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On Existence And Uniqueness Of A Solution Of An Integral Equation Using Contractive Mapping*

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Abstract

In the present manuscript, we introduced $\alpha-\psi$ type contractive mapping in C^*-algebra valued partial metric space for unital C^*-algebra $\mathbb{A}$, and proved some fixed point theorems. To discuss the usability of the proved results, we established the existence and uniqueness of a solution of an integral equations. .

1 Introduction

The Banach contraction principle [12] is the cornerstone of the development of fixed point theory. The theory has numerous applications not only in the different branches of mathematics but also in economics, chemistry, biology, computer science, engineering and others. Fixed point and common fixed point results for different types of contractions in various spaces have been investigated extensively by various researchers (see [1], [11], [13], [18], [19]).

Matthews [14] introduced the notion of partial metric space, which is a generalization of metric space. Ma et al. [18] introduced the notion of C^*-algebra valued metric space by replacing the set of real numbers with the positive elements of C^*-algebra. Various researchers discussed number of fixed point results in C^*-algebra valued metric space by using different approaches and definitions (see [5], [6], [7], [8], [9], [10], [15], [16]).

Recently, Chandok et al. [13] generalized the class of C^*-algebra valued metric space with the notion of C^*-algebra valued partial metric space by combining partial metric space and C^*-algebra valued metric space and proved some fixed point results. Later, Tomar et al. [2] introduced the concepts of contractiveness and expansiveness in a C^*-algebra valued partial metric space and proved the existence of a fixed point for self-mappings. Samet [3, 4] introduced the concept of $\alpha-\psi$ contractive mapping in b-metric space. Omran et al. [17] gave $(\alpha-\psi)$ type contractive mapping in C^*-algebra valued b-metric space and proved the Banach version of fixed point theorem.

In the present manuscript, we introduce $\alpha-\psi$ type contractive mapping in C^*-algebra valued partial metric space for unital C^*-algebra $\mathbb{A}$, and prove some fixed point results. Our results generalize well-known results in the literature. Also, we provide an example and an application to illustrate our results.

2 Preliminaries

We begin the section by providing some basic concepts of C^*-algebra followed by some lemmas and definitions given in [19]. Throughout the paper, by $\mathbb{A}$, we denote an unital C^*-algebra with the unity element $I_{\mathbb{A}}$.

Lemma 1 ([17]) *Suppose $\mathbb{A}$ be unital C^*- algebra with unity $I_{\mathbb{A}}$. Then, the following hold:*

(*i*) *If $a\in\mathbb{A}$ with $||a||\leq 1/2$, then $(I_{\mathbb{A}}-a)$ is invertible and $||a(1-a)^{-1}||\leq 1$.*

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(ii) *For any* $x \in \mathbb{A}$ *and* $a, b \in \mathbb{A}^+$ *such that* $a \preceq b$*, we have* x^*ax *and* x^*bx *are positive element and* $x^*ax \preceq x^*bx$.

(iii) *If* $\theta \preceq a \preceq b$ *then* $||a|| \leq ||b||$.

(iv) *If* $a, b \in \mathbb{A}^+$ *and* $ab = ba$*, then* $a.b \succeq \theta_{\mathbb{A}}$.

(v) *Let* $\mathbb{A}'$ *denote the set* $\{a \in \mathbb{A} : ab = ba\ \forall\ b \in \mathbb{A}\}$ *and let* $a \in \mathbb{A}'$. *If* $b, c \in \mathbb{A}$ *with* $b \succeq c \succeq \theta$ *and* $(I_{\mathbb{A}} - a) \in (\mathbb{A}')^+$ *is an invertible element, then* $(I_{\mathbb{A}} - a)^{-1}b \preceq (I_{\mathbb{A}} - a)^{-1}c$.

Definition 1 ([13]) *Let* X *be a nonempty set and* $p_{\mathbb{A}} : X \times X \to \mathbb{A}$ *satisfying:*

(i) $\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x, y)$ *and* $p_{\mathbb{A}}(x, x) = p_{\mathbb{A}}(y, y) = p_{\mathbb{A}}(x, y)$ *if and only if* $x = y$*;*

(ii) $p_{\mathbb{A}}(x, x) \preceq p_{\mathbb{A}}(x, y)$*;*

(iii) $p_{\mathbb{A}}(x, y) = p_{\mathbb{A}}(y, x)$*;*

(iv) $p_{\mathbb{A}}(x, y) \preceq p_{\mathbb{A}}(x, z) + p_{\mathbb{A}}(z, y) - p_{\mathbb{A}}(z, z)$,

for any $x, y, z \in X$. *Then,* $p_{\mathbb{A}}$ *is called a* C^**-algebra valued partial metric and* $(X, \mathbb{A}, p_{\mathbb{A}})$ *is called a unital* C^**-algebra valued partial metric space.*

Definition 2 ([13]) *Let* $(X, \mathbb{A}, p_{\mathbb{A}})$ *be a* C^**-algebra valued partial metric space,* $x \in X$ *and* $\{x_n\}$ *a sequence in* X. *Then*

(i) $\{x_n\}$ *converges to* x *with respect to* $\mathbb{A}$*, whenever for every* $\epsilon > 0$ *there is a natural number* $\mathbb{N}$ *such that* $||p_{\mathbb{A}}(x_n, x) - p_{\mathbb{A}}(x, x)|| \leq \epsilon$ *for all* $n \geq N$. *We denote it by*

$$\lim_{n\to\infty} \big(p_{\mathbb{A}}(x_n, x) - p_{\mathbb{A}}(x, x)\big) = \theta_{\mathbb{A}}.$$

