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Controllability Radius For Infinite Dimensional Systems*

Amel Merghad†, Salah Eddine Rebiai‡

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Abstract

In this paper, we establish a formula for the exact controllability radius for a class of infinite dimensional systems.

1 Introduction

Let X and U be two complex Hilbert spaces. In this paper we consider the linear control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{if } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

where $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$. X is called the state space, U the control space and $u(.) \in L^2(0, T; U)$ the control function. The mild solution of (1) is given by

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds.$$

We will denote the system (1) by (A, B).

Definition 1 *The system (A, B) is called exactly controllable if for every $(x_0, x_1) \in X^2$, there exists a control $u(.) \in L^2(0, T; U)$ and a time $T > 0$ such that*

$$e^{TA}x_0 + \int_0^T e^{(T-s)A}Bu(s)ds = x_1.$$

Define the following bounded linear operator

$$\begin{aligned} [A, B] : X \times U &\longrightarrow X \\ (x, u) &\longmapsto Ax + Bu. \end{aligned}$$

Then, according to [4] the system (A, B) is exactly controllable if and only if for each $\lambda \in \mathbb{C}$ the linear operator $[A - \lambda I, B]$ is surjective.

Since the subset of all exactly controllable pairs (A, B) is open (see [6]), it is interesting to study the robustness of the exact controllability property. The exact controllability radius is defined as the smallest perturbation of (A, B) that makes the system uncontrollable, that is

$$r_{(A,B)} = \inf_{(\Delta_A, \Delta_B) \in \mathcal{L}(X) \times \mathcal{L}(U,X)} \{\|[\Delta_A, \Delta_B]\|, \ (A + \Delta_A, B + \Delta_B) \text{ is not exactly controllable}\}. \tag{2}$$

The problem of estimating (2) is of great importance in mathematical systems theory, and there have been several works in this direction over the last decades, see for example [8], [9], [1], [3] and the references therein. However the attention has mainly been devoted to this problem for finite-dimensional systems and very little is known for systems in infinite-dimensional spaces. Our main purpose in this paper is to derive a formula for the exact controllability radius for a class of infinite dimensional systems described by (1), and this will be done in Section 2. In Section one, we will recall for a later use some known results from the theory of linear multi-valued operators, for more details see [2, 9].

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†LTM, University of Batna 2, Batna, Algeria
‡LTM, University of Batna 2, Batna, Algeria

2 Preliminaries

Let X and Y be Hilbert spaces over the field $K = \mathbb{R}$ or $\mathbb{C}$. The notation $\mathcal{T} : X \rightrightarrows Y$ indicates that $\mathcal{T}$ is a set valued operator, that is, for each $x \in \mathcal{T}$ is a subset of Y. the inverse of $\mathcal{T}$ is the set-valued operator $\mathcal{T}^{-1} : Y \rightrightarrows X$ defined by $x \in \mathcal{T}^{-1} \Leftrightarrow y \in \mathcal{T}(x)$. The domain, range, the graph, and the kernel of $\mathcal{T}$ are defined, respectively by

$$D(\mathcal{T}) = \{x \in X : \mathcal{T}(x) \neq \emptyset\},$$
$$\operatorname{Im} \mathcal{T} = \cup_{x \in D(\mathcal{T})} \mathcal{T}(x),$$
$$Gr(\mathcal{T}) = \{(x, y) \in X \times Y : x \in D(\mathcal{T}), y \in \mathcal{T}(x)\},$$
$$\ker(\mathcal{T}) = \{x \in D(\mathcal{T}) : 0 \in \mathcal{T}(x)\}.$$

A multivalued operator $\mathcal{T}$ is called linear if for all $x, y \in D(\mathcal{T})$ and non zero scalars α we have

$$\mathcal{T}x + \mathcal{T}z = \mathcal{T}(x + z) \quad \text{and} \quad \alpha \mathcal{T}x = \mathcal{T}(\alpha x).$$

(Obviously the domain of a multivalued linear operator is a linear subspace). The norm of $\mathcal{T}$ is defined as follows

$$\|\mathcal{T}\| = \sup\left\{\inf_{y \in \mathcal{T}} \|y\| : x \in D(\mathcal{T}), \|x\| = 1\right\}.$$

It follows from the definition that

$$\inf_{y \in \mathcal{T}(x)} \|y\| \leq \|\mathcal{T}\| \|x\| \quad \text{for all } x \in D(\mathcal{T}).$$

We also assert that

$$\text{if } y \in \mathcal{T}(x) \text{ and } y \in (\mathcal{T}(0))^{\perp}, \text{ then } d(0, \mathcal{T}(x)) = \inf_{z \in \mathcal{T}(x)} \|z\| = \|y\|.$$

