned}
&= 2|y|^2|3x - x + x - y| \\
&\leq 2|y|^2\big(|2x| + |x - y|\big) \\
&\leq |x|^1|y|^2\big(4|x + y| + 2|x - y|\big),
\end{aligned}
$$

for all $x, y \in U$. Then, for $p = 1$, $q = 2$, $r = 1$, $\epsilon = 4$ and $\theta = 2$, hypothesis (4) holds. But f is not a quadratic function.

Remark 5 *The result proved in Theorem 2 and Theorem 5 is associated to various hyperstability results concerning linear and quadratic functional equations in the existing literature. For instance, see Theorem 2 of [10].*

Theorem 6 *Consider two normed spaces V and W and take a non-empty subset $U \subset V - \{0\}$. Choose $\epsilon, \theta \geq 0$ and real numbers p, q and r satisfying $p+q+r < 0$ and $p+q+2r < 0$. Assume that for each $x \in U$, there exists a natural number n_x such that for all $n \geq n_x$, we have $nx \in U$. If a function $f : V \to W$ defined on V satisfies*

$$
\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right\| \leq \|x\|^p\|y\|^q(\epsilon\|x+y\|^r + \theta\|x-y\|^r), \tag{5}
$$

for all x, y, $\frac{x+y}{2}$, $\frac{x-y}{2} \in U$, then f is quadratic of Jensen type on U, i.e.,

$$
2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y),
$$

for all x, y, $\frac{x+y}{2}$, $\frac{x-y}{2} \in U$.

Proof. It is given that $p + q + r < 0$ and $p + q + 2r < 0$. Without loss of generality, let $p + r < 0$. Let $x, y \in U$ with $\frac{x+y}{2}$, $\frac{x-y}{2} \in U$. Hence, by the given hypothesis there exists a natural number m such that $\frac{nx}{2}$, $\frac{ny}{2}$, $\frac{n(x+y)}{2}$ and $\frac{n(x-y)}{2} \in U$ for every $n \geq m$. Let

$$
\phi(x, y) = \|x\|^p\|y\|^q(\epsilon\|x+y\|^r + \theta\|x-y\|^r).
$$

On similar line of Theorem 3, we have

$$
f(x) = \lim_{n\to\infty}\left(2f\left(\frac{(n+1)x}{2}\right) + 2f\left(\frac{(n-1)x}{2}\right) - f(nx)\right),
$$

$$
f(y) = \lim_{n\to\infty}\left(2f\left(\frac{(n+1)y}{2}\right) + 2f\left(\frac{(n-1)y}{2}\right) - f(ny)\right),
$$

$$
f\left(\frac{x+y}{2}\right) = \lim_{n\to\infty}\left(2f\left(\frac{(n+1)(x+y)}{4}\right) + 2f\left(\frac{(n-1)(x+y)}{4}\right) - f\left(\frac{n(x+y)}{2}\right)\right)
$$

and

$$
f\left(\frac{x-y}{2}\right) = \lim_{n\to\infty}\left(2f\left(\frac{(n+1)(x-y)}{4}\right) + 2f\left(\frac{(n-1)(x-y)}{4}\right) - f\left(\frac{n(x-y)}{2}\right)\right).
$$