(ii) $\{x_n\}$ *is a partial Cauchy sequence with respect to* $\mathbb{A}$ *whenever for every* $\epsilon > 0$ *there is a natural number* $\mathbb{N}$ *such that*

$$\left(p_{\mathbb{A}}(x_n, x_m) - \frac{1}{2}p_{\mathbb{A}}(x_n, x_n) - \frac{1}{2}p_{\mathbb{A}}(x_m, x_m)\right)\left(p_{\mathbb{A}}(x_n, x_m) \right.$$
$$\left. - \frac{1}{2}p_{\mathbb{A}}(x_n, x_n) - \frac{1}{2}p_{\mathbb{A}}(x_m, x_m)\right)^* \preceq \epsilon^2.$$

for all $m, n \geq N$.

(iii) $(X, \mathbb{A}, p_{\mathbb{A}})$ *is said to be complete with respect to* $\mathbb{A}$ *if every partial Cauchy sequence converges to a point* $x \in X$ *with respect to* $\mathbb{A}$ *such that*

$$\lim_{n\to\infty}\left(p_{\mathbb{A}}(x_n, x) - \frac{1}{2}p_{\mathbb{A}}(x_n, x_n) - \frac{1}{2}p_{\mathbb{A}}(x, x)\right) = \theta_{\mathbb{A}}.$$

Example 1 *Let* $X = [0, 1)$ *and* $x \in \mathbb{A} = \mathbb{C}$ *be a non zero element. Define* $p_{\mathbb{A}}(q, r) = \max\{1 - q, 1 - r\}xx^*$ *Then, we can easily show that* $p_{\mathbb{A}} : X \times X \to \mathbb{A}$ *is a complete* C^**-algebra valued partial metric space. But* $p_{\mathbb{A}} : X \times X \to \mathbb{A}$ *is not a* C^**-algebra valued metric space, since* $p_{\mathbb{A}}(q, q) = (1 - q)xx^* \neq \theta_{\mathbb{A}}$.

Example 2 *Let* $X = \mathbb{R}$ *and* $\mathbb{A} = M_2(\mathbb{C})$ *be a set of all* 2×2 *matrices with entries in* $\mathbb{C}$ *and* $M_2(\mathbb{C})$ *is a* C^**-algebra with the matrix norm. Define*

$$p_{\mathbb{A}}(s, t) = \begin{pmatrix} \gamma_1|x_{11} - y_{11}| & 0 \\ 0 & \gamma_1|x_{11} - y_{11}| \end{pmatrix},$$

where $s = (x_{ij})^2{}_{i,j=1}$ *and* $t = (y_{ij})^2{}_{i,j=1}$ *are* 2×2*-matrices entry from* $\mathbb{C}$ *for all* $i, j = 1, 2$ *and* $\gamma_1, \gamma_2 > 0$. *One can define the partial ordering* $(\preceq)$ *on* $M_2(\mathbb{C})$ *as* $s \preceq t$ *if and only if* $|x_{ij}| \preceq |y_{ij}|$. *Then,* $(X, M_2(\mathbb{C}), p_{\mathbb{A}})$ *is a* C^**-algebra valued partial metric space.*

Example 3 *Let $S(X)$ be the collection of all convergent sequences such that*

$$F_\epsilon(f_n) = \{f : d(f_n, f) \leq \epsilon,\ \epsilon > 0 \text{ and } f_n, f \in X\}$$

for all $n \in \mathbb{N}$ and $\mathbb{A} = M_2(\mathbb{C})$ be C^*-algebra complex matrices. If $A = [a_{ij}] \in \mathbb{A}$, then $A^* = [\overline{a_{ji}}]$ is non zero element of $\mathbb{A}$.

Norm is defined as

$$\|A\| = \sup\{||A\gamma||_p : \gamma \in \mathbb{C}^n,\ ||\gamma||_p \leq 1\},$$

where $||.||_p$ norm is usual l^p-norm on $\mathbb{C}^n$. Define $p_{\mathbb{A}} : S(X) \times S(X) \to \mathbb{A}$ such that

$$p_{\mathbb{A}}(F_{\epsilon_1} f_n, F_{\epsilon_2} f_m) = |f_1 - f_2|AA^* + \max\{\epsilon_1, \epsilon_2\} I_A.$$

Then, $p_{\mathbb{A}}$ is a complete C^*-algebra valued partial metric space which is not a C^*-algebra valued metric space, since $p_{\mathbb{A}}(F_{\epsilon_1} f_n, F_{\epsilon_1} f_n) = \epsilon \neq \theta_{\mathbb{A}}$ and

$$\begin{aligned} p_{\mathbb{A}}(F_{\epsilon_1} f_n, F_{\epsilon_2} f_m) &= |f_1 - f_2|AA^* + \max\{\epsilon_1, \epsilon_2\} I_A \\ &\leq [|f_1 - f_3| + |f_3 - f_2|]AA^* + [max\{\epsilon_1, \epsilon_3\} \\ &\quad + \max\{\epsilon_3, \epsilon_2\} - \epsilon_3] I_A \\ &= p_{\mathbb{A}}(F_{\epsilon_1} f_n, F_{\epsilon_3} f_r) + p_{\mathbb{A}}(F_{\epsilon_3} f_r, F_{\epsilon_2} f_m) - p_{\mathbb{A}}(F_{\epsilon_3} f_r, F_{\epsilon_3} f_r). \end{aligned}$$