Indeed, if $y \in \mathcal{T}(x)$, then $\mathcal{T}(x) = y = \mathcal{T}(0)$. Let $z \in \mathcal{T}(x)$. Then there exists $w \in \mathcal{T}(0)$ such that $z = y + w$ and

$$\begin{aligned} d(0, \mathcal{T}(x)) &= \inf_{z \in \mathcal{T}(x)} \|z\| = \inf_{w \in \mathcal{T}(0)} \|y + w\| = \inf_{w \in \mathcal{T}(0)} \left[\|y\|^2 + \|w\|^2\right]^{\frac{1}{2}} \\ &= \|y\| \qquad (0 \in \mathcal{T}(0)). \end{aligned}$$

$$(\mathcal{T}^*)^* = \mathcal{T}, (\mathcal{T}^*)^{-1} = (\mathcal{T}^{-1})^*, \|\mathcal{T}^*\| = \|\mathcal{T}\|. \tag{3}$$

Lemma 1 ([5]) *Let X and Y be Banach spaces. If $\phi : X \to Y$ is a bounded linear operator and surjective, then*

$$\inf\{\|P\| : P \in \mathcal{L}(X), \quad \phi + P \quad \textit{is not surjective}\} = \|\phi^{-1}\|^{-1}, \tag{4}$$

where ϕ^{-1} is a linear multivalued operator.

Now, we follow the approach adopted by Son and Thuan [8] to prove that $\|A^{-1}\| = \|A^{\dagger}\|$ (where $A^{\dagger}$ is the pseudo-inverse of A) if A is a surjective bounded linear operator in a Hilbert space.

Lemma 2 *Let $A : X \to Y$ be a surjective bounded linear operator where X and Y are Hilbert spaces. Then $\|A^{-1}\| = \|A^{\dagger}\|$.*

Proof. Since A is surjective, we see that AA^* is invertible and we have

$$A^{\dagger} = A^*(AA^*)^{-1}.$$

Let $u = A^{\dagger}(y)$ for $y \in Y$. Then $Au = AA^{\dagger}y = (AA^*)(AA^*)^{-1}y = y$. Therefore $u \in A^{-1}(y)$. It follows that $A^{-1}(y) = u + A^{-1}(0)$. An easy computation shows that $u \in (A^{-1}(0))^{\perp}$. Since $u \in A^{-1}(y)$ and $u \in (A^{-1}(0))^{\perp}$, we conclude that $d(0, A^{-1}(y)) = \|u\| = \|A^{\dagger}(y)\|$. By definition, then

$$\|A^{\dagger}\| = \sup_{\|y\|=1} \|A^{\dagger}(y)\|.$$

■

3 Main Result

Theorem 1 *Assume that the system (A, B) is exactly contollable. Then*

$$r_{(A,B)} = \frac{1}{\sup_{\lambda\in\mathbb{C}} \|[A-\lambda I, B]^{\dagger}\|}.$$

Proof. If the system (A, B) is exactly controllable, then

$$[A-\lambda I, B]U = X, \quad \forall\lambda \in \mathbb{C}.$$

Assume that the perturbed control system is not exactly controllable for some $[\Delta_A^0, \Delta_B^0]$. Then there exists

$$\lambda_0 \in \mathbb{C}$$

such that

$$[A+\Delta_A^0 - \lambda_0 I, B+\Delta_B^0] = [A-\lambda_0 I, B] + [\Delta_A^0, \Delta_B^0]$$

is not surjective. So by (4) we have

$$\begin{aligned}\frac{1}{\|[A-\lambda_0 I, B]^{-1}\|} &= \inf\{\|[\Delta_A, \Delta_B]\|, [A-\lambda_0 I, B] + [\Delta_A, \Delta_B]\text{is not surjective}\}\\ &\geq \inf\{\|[\Delta_A, \Delta_B]\|, (A+\Delta_A, B+\Delta_B]\text{is not exactly contollable }\}\\ &\geq r_{(A,B)}.\end{aligned}$$

It follows that

$$\frac{1}{\sup_{\lambda\in\mathbb{C}} \|[A-\lambda I, B]^{-1}\|} \geq r_{(A,B)}.$$

To prove the converse, we first note that for any operator $\Delta \in \mathcal{L}(X\times U, X)$, there exists $\Delta_1 \in \mathcal{L}(X)$ and $\Delta_2 \in \mathcal{L}(U, X)$ such that $\Delta = [\Delta_1, \Delta_2]$.