Consider

$$
\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right\| \\
= \; &\lim_{n\to\infty}\Bigg\|2\left(2f\left(\frac{(n+1)(x+y)}{4}\right) + 2f\left(\frac{(n-1)(x+y)}{4}\right) - f\left(\frac{n(x+y)}{2}\right)\right) \\
&+2\left(2f\left(\frac{(n+1)(x-y)}{4}\right) + 2f\left(\frac{(n-1)(x-y)}{4}\right) - f\left(\frac{n(x-y)}{2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(2f\left(\frac{(n+1)x}{2}\right)+2f\left(\frac{(n-1)x}{2}\right)-f(nx)\right)\\
&-\left(2f\left(\frac{(n+1)y}{2}\right)+2f\left(\frac{(n-1)y}{2}\right)-f(ny)\right)\Bigg\|\\
\leq\ & \lim_{n\to\infty} 2\left\|2f\left(\frac{(n+1)(x+y)}{4}\right)+2f\left(\frac{(n+1)(x-y)}{4}\right)-f\left(\frac{(n+1)x}{2}\right)-f\left(\frac{(n+1)y}{2}\right)\right\|\\
&+2\left\|2f\left(\frac{(n-1)(x+y)}{4}\right)+2f\left(\frac{(n-1)(x-y)}{4}\right)-f\left(\frac{(n-1)x}{2}\right)-f\left(\frac{(n-1)y}{2}\right)\right\|\\
&+\left\|2f\left(\frac{n(x+y)}{2}\right)+f\left(\frac{n(x-y)}{2}\right)-f(nx)-f(ny)\right\|\\
\leq\ & \lim_{n\to\infty}\left(2\left(\frac{n+1}{2}\right)^{p+q+r}+2\left(\frac{n-1}{2}\right)^{p+q+r}+n^{p+q+r}\right)\phi(x,y)\\
=\ & 0,
\end{aligned}
$$

implies

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y),$$

for all $x,\ y,\ \frac{x+y}{2},\ \frac{x-y}{2}\in U$. ■

Remark 6 *The conditions ($p+q+r<0$ and $p+q+2r<0$) specified in Theorem 6 are necessary for the hyperstability result. In Example 5, we illustrated that if the criteria ($p+q+r<0$ and $p+q+2r<0$) is not met, then the function may not be quadratic of Jensen type.*

Example 5 *Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=x^3$, and let $U=[1,\infty)$. Consider*

$$
\begin{aligned}
\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{3}\right)-f(x)-f(y)\right\| &= \left|2\left(\frac{x+y}{2}\right)^3+2\left(\frac{x-y}{2}\right)^3-x^3-y^3\right|\\
&= \left|\frac{1}{4}\left(-2x^3+6xy^2-4y^3\right)\right|\\
&= \frac{1}{2}\left|x^3-3xy^2+2y^3\right|\\
&= \frac{1}{2}\left|x^3-xy^2+2y^3-2xy^2\right|\\
&= \frac{1}{2}\left|x(x^2-y^2)+2y(x^2-y^2)\right|\\
&\leq \frac{1}{2}\left|x^2-y^2\right||x+2y|\\
&\leq |x|^2|y|^2\left(2|x+y|+|x-y|\right),
\end{aligned}
$$

for all $x,y\in U$. Then, for $p=2,\ q=2,\ r=1,\ \epsilon=2,\ \theta=1$, hypothesis (5) holds but f is not a quadratic type Jensen function.

Remark 7 *The result proved in Theorem 3 and Theorem 6 is associated to various hyperstability results concerning linear and quadratic functional equations in the existing literature. For instance, see Corollaries 1–3 of [10], Theorems 2 and 5 of [7] and many more.*

Theorem 7 *Consider two normed spaces V and W and a non-empty subset $U\subset V-\{0\}$. Choose $\epsilon\geq 0$ and real numbers $p,\ q,\ r$ and s satisfying $p+q+r+s<0$. Assume that for each $x\in U$, there exists a natural number n_x such that for all $n\geq n_x$, we have $nx\in U$. If a function $f:V\to W$ defined on V satisfies*

$$\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(z+x)\|\leq\|x\|^p\|y\|^q\|z\|^r\epsilon\|x+y+z\|^s, \quad (6)$$

for all $x,\ y,\ z,\ x+y,\ y+z,\ x+z,\ x+y+z \in U$*, then* f *is a quadratic type functional equation satisfying*

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x),$$

for all $x,\ y,\ z,\ x+y,\ y+z,\ x+z,\ x+y+z \in U$.