Definition 3 ([17]) *If $\psi : \mathbb{A} \to \mathbb{B}$ is a linear mapping in C^*-algebra, it is said to be positive if $\psi(\mathbb{A}^+) \subseteq \mathbb{B}^+$. In this case, $\psi(\mathbb{A}_h) \subseteq \mathbb{B}_h$ and the restriction map: $\psi : \mathbb{A}_h \to \mathbb{B}_h$ is increasing.*

Definition 4 ([17]) *Suppose $\mathbb{A}$ and $\mathbb{B}$ are two C^*-algebras. A mapping $\psi : \mathbb{A} \to \mathbb{B}$ is said to be a C^*-homomorphism if*

(i) *$\psi(ax + by) = a\psi(x) + b\psi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathbb{A}$;*

(ii) *$\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in \mathbb{A}$;*

(iii) *$\psi(x^*) = \psi(x)^*$ for all $x \in \mathbb{A}$;*

(iv) *ψ maps unit in $\mathbb{A}$ to unit in $\mathbb{B}$.*

Corollary 1 ([17]) *Every C^*-homomorphism is contractive and hence bounded.*

Lemma 2 ([17]) *Every $*$ homomorphism is positive.*

Definition 5 ([17]) *Let $\Psi_{\mathbb{A}}$ be the set of positive functions, $\psi_{\mathbb{A}} : \mathbb{A}^+ \to \mathbb{A}^+$ satisfying the following conditions :*

(i) *$\psi_{\mathbb{A}}(a)$ is continuous and nondecreasing;*

(ii) *$\psi_{\mathbb{A}}(a) = \theta_{\mathbb{A}}$ if and only if $a = \theta_{\mathbb{A}}$;*

(iii) *$\sum_{n=1}^{\infty} \psi_{\mathbb{A}}^n(a) < \infty$, $\lim_{n\to\infty} \psi_{\mathbb{A}}^n(a) = \theta_{\mathbb{A}}$ for each $a \succ \theta_{\mathbb{A}}$, where $\psi_{\mathbb{A}}^n$ is n^{th} iterate of $\psi_{\mathbb{A}}$;*

(iv) *the series $\sum_{k=0}^{\infty} \psi_{\mathbb{A}}(a) < \infty$ for each $a \succ \theta_{\mathbb{A}}$ is increasing and continuous at $\theta_{\mathbb{A}}$.*

3 Main Results

In this section, we prove some fixed point theorems using $\alpha - \psi_A$-type contractive mapping in C^*-algebra valued partial metric space.

Definition 6 ([3]) *Let $T : X \to X$ be a self map and $\alpha : X \times X \to \mathbb{A}^+$. Then, T is called α-admissible if for $x, y \in X$ with $\alpha(x, y) \succeq I_{\mathbb{A}}$, then $\alpha(Tx, Ty) \succeq I_{\mathbb{A}}$.*

Definition 7 ([17]) *Let X be non-empty set and $\alpha_{\mathbb{A}} : X \times X \to (\mathbb{A}')^+$ be a function, we say that a self map T on X is $\alpha_{\mathbb{A}}$-admissible if*

$$(x, y) \in X \times X, \alpha_{\mathbb{A}}(x, y) \succeq I_{\mathbb{A}} \quad \Rightarrow \quad \alpha_{\mathbb{A}}(Tx, Ty) \succeq I_{\mathbb{A}},$$

where $I_{\mathbb{A}}$ is the unit of $\mathbb{A}$.

Definition 8 *Let $(X, \mathbb{A}, p_{\mathbb{A}})$ be a C^*-algebra valued partial metric space and $T : X \to X$ be a self mapping, we say that T is $\alpha_{\mathbb{A}} - \psi_{\mathbb{A}}$ type contractive mapping if there exist two functions $\alpha_{\mathbb{A}} : X \times X \to \mathbb{A}^+$ and $\psi_{\mathbb{A}} \in \Psi_{\mathbb{A}}$ such that*

$$\alpha_{\mathbb{A}}(x, y) p_{\mathbb{A}}(Tx, Ty) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x, y)) \text{ for all } x, y \in X. \tag{1}$$

Theorem 1 *Let $(X, \mathbb{A}, p_{\mathbb{A}})$ be a complete C^*-algebra valued partial metric space and $T : X \to X$ be an $\alpha_{\mathbb{A}} - \psi_{\mathbb{A}}$-contractive mapping satisfying the following conditions*

(i) T is $\alpha_{\mathbb{A}}$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}}$;

(iii) T is continuous.

Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}}$. We construct a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Now, suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is $\alpha_{\mathbb{A}}$-admissible, therefore we get

$$\alpha_{\mathbb{A}}(x_0, x_1) = \alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}} \Rightarrow \alpha_{\mathbb{A}}(Tx_0, T^2x_0) = \alpha_{\mathbb{A}}(x_1, x_2) \succeq I_{\mathbb{A}}. \tag{2}$$

By mathematical induction, we have

$$\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}} \text{ for all } n \in \mathbb{N}. \tag{3}$$

Using (1) and (3), we get

$$\begin{aligned} p_{\mathbb{A}}(x_n, x_{n+1}) &= p_{\mathbb{A}}(Tx_{n-1}, Tx_n) \\ &\preceq \alpha_{\mathbb{A}}(x_{n-1}, x_n) p_{\mathbb{A}}(Tx_{n-1}, Tx_n) \\ &\preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_{n-1}, x_n)). \end{aligned}$$