For any small $\epsilon > 0$, we have

$$\sup_{\lambda\in\mathbb{C}} \|[A-\lambda I, B]^{-1}\| - 2\epsilon > 0.$$

Then there exists $\lambda_\epsilon \in \mathbb{C}$ such that

$$\|[A-\lambda_\epsilon I, B]^{*-1}\| = \|[A-\lambda_\epsilon I, B]^{-1}\| > \sup_{\lambda\in\mathbb{C}} \|[A-\lambda I, B]^{-1}\| - \epsilon.$$

Since $[A-\lambda_\epsilon I, B]^{*-1}$ is single-valued (because $[A-\lambda_\epsilon I, B]$ is surjective) its norm is the operator norm and thus there exists $(x_\epsilon, u_\epsilon) \in X\times U$ with $\|(x_\epsilon, u_\epsilon)\|_{X\times U} = 1$ and

$$\|[A-\lambda_\epsilon I, B]^{*-1}(x_\epsilon, u_\epsilon)\| > \sup_{\lambda\in\mathbb{C}} \|[A-\lambda I, B]^{*-1}\| - 2\epsilon.$$

Let $x_\epsilon^* = -[A-\lambda_\epsilon I, B]^{-1*}(x_\epsilon, u_\epsilon)$. Then $[A-\lambda_\epsilon I, B]^*(x_\epsilon^*) = -(x_\epsilon, u_\epsilon)$. By the Hahn-Banach theorem, there exists $z_\epsilon \in X$ such that $\|z_\epsilon\| = 1$, $\langle z_\epsilon, x_\epsilon^*\rangle = \|x_\epsilon^*\|$, by setting

$$\Delta_\epsilon(x, u) = \frac{1}{\|x_\epsilon^*\|^2}\langle (x, u), (x_\epsilon, u_\epsilon)\rangle x_\epsilon^*,$$

it is clear that Δ_ϵ is a bounded linear map with norm

$$\|\Delta_\epsilon\| = \frac{1}{\|x_\epsilon^*\|} = \frac{1}{\|[A-\lambda_\epsilon I, B]^{-1*}(x_\epsilon, u_\epsilon)\|}.$$

On the other hand,

$$[A-\lambda_\epsilon I, B]^*(x_\epsilon^*) + \Delta_\epsilon^*(x_\epsilon^*) = 0$$

or equivalently $[A-\lambda_\epsilon I, B]+\Delta_\epsilon$ is not surjective. It follows that the perturbed system $(A,B)+\Delta_\epsilon$ is not exactly controllable. Thus by definition

$$r_{(A,B)} \leq \|\Delta_\epsilon\| < \frac{1}{\sup_{\lambda\in\mathbb{C}} \|[A-\lambda I, B]^{-1}\| - 2\epsilon}.$$

By letting $\epsilon \to 0$ we obtain the converse inequality. The proof is finished. ■

Remark 1 (Extension to fractional systems) *From a combination of the theorem in [4] page 537 and Theorem 2.1 in [7], we can show in the same way that this result remains valid for time fractional systems described by*

$$\begin{cases} {}^cD^\alpha_{0\ t} = Ax(t) + Bu(t) & \text{if } t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $\frac{1}{2} < \alpha < 1$, $A : X \to X$, $B : U \to X$ *are bounded linear operators, and* $u \in L^2(0,T;U)$.

Example 1 *It is proved in [10] that the system* (A,B) *defined by*

$$(Bf)(x) = \begin{cases} f(x), & \frac{1}{2} \leq x \leq 1, \\ 0, & 0 \leq x < \frac{1}{2}, \end{cases} \qquad \text{and} \quad (Af)(x) = \begin{cases} 0, & \frac{1}{2} \leq x \leq 1, \\ f(1-x), & 0 \leq x < \frac{1}{2}, \end{cases}$$

where $f \in X = L_2(0,1)$ *is exactely controllable on* X *with control space* $U = X$*. Then*

$$[A-\lambda I, B]^\dagger f = \begin{pmatrix} H^1_\lambda f \\ H^2_\lambda f \end{pmatrix}$$

where

$$H^1_\lambda f(x) = \frac{1+|\lambda|^2}{|\lambda|^4+|\lambda|^2+1} \begin{cases} (\frac{\lambda}{1+|\lambda|^2} - \bar\lambda) f(x) + (1 - \frac{|\lambda|^2}{1+|\lambda|^2}) f(1-x), & \frac{1}{2} \leq x \leq 1, \\ -\bar\lambda f(x) - \frac{\bar\lambda^2}{1+|\lambda|^2} f(1-x), & 0 \leq x < \frac{1}{2}, \end{cases}$$

$$H^2_\lambda f(x) = \frac{1+|\lambda|^2}{|\lambda|^4+|\lambda|^2+1} \begin{cases} f(x) + \frac{\lambda}{1+|\lambda|^2} f(1-x), & \frac{1}{2} \leq x \leq 1, \\ 0, & 0 \leq x < \frac{1}{2}, \end{cases}$$

$$r_{(A,B)} \leq \frac{1}{\sup_{\lambda\in\mathbb{C}} \|H^2_\lambda\|} = \frac{1}{\sup_{\lambda\in\mathbb{C}} \frac{1+|\lambda|^2}{|\lambda|^4+|\lambda|^2+1} \sqrt{1 + [\frac{|\lambda|}{1+|\lambda|^2}]^2}} \approx \frac{1}{1,035}.$$

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