Proof. *It is given that* $p+q+r+s<0$. *Without loss of generality let* $q+r+s<0$. *Let* $x,y,z \in U$ *with* $x+y,\ y+z,\ x+z,\ x+y+z \in U$. *Hence, by our assumption there exists a natural number* m *such that* nx, ny, nz, $n(x+y)$, $n(y+z)$, $n(x+z)$ *and* $n(x+y+z) \in U$ *for every* $n \geq m$. *Let*

$$\phi(x,y,z)=\|x\|^p\|y\|^q\|z\|^r\epsilon\|x+y+z\|^s.$$

On similar line of Theorem 4, we have

$$f(x)=\lim_{n\to\infty}\Big(2f\big((n+1)(x)\big)+f(2nx)-f\big((2n+1)x\big)-2f(nx)\Big),$$

$$f(y)=\lim_{n\to\infty}\Big(2f\big((n+1)(y)\big)+f(2ny)-f\big((2n+1)y\big)-2f(ny)\Big),$$

$$f(z)=\lim_{n\to\infty}\Big(2f\big((n+1)(z)\big)+f(2nz)-f\big((2n+1)z\big)-2f(nz)\Big),$$

$$f(x+y)=\lim_{n\to\infty}\Big(2f\big((n+1)(x+y)\big)+f\big(2n(x+y)\big)-f\big((2n+1)(x+y)\big)-2f\big(n(x+y)\big)\Big),$$

$$f(y+z)=\lim_{n\to\infty}\Big(2f\big((n+1)(y+z)\big)+f\big(2n(y+z)\big)-f\big((2n+1)(y+z)\big)-2f\big(n(y+z)\big)\Big),$$

$$f(x+z)=\lim_{n\to\infty}\Big(2f\big((n+1)(x+z)\big)+f\big(2n(x+z)\big)-f\big((2n+1)(x+z)\big)-2f\big(n(x+z)\big)\Big)$$

and

$$f(x+y+z)=\lim_{n\to\infty}\Big(2f\big((n+1)(x+y+z)\big)+f\big(2n(x+y+z)\big)-f\big((2n+1)(x+y+z)\big)-2f\big(n(x+y+z)\big)\Big).$$

Now,

$$\begin{aligned}
& \|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(y+z)-f(x+z)\| \\
= & \lim_{n\to\infty}\Big\|\Big(2f\big((n+1)(x+y+z)\big)+f\big(2n(x+y+z)\big)-f\big((2n+1)(x+y+z)\big)-2f\big(n(x+y+z)\big)\Big) \\
& +\Big(2f\big((n+1)(x)\big)+f(2nx)-f\big((2n+1)x\big)-2f(nx)\Big) \\
& +\Big(2f\big((n+1)(y)\big)+f(2ny)-f\big((2n+1)y\big)-2f(ny)\Big) \\
& +\Big(2f\big((n+1)(z)\big)+f(2nz)-f\big((2n+1)z\big)-2f(nz)\Big) \\
& -\Big(2f\big((n+1)(x+y)\big)+f\big(2n(x+y)\big)-f\big((2n+1)(x+y)\big)-2f\big(n(x+y)\big)\Big) \\
& -\Big(2f\big((n+1)(y+z)\big)+f\big(2n(y+z)\big)-f\big((2n+1)(y+z)\big)-2f\big(n(y+z)\big)\Big) \\
& -\Big(2f\big((n+1)(x+z)\big)+f\big(2n(x+z)\big)-f\big((2n+1)(x+z)\big)-2f\big(n(x+z)\big)\Big)\Big\|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \lim_{n\to\infty} 2\Big\| f\big((n+1)(x+y+z)\big) + f\big((n+1)x\big) + f\big((n+1)y\big) + f\big((n+1)z\big) \\
& -f\big((n+1)(x+y)\big) - f\big((n+1)(y+z)\big) - f\big((n+1)(x+z)\big)\Big\| \\
& +\Big\| f\big(2n(x+y+z)\big) + f(2nx) + f(ny) + f(2nz) - f\big(2n(x+y)\big) - f(2n(y+z)) - f\big(2n(x+z)\big)\Big\| \\
& +\Big\| f\big((2n+1)(x+y+z)\big) + f\big((2n+1)x\big) + f\big((2n+1)y\big) + f\big((2n+1)z\big) - f\big((2n+1)(x+y)\big) \\
& -f\big((2n+1)(y+z)\big) - f\big((2n+1)(x+z)\big)\Big\| + 2\Big\| f\big(n(x+y+z)\big) + f(nx) + f(ny) + f(nz) \\
& -f\big(n(x+y)\big) - f\big(n(y+z)\big) - f\big(n(x+z)\big)\Big\| \\
\leq\ & \lim_{n\to\infty}\Big(2(n+1)^{p+q+r+s} + (2n)^{p+q+r+s} + (2n+1)^{p+q+r+s} + 2n^{p+q+r+s}\Big)\phi(x,y,z) \\
=\ & 0,
\end{aligned}
$$