By mathematical induction, we get

$$p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \psi^n_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)) \text{ for all } n \in \mathbb{N}. \tag{4}$$

For $m \geq 1$ and $n \geq 1$, we have

$$\begin{aligned} p_{\mathbb{A}}(x_m, x_{m+n}) &\preceq p_{\mathbb{A}}(x_m, x_{m+1}) + p_{\mathbb{A}}(x_{m+1}, x_{m+n}) - p_{\mathbb{A}}(x_{m+1}, x_{m+1}) \\ &\preceq p_{\mathbb{A}}(x_m, x_{m+1}) + p_{\mathbb{A}}(x_{m+1}, x_{m+2}) + p_{\mathbb{A}}(x_{m+2}, x_{m+3}) \\ &\quad + \cdots + p_{\mathbb{A}}(x_{m+n-1}, x_{m+n}) - p_{\mathbb{A}}(x_{m+1}, x_{m+1}) \\ &\quad - p_{\mathbb{A}}(x_{m+2}, x_{m+2}) - \cdots - p_{\mathbb{A}}(x_{m+n-1}, x_{m+n-1}) \\ &\preceq \psi_{\mathbb{A}}^{m}(p_{\mathbb{A}}(x_0, x_1)) + \psi_{\mathbb{A}}^{m+1}(p_{\mathbb{A}}(x_0, x_1)) + \cdots \\ &\quad + \psi_{\mathbb{A}}^{m+n-1}(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i) \\ &= \sum_{i=m}^{m+n-1} \psi_{\mathbb{A}}^{i}(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i). \end{aligned} \tag{5}$$

Using $\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x_n, x_n) \preceq p_{\mathbb{A}}(x_n, x_{n+1})$ and (4), we get

$$\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x_n, x_n) \preceq p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \psi_{\mathbb{A}}^{n}(p_{\mathbb{A}}(x_0, x_1)). \tag{6}$$

Taking limit as $n \to \infty$ and using Definition 5, we get

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x_n) = \theta_{\mathbb{A}}. \tag{7}$$

Using (7) in (5) and Definition 5, we get

$$p_{\mathbb{A}}(x_m, x_{m+n}) \preceq \sum_{i=m}^{m+n-1} \psi_{\mathbb{A}}^{i}(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i) \to \theta_{\mathbb{A}} \ as \ n \to \infty. \tag{8}$$

Thus, $\{x_n\}$ is a Cauchy sequence in a complete C^*-algebra valued partial metric space $(X, \mathbb{A}, p_{\mathbb{A}})$. Thus there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x_m) = \lim_{n\to\infty} p_{\mathbb{A}}(x_n, x) = p_{\mathbb{A}}(x, x). \tag{9}$$

By using (8), we have

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x) = p_{\mathbb{A}}(x, x) = \theta_{\mathbb{A}}. \tag{10}$$

Now, we shall show that x is a fixed point of T. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} p_{\mathbb{A}}(Tx, x) &= \lim_{n\to\infty} p_{\mathbb{A}}(Tx, x_{n+1}) \\ &\preceq \lim_{n\to\infty} \Big(p_{\mathbb{A}}(Tx, x_n) + p_{\mathbb{A}}(x_n, x_{n+1}) - p_{\mathbb{A}}(x_{n+1}, x_{n+1}) \Big) \\ &= \lim_{n\to\infty} \Big(p_{\mathbb{A}}(Tx, Tx_{n-1}) + p_{\mathbb{A}}(x_n, x_{n+1}) - p_{\mathbb{A}}(x_{n+1}, x_{n+1}) \Big) \\ &\preceq \lim_{n\to\infty} \Big(\alpha_{\mathbb{A}}(x, x_{n-1}) p_{\mathbb{A}}(Tx, Tx_{n-1}) + p_{\mathbb{A}}(x_n, x_{n+1}) \\ &\quad - p_{\mathbb{A}}(x_{n+1}, x_{n+1}) \Big) \\ &\preceq \lim_{n\to\infty} \Big(\psi_{\mathbb{A}}(p_{\mathbb{A}}(x, x_{n-1})) + p_{\mathbb{A}}(x_n, x_{n+1}) - p_{\mathbb{A}}(x_{n+1}, x_{n+1}) \Big) \\ &= \psi_{\mathbb{A}}(p_{\mathbb{A}}(x, x)) + p_{\mathbb{A}}(x, x) - p_{\mathbb{A}}(x, x) \to \theta_{\mathbb{A}}. \end{aligned}$$

Hence $Tx = x$, i.e. x is a fixed point of T.

To prove the uniqueness, we will consider the following condition : for all $x, y \in X$, there exist $z \in X$ such that $\alpha_{\mathbb{A}}(x,z) \succeq I_{\mathbb{A}}$ and $\alpha_{\mathbb{A}}(y,z) \succeq I_{\mathbb{A}}$. Consider,

$$\begin{aligned} p_{\mathbb{A}}(x, T^n z) &= p_{\mathbb{A}}(Tx, T(T^{n-1}z)) \\ &\preceq \alpha_{\mathbb{A}}(x, T^{n-1}z) p_{\mathbb{A}}(Tx, T(T^{n-1}z)) \\ &\preceq \psi_{\mathbb{A}}^n(p_{\mathbb{A}}(x,z)) \to \theta_{\mathbb{A}} \text{ as } n \to \infty. \end{aligned}$$

Thus, $T^n z = x$. Similarly, $T^n z = y$ as $n \to \infty$. So, the uniqueness of the limit gives $x = y$. This completes the proof. ■

Theorem 2 *Let $(X, \mathbb{A}, p_{\mathbb{A}})$ be a complete C^*-algebra valued partial metric space and $T : X \to X$ be a mapping satisfying*

$$\alpha_{\mathbb{A}}(x,y) p_{\mathbb{A}}(Tx, Ty) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x, Tx) + p_{\mathbb{A}}(y, Ty)) \text{ for all } x, y \in X; \tag{11}$$

where $\alpha_{\mathbb{A}} : X \times X \to \mathbb{A}^+$, $\psi_{\mathbb{A}} \in \Psi_{\mathbb{A}}$ and the following conditions hold:

(i) T is $\alpha_{\mathbb{A}}$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}}$;

(iii) T is continuous.

Then, T has a unique fixed point.

Proof. On the similar lines of Theorem 1, we get

$$\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}} \text{ for all } n \in \mathbb{N}. \tag{12}$$

Using (11) and (12), we get

$$\begin{aligned} p_{\mathbb{A}}(x_n, x_{n+1}) &= p_{\mathbb{A}}(Tx_{n-1}, Tx_n) \\ &\preceq \alpha_{\mathbb{A}}(x_{n-1}, x_n) p_{\mathbb{A}}(Tx_{n-1}, Tx_n) \\ &\preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(Tx_{n-1}, x_{n-1}) + p_{\mathbb{A}}(Tx_n . x_n)) \\ &= \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n-1}) + p_{\mathbb{A}}(x_{n+1}, x_n)) \\ &= \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n-1})) + \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n+1})). \end{aligned} \tag{13}$$

From (13), we get

$$(1 - \psi_{\mathbb{A}}) p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n-1})),$$
$$p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \psi_{\mathbb{A}}(1 - \psi_{\mathbb{A}})^{-1} p_{\mathbb{A}}(x_n, x_{n-1}).$$

From Lemma 1 and Definition 5, letting

$$\phi_{\mathbb{A}} = \psi_{\mathbb{A}}(1 - \psi_{\mathbb{A}})^{-}1 = \psi_{\mathbb{A}} \sum_{n=0}^{\infty} \psi_{\mathbb{A}}^n = \sum_{n=0}^{\infty} \psi_{\mathbb{A}}^{n+1} < \infty,$$

we get $p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \phi_{\mathbb{A}}(p_{\mathbb{A}}(x_{n-1}, x_n))$. Using mathematical induction, we get

$$p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \phi_{\mathbb{A}}^n(p_{\mathbb{A}}(x_0, x_1)) \text{ for all } n \in \mathbb{N}. \tag{14}$$

For $m \geq 1$ and $n \geq 1$, follow from similar calculations in Theorem 1 that

$$p_{\mathbb{A}}(x_m, x_{m+n}) \preceq \sum_{i=m}^{m+n-1} \phi_{\mathbb{A}}^i(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i). \tag{15}$$

Using $\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x_n, x_n) \preceq p_{\mathbb{A}}(x_n, x_{n+1})$ and (14), we get

$$\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x_n, x_n) \preceq p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \phi^n_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)). \tag{16}$$

Taking limit as $n \to \infty$ in (16) and Definition 5, we get

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x_n) = \theta_{\mathbb{A}}. \tag{17}$$

Using (17) in (15) and Definition 5, we get

$$p_{\mathbb{A}}(x_m, x_{m+n}) \preceq \sum_{i=m}^{m+n-1} \phi^i_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i) \to \theta_{\mathbb{A}} \text{ as } n \to \infty. \tag{18}$$

Thus, $\{x_n\}$ is a Cauchy sequence in a complete C^*-algebra valued partial metric space $(X, \mathbb{A}, p_{\mathbb{A}})$. So there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Thus, there exists $x \in X$ such that

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x_m) = \lim_{n\to\infty} p_{\mathbb{A}}(x_n, x) = p_{\mathbb{A}}(x, x). \tag{19}$$

By using equation (18), we get

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x) = p_{\mathbb{A}}(x, x) = \theta_{\mathbb{A}}. \tag{20}$$

Now, we shall show that x is fixed point of T. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
p_{\mathbb{A}}(Tx, x) &= \lim_{n\to\infty} p_{\mathbb{A}}(Tx_n, x_n) \\
&= \lim_{n\to\infty} p_{\mathbb{A}}(Tx_n, Tx_{n-1}) \\
&\preceq \lim_{n\to\infty} \alpha_{\mathbb{A}}(x_n, x_{n-1}) p_{\mathbb{A}}(Tx_n, Tx_{n-1}) \\
&\preceq \lim_{n\to\infty} \psi_{\mathbb{A}}\big(p_{\mathbb{A}}(Tx_n, x_n) + p_{\mathbb{A}}(Tx_{n-1}, x_{n-1})\big) \\
&= \lim_{n\to\infty} \psi_{\mathbb{A}}\big(p_{\mathbb{A}}(x_{n+1}, x_n) + p_{\mathbb{A}}(x_n, x_{n-1})\big) \\
&= \psi_{\mathbb{A}}\big(p_{\mathbb{A}}(x, x) + p_{\mathbb{A}}(x, x)\big) \to \theta_{\mathbb{A}}.
\end{aligned}$$

Hence, $Tx = x$, i.e., x is a fixed point of T.