implies

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z)f(x+z),$$

for all $x,\ y,\ x+y,\ y+z,\ x+z,\ x+y+z \in U$. ■

Remark 8 *The condition* $(p+q+r+s<0)$ *specified in Theorem 7 is necessary for the hyperstability result. In Example 6, we illustrated that if the criterion* $(p+q+r+s<0)$ *is not met, then the function may not be of quadratic type.*

Example 6 *Let* $f:\mathbb{R}\to\mathbb{R}$ *defined as* $f(x)=x^4$. *Consider*

$$
\begin{aligned}
& \Big\| f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x)\Big\| \\
=\ & \Big|(x+y+z)^4 + x^4 + y^4 + z^4 - (x+y)^4 - (y+z)^4 - (x+z)^4\Big| \\
=\ & \Big|8\left(x^2yz + xy^2z + xyz^2\right)\Big| \\
\leq\ & 8|xyz||x+y+z| \\
\leq\ & 8|x||y||z||x+y+z|\,.
\end{aligned}
$$

Then, for $\epsilon=8,\ p=q=r=s=1$, *hypothesis (6) holds. But* f *is not a quadratic function.*

Remark 9 *Theorem 4 and Theorem 7 is associated to various hyperstability results concerning linear and quadratic functional equations in the existing literature. For instance, see Proposition 1.10 of [36], Corollary 3.7, Corollary 3.10, Corollary 3.12 of [5].*

Theorem 8 *Consider two normed spaces* V *over field* $\mathbb{F}$ *and* W *over field* $\mathbb{K}$. *Let* $a,b\in\mathbb{F}-\{0\}$ *and* $\phi: X\times X\to[0,\infty)$ *be a mapping such that*

$$\lim_{n\to\infty}\phi(a^{-1}(n+1)x, -b^{-1}nx) = 0, \qquad \lim_{n\to\infty}\phi(nx,ny) = 0$$

for all $x,y\in V-\{0\}$. *Let* $A,B\in\mathbb{K}$ *and* $C\in W$. *If* $f:V\to W$ *satisfies*

$$\|f(ax+by) + f(ax-by) - Af(x) - Bf(y) - C\| \leq \phi(x,y) \tag{7}$$

for all $x,y\in\{z\in V:\|z\|\geq d\}$ *for some* $d>0$, *then* f *satisfies*

$$f(ax+by) + f(ax-by) = Af(x) + Bf(y) + C$$

for all $x,y\in V$.

Proof. By substituting $x = \frac{(n+1)x}{a}$ and $y = \frac{nx}{b}$ in (7), we have

$$\left\|f(x) + f\left((2n+1)x\right) - Af\left(a^{-1}(n+1)x\right) - Bf\left(-b^{-1}nx\right) - C\right\| \leq \phi\left(a^{-1}(n+1)x, -b^{-1}nx\right),$$

for all $x \in V - \{0\}$ and $n \in \mathbb{N}$, where $a^{-1}(n+1)x, b^{-1}nx \in \{z \in V : \|z\| \geq d\}$. Taking limit as $n \to \infty$ on both sides, we have

$$\lim_{n\to\infty}\left\|f(x)+f\left((2n+1)x\right)-Af\left(a^{-1}(n+1)x\right)-Bf\left(-b^{-1}nx\right)-C\right\| \leq \lim_{n\to\infty}\phi\left(a^{-1}(n+1)x, -b^{-1}nx\right) = 0.$$

or

$$f(x) = \lim_{n\to\infty}\Big(Af\left(a^{-1}(n+1)x\right) + Bf\left(-b^{-1}nx\right) + C - f\left((2n+1)x\right)\Big).$$