To prove the uniqueness of fixed point, let us assume that y is another fixed point of T. In a similar way, we can construct a Cauchy sequence $\{y_n\}$ such that

$$\lim_{n\to\infty} y_n = y \text{ for } y \in X.$$

Consider

$$\begin{aligned}
\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x, y) &= \lim_{n\to\infty} p_{\mathbb{A}}(Tx_n, Ty_n) \\
&\preceq \lim_{n\to\infty} \alpha_{\mathbb{A}}(x_n, y_n) p_{\mathbb{A}}(Tx_n, Ty_n) \\
&\preceq \lim_{n\to\infty} \phi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, Tx_n) + p_{\mathbb{A}}(y_n, Ty_n)) \\
&= \lim_{n\to\infty} \phi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n+1}) + p_{\mathbb{A}}(y_n, y_{n+1})) \\
&= \lim_{n\to\infty} \phi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n+1})) + \lim_{n\to\infty} \phi_{\mathbb{A}}(p_{\mathbb{A}}(y_n, y_{n+1})) \\
&\preceq \lim_{n\to\infty} \phi^{n+1}_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)) + \lim_{n\to\infty} \phi^{n+1}_{\mathbb{A}}(p_{\mathbb{A}}(y_0, y_1)) \\
&\to \theta_{\mathbb{A}} \text{ as } n \to \infty.
\end{aligned}$$

Hence, $x = y$. This completes the proof. ∎

Theorem 3 *Let $(X, \mathbb{A}, p_{\mathbb{A}})$ be a complete C^*-algebra valued partial metric space and $T : X \to X$ be a mapping satisfying*

$$\alpha_{\mathbb{A}}(x,y)p_{\mathbb{A}}(Tx,Ty) \preceq \psi_{\mathbb{A}}(p(x,y) + p_{\mathbb{A}}(x,Tx) + p_{\mathbb{A}}(y,Ty)) \text{ for all } x, y \in X; \tag{21}$$

where $\alpha_{\mathbb{A}} : X \times X \to \mathbb{A}^+$, $\psi_{\mathbb{A}} \in \Psi_{\mathbb{A}}$ and the following conditions hold:

(i) T is $\alpha_{\mathbb{A}}$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}}$;

(iii) T is continuous.

Then, T has a unique fixed point.

Proof. Similar to the proof of Theorem 1, we get

$$\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}} \text{ for all } n \in \mathbb{N}. \tag{22}$$

Using (21) and (22), we get

$$\begin{aligned} p_{\mathbb{A}}(x_n, x_{n+1}) &= p_{\mathbb{A}}(Tx_{n-1}, Tx_n) \\ &\preceq \alpha_{\mathbb{A}}(x_{n-1}, x_n)p_{\mathbb{A}}(Tx_{n-1}, Tx_n) \\ &\preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_{n-1}, x_n) + p_{\mathbb{A}}(Tx_{n-1}, x_{n-1}) + p_{\mathbb{A}}(Tx_n, x_n)) \\ &= \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_{n-1}, x_n) + p_{\mathbb{A}}(x_n, x_{n-1}) + p_{\mathbb{A}}(x_{n+1}, x_n)) \\ &= \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n-1}))2I_A + \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n+1})). \end{aligned} \tag{23}$$

From (23), we get

$$(1 - \psi_{\mathbb{A}})p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x_n, x_{n-1}))2I_{\mathbb{A}},$$

$$p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \psi_{\mathbb{A}}(1 - \psi_{\mathbb{A}})^{-1}2I_A p_{\mathbb{A}}(x_n, x_{n-1}).$$

From Lemma 1 and Definition 5, letting $\frac{\phi_{\mathbb{A}}}{2I_{\mathbb{A}}} = \psi_{\mathbb{A}}(1-\psi_{\mathbb{A}})^{-1}$, we get $p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \phi_{\mathbb{A}}(p_{\mathbb{A}}(x_{n-1}, x_n))$. Using mathematical induction, we get

$$p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \phi^n_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)) \; for \; all \; n \in \mathbb{N}. \tag{24}$$

For $m \geq 1$ and $n \geq 1$, follows from the similar calculations in Theorem 1 that

$$p_{\mathbb{A}}(x_m, x_{m+n}) \preceq \sum_{i=m}^{m+n-1} \phi^i_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i). \tag{25}$$

Using $\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x_n, x_n) \preceq p_{\mathbb{A}}(x_n, x_{n+1})$ and by (24), we get

$$\theta_{\mathbb{A}} \preceq p_{\mathbb{A}}(x_n, x_n) \preceq p_{\mathbb{A}}(x_n, x_{n+1}) \preceq \phi^n_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)). \tag{26}$$

Taking limit as $n \to \infty$ in (26) and Definition 5, we get

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x_n) = \theta_{\mathbb{A}}. \tag{27}$$

Using (27) in (25) and Definition 5, we get

$$\begin{aligned} p_{\mathbb{A}}(x_m, x_{m+n}) &\preceq \sum_{i=m}^{m+n-1} \phi^i_{\mathbb{A}}(p_{\mathbb{A}}(x_0, x_1)) - \sum_{i=m+1}^{m+n-1} p_{\mathbb{A}}(x_i, x_i) \\ &\to \theta_{\mathbb{A}} \; as \; n \to \infty. \end{aligned} \tag{28}$$

Thus, $\{x_n\}$ is a Cauchy sequence in complete C^*-algebra valued partial metric space $(X, \mathbb{A}, p_{\mathbb{A}})$. So there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Thus, there exist $x \in X$ such that

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x_m) = \lim_{n\to\infty} p_{\mathbb{A}}(x_n, x) = p_{\mathbb{A}}(x, x). \tag{29}$$

Using (28), we have

$$\lim_{n\to\infty} p_{\mathbb{A}}(x_n, x) = p_{\mathbb{A}}(x, x) = \theta_{\mathbb{A}}. \tag{30}$$

Now, we shall show that x is fixed point of T. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
p_{\mathbb{A}}(Tx, x) &= \lim_{n\to\infty} p_{\mathbb{A}}(Tx_n, x_n) \\
&= \lim_{n\to\infty} p_{\mathbb{A}}(Tx_n, Tx_{n-1}) \\
&\preceq \lim_{n\to\infty} \alpha_{\mathbb{A}}(x_n, x_{n-1}) p_{\mathbb{A}}(Tx_n, Tx_{n-1}) \\
&\preceq \lim_{n\to\infty} \psi_{\mathbb{A}}\big(p_{\mathbb{A}}(x_n, x_{n-1}) + p_{\mathbb{A}}(Tx_n, x_n) + p_{\mathbb{A}}(Tx_{n-1}, x_{n-1})\big) \\
&= \lim_{n\to\infty} \psi_{\mathbb{A}}\big(p_{\mathbb{A}}(x_n, x_{n-1}) + p_{\mathbb{A}}(x_{n+1}, x_n) + p_{\mathbb{A}}(x_n, x_{n-1})\big) \\
&= \psi_{\mathbb{A}}\big(p_{\mathbb{A}}(x, x) + p_{\mathbb{A}}(x, x) + p_{\mathbb{A}}(x, x)\big) \to \theta_{\mathbb{A}}.
\end{aligned}$$

Hence, $Tx = x$, i.e. x is a fixed point of T.

To prove the uniqueness of fixed point, let us assume that y is another fixed point of T. Consider,

$$\begin{aligned}
p_{\mathbb{A}}(x, y) &= p_{\mathbb{A}}(T^n x, T^n y) \\
&\preceq \alpha_{\mathbb{A}}(T^{n-1}x, T^{n-1}y) p_{\mathbb{A}}(T^n x, T^n y) \\
&\preceq \phi_{\mathbb{A}}(p_{\mathbb{A}}(T^{n-1}x, T^{n-1}y) + p_{\mathbb{A}}(T^{n-1}x, T^n x) + p_{\mathbb{A}}(T^{n-1}y, T^n y)) \\
&= \phi_{\mathbb{A}}^{n-1}(p_{\mathbb{A}}(x, y)) + \phi_{\mathbb{A}}^{n-1}(p_{\mathbb{A}}(x, x)) + \phi_{\mathbb{A}}^{n-1}(p_{\mathbb{A}}(y, y)) \\
&= \phi_{\mathbb{A}}^{n-1}(p_{\mathbb{A}}(x, y)) \to \theta_{\mathbb{A}} \textit{ as } n \to \infty.
\end{aligned}$$

Hence, $x = y$. This completes the proof. ■

By the above results, similarly one can also obtain the following corollaries mentioned below:

Corollary 2 *Let $(X, \mathbb{A}, p_{\mathbb{A}})$ be a complete C^*-algebra valued partial metric space and $T : X \to X$ a mapping satisfying*

$$\alpha_{\mathbb{A}}(x, y) p_{\mathbb{A}}(Tx, Ty) \preceq \psi_{\mathbb{A}}(p(x, y) + p_{\mathbb{A}}(x, Ty) + p_{\mathbb{A}}(y, Tx)) \ \textit{ for all } x, y \in X,$$

where $\alpha_{\mathbb{A}} : X \times X \to \mathbb{A}^+$, $\psi_{\mathbb{A}} \in \Psi_{\mathbb{A}}$ and the following conditions hold:

(i) T is $\alpha_{\mathbb{A}}$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}}$;

(iii) T is continuous.

Then, T has a unique fixed point.

Corollary 3 *Let $(X, \mathbb{A}, p_{\mathbb{A}})$ be a complete C^*-algebra valued partial metric space and $T : X \to X$ a mapping satisfying*

$$\begin{aligned}
\alpha_{\mathbb{A}}(x, y) p_{\mathbb{A}}(Tx, Ty) \ \preceq \ & \psi_{\mathbb{A}}(p(x, y) + p_{\mathbb{A}}(x, Tx) + p_{\mathbb{A}}(y, Ty) \\
& + p_{\mathbb{A}}(x, Ty) + p_{\mathbb{A}}(y, Tx)) \textit{ for all } x, y \in X;
\end{aligned}$$

where $\alpha_{\mathbb{A}} : X \times X \to \mathbb{A}^+$, $\psi_{\mathbb{A}} \in \Psi_{\mathbb{A}}$ and the following conditions hold:

(i) T is $\alpha_{\mathbb{A}}$-admissible;

(ii) *there exists* $x_0 \in X$ *such that* $\alpha_{\mathbb{A}}(x_0, Tx_0) \succeq I_{\mathbb{A}}$;

(iii) T *is continuous.*

Then, T *has a unique fixed point.*

Example 4 *As given in Example 1, define* $T : X \to X$ *by* $Tx = x/3$ *and* $\alpha_{\mathbb{A}} : X \times X \to \mathbb{A}$ *such that* $\alpha_{\mathbb{A}}(x, y) = 1$. *So* $\alpha_{\mathbb{A}}(Tx, Ty) = 1$ *implies that* T *is* $\alpha_{\mathbb{A}}$*-admissible. Define* $\psi_{\mathbb{A}} : \mathbb{C}^+ \to \mathbb{C}^+, \psi_{\mathbb{A}}(a) = 2a$. *Clearly,* T *is* $\alpha_{\mathbb{A}} - \psi_{\mathbb{A}}$*-contractive mapping and satisfies* $\alpha_{\mathbb{A}}(x, y)p_{\mathbb{A}}(Tx, Ty) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(x, y))$ *for all* $x, y \in X$.