Similarly, we have

$$f(y) = \lim_{n\to\infty}\Big(Af\left(a^{-1}(n+1)y\right) + Bf\left(-b^{-1}ny\right) + C - f\left((2n+1)y\right)\Big),$$

$$f(ax+by) = \lim_{n\to\infty}\Big(Af\left(a^{-1}(n+1)(ax+by)\right) + Bf\left(-b^{-1}n(ax+by)\right) + C - f\left((2n+1)(ax+by)\right)\Big)$$

and

$$f(ax-by) = \lim_{n\to\infty}\Big(Af\left(a^{-1}(n+1)(ax-by)\right) + Bf\left(-b^{-1}n(ax-by)\right) + C - f\left((2n+1)(ax-by)\right)\Big).$$

Now,

$$\begin{aligned}
& \|f(ax+by) + f(ax-by) - Af(x) - Bf(y) - C\| \\
= & \lim_{n\to\infty}\Big\|\Big(Af\left(a^{-1}(n+1)(ax+by)\right) + Bf\left(-b^{-1}n(ax+by)\right) + C - f\left((2n+1)(ax+by)\right)\Big) \\
& +\Big(Af\left(a^{-1}(n+1)(ax-by)\right) + Bf\left(-b^{-1}n(ax-by)\right) + C - f\left((2n+1)(ax-by)\right)\Big) \\
& -A\Big(Af\left(a^{-1}(n+1)x\right) + Bf\left(-b^{-1}nx\right) + C - f\left((2n+1)x\right)\Big) \\
& -B\Big(Af\left(a^{-1}(n+1)y\right) + Bf\left(-b^{-1}ny\right) + C - f\left((2n+1)y\right)\Big) - C\Big\| \\
\leq & \lim_{m\to\infty} A\Big\|f\left(a^{-1}(n+1)(ax+by)\right) + f\left(a^{-1}(n+1)(ax-by)\right) - Af(a^{-1}(n+1)x) \\
& -Bf(a^{-1}(n+1)y) - C\Big\| + B\Big\|f\left(-b^{-1}n(ax+by)\right) + f\left(-b^{-1}n(ax-by)\right) - Af\left(-b^{-1}nx\right) \\
& -Bf\left(-b^{-1}ny\right) - C\Big\| + \Big\|f\left((2n+1)(ax+by)\right) + f\left((2n+1)(ax-by)\right) \\
& -Af\left((2n+1)x\right) - Bf\left((2n+1)y\right) - C\Big\| \\
\leq & \lim_{m\to\infty}\Big(A\phi\left(a^{-1}(n+1)x, a^{-1}(n+1)y\right) + B\phi\left(-b^{-1}nx, -b^{-1}ny\right) + \phi\left((2n+1)x, (2n+1)y\right)\Big) \\
= & 0.
\end{aligned}$$

Therefore,

$$f(ax+by) + f(ax-by) = Af(x) + Bf(y) + C,$$

for all $x, y \in V$. ■

Remark 10 *Theorem 8 is associated to various hyperstability results concerning linear and quadratic functional equations in the existing literature. For instance, see Theorem 2, Corollary 3 of [29], Theorem 2.1, Theorem 2.3 of [30], Theorem 2.1 of [28], Theorem 2 of [10] and many more.*

3 Conclusion

In this manuscript, we have presented some hyperstability results for Jensen type quadratic functional equations as well as quadratic functional equations. In addition, we provided some examples to illustrate the importance of the presumptions to discuss the stability results for the quadratic functional equations.

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