Example 5 *As given in Example 3, define* $T : S(X) \to S(X)$ *by* $T(F(f_n)) = \{f_n : d(f_n, f) \leq \epsilon/3\}$ *and* $\alpha_{\mathbb{A}} : S(X) \times S(X) \to \mathbb{A}$ *such that* $\alpha_{\mathbb{A}}(f_n, f_m) = I_f$. *So* $\alpha_{\mathbb{A}}(Tf_n, Tf_m) = I_f$ *implies that* T *is* $\alpha_{\mathbb{A}}$*-admissible. Define* $\psi_{\mathbb{A}} : \mathbb{C}^+ \to \mathbb{C}^+, \psi_{\mathbb{A}}(a) = 5a$. *Clearly,* T *is* $\alpha_{\mathbb{A}} - \psi_{\mathbb{A}}$*-contractive mapping and satisfies*

$$\alpha_{\mathbb{A}}(f_n, f_m)p_{\mathbb{A}}(TF_{\epsilon_1}f_n, TF_{\epsilon_1}f_m) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(F_{\epsilon_1}f_n, F_{\epsilon_1}f_m)) \quad \text{for all } x, y \in X.$$

4 Application

In this section, we check the existence and uniqueness of a solution of an integral equation by using Theorem 1.

Theorem 4 *Consider an integral equation* $m(t) = \int_E F(t, m(s))ds + g(t)$, *where* $t, s \in E$, *a Lebesgue measurable set,* $F : E \times R \to R$ *and* $g \in L^\infty(E)$ *such that there exist a continuous function* $\phi : E \times E \to R$ *and* $\theta_{\mathbb{A}} \prec k \prec I_{\mathbb{A}}$ *satisfying*

(i) $|F(t, m(s)) - F(t, n(s))| < k|\phi(t, s)(m - n)|$ *for* $m, n \in R$;

(ii) $\sup_{t\in E} \int_E ||\phi(t, s)||ds < 1$.

Then, the integral equation has a unique solution, i.e., $m^* \in L^\infty(E)$.

Proof. Let $L^\infty(E)$ be the set of bounded measurable functions on E and $H = L^2(E)$ be a Hilbert space. The set of bounded linear operator $L(H)$ is C^*-algebra with the usual operator norm.

Define $p_{\mathbb{A}} : L^\infty(E) \times L^\infty(E) \to L(H)$ by $p_{\mathbb{A}}(m, n) = \pi_{|m-n|}$ where $\pi_h : H \to H$ is multiplication operator defined by $\pi_h(\phi) = h.\phi, \phi \in H$. Then, $(L^\infty(E), L(H), p_{\mathbb{A}})$ is a complete C^*-algebra partial valued metric space.

Let $T : L^\infty(E) \to L^\infty(E)$, $\alpha : L^\infty(E) \times L^\infty(E) \to L(H)$ and $\psi : L(H) \to L(H)$ be defined by

$$Tm(t) = \int_E F(t, m(s))ds + g(t), \quad \text{for } t \in E,$$

$$\alpha(m, n) = 1 \text{ and } \psi(m) = m,$$

respectively. The solution of an integral equation is fixed point. Consider

$$\begin{aligned}
\|p_{\mathbb{A}}(Tm,Tn)\| &= \sup_{\|h\|=1}(\pi_{|Tm-Tn|h},h)\\
&= \sup_{\|h\|=1}\int_E |Tm-Tn|h(t)\overline{h(t)}dt\\
&= \sup_{\|h\|=1}\int_E\int_E (F(t,m(s))-F(t,n(s)))dsh(t)\overline{h(t)}dt\\
&< \sup_{\|h\|=1}\int_E\int_E (F(t,m(s))-F(t,n(s)))ds\|h(t)\|^2dt\\
&< \sup_{\|h\|=1}\int_E\int_E k|\phi(t,s)(m(s)-n(s))|ds\|h(t)\|^2dt\\
&< \|k\|\|m-n\|_\infty \sup_{t\in E}\int_E \|\phi(t,s)\|ds\\
&= \|k\|.\|p_{\mathbb{A}}(m,n)\|.
\end{aligned}$$

It is given that $\theta_{\mathbb{A}} \prec k \prec I_{\mathbb{A}}$, therefore $\|k\| \leq I_{\mathbb{A}}$. Thus

$$\alpha_{\mathbb{A}}(m,n)p_{\mathbb{A}}(Tm,Tn) \leq \|k\|\psi_{\mathbb{A}}(p_{\mathbb{A}}(m,n)) \preceq \psi_{\mathbb{A}}(p_{\mathbb{A}}(m,n)).$$

Hence, T is an $\alpha_{\mathbb{A}} - \psi_{\mathbb{A}}$ type contractive mapping and satisfies all the conditions of Theorem 1. So T has a unique fixed point and the integral equation has a unique solution, i.e. $m^* \in L^\infty$. ■

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