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A New Characterization For Finding A Common Zero Of Set-Valued Accretive Operators In Banach Spaces*

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Abstract

In this paper, we obtain a new characterization for finding a common zero of set-valued accretive operators in Banach spaces by using properties of resolvent composition operators. We next present convergence theorem to solution of system of equilibrium problems in Banach spaces. Our technique of proof is of independent interest.

1 Introduction

Let E be a normed linear space. For a multivalued map $A : E \to 2^E$, the domain of A, $D(A)$, the image of a subset S of E, $A(S)$, the range of $A, R(A)$ and the graph of $A, G(A)$ are defined as follows:

$$D(A) := \{x \in H : Ax \neq \emptyset\},\ A(S) := \cup\{Ax : x \in S\},$$

$$R(A) := A(H),\ G(A) := \{(x, u) :\ x \in D(A),\ u \in Ax\}.$$

Let $\langle ., . \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}$$

for all $x \in E$. In sequel, we use j to denote the single-valued normalized duality mapping. An operator A is said to be accretive if for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0$$

for all $u \in Ax$ and $v \in Ay$. An accretive operator A is said to be maximal if there is no proper accretive extension of A, and m-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$, where I is the identity operator on E. If A is m-accretive, then it is maximal accretive, but the reverse is not true. For an accretive operator A, we can define for each $\lambda > 0$, a single-valued mapping $J^A_\lambda : R(I + \lambda A) \to D(A)$ by

$$J^A_\lambda = (I + \lambda A)^{-1}.$$

It is called the resolvent of A. An accretive operator A defined on a Banach space E is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$. We know that for an accretive operator A which satisfies the range condition, $A^{-1}0 = Fix(J^A_\lambda)$ for all $\lambda > 0$, where, $Fix(J^A_\lambda)$ the set of fixed points of the mapping J^A_λ, that is $Fix(J^A_\lambda) := \{x \in E : J^A_\lambda x = x\}$. For E a real Banach space, a fundamental problem is that of finding an element $u \in A^{-1}0$, where $A : E \to 2^E$ is a multivalued map defined on E. This problem has been investigated by many researchers, see for instance, Brézis and Lions [10], Martinet [9], Minty [12], Reich [11], Rockafellar [6], Takahashi and Ueda [8], and others. Such a problem is connected with the convex minimization problem.

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Now, we consider the following problem: Find an element

$$x^* \in S = A^{-1}0 \cap B^{-1}0, \tag{1}$$

where $A : D(A) \to 2^E$ and $B : D(B) \to 2^E$ are two accretive operators.

Most existing results for solving (1) require that the resolvents of underlying operators must be commuting and also, the intersection of the fixed point sets $Fix(J_\lambda^A) \cap Fix(J_\lambda^B)$ must be nonempty. Above discussion suggests the following questions.

Question 1 *Is it always true that $Fix(J_\lambda^A) \cap Fix(J_\lambda^B) = Fix(J_\lambda^A \circ J_\lambda^B)$ without commuting assumptions?*

Question 2 *Can we use our results and a modified Mann algorithm such that it converges strongly to a solution of system of equilibrium problems in real Banach spaces without compactness assumption?*

The purpose of this paper is to give affirmative answers to these questions mentioned above.

2 Preliminaries

A normed linear space E is said to be strictly convex if the following holds:

$$\|x\| = \|y\| = 1, \ x \neq y \ \Rightarrow \ \left\|\frac{x+y}{2}\right\| < 1.$$

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by:

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{1}{2}\|x + y\| \ : \|x\| = \|y\| = 1, \ \|x - y\| \geq \epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth,* if the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J.

Let E be a smooth real Banach space with dual space E. We introduce the Lyapunov functional $\phi : E \times E \to \mathbb{R}$, defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \ \ \forall\, x, y \in E. \tag{2}$$

It was introduced by Alber in [1] and has been studied by Alber and Guerre-Delabriere [2], Kamimura and Takahashi [4] and a host of other authors. Note that if $E = H$, a real Hilbert space, then the normalized duality map J is the identity map. Hence, equation (2) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$.

In the sequel, the following result will be useful.

Lemma 1 ([1]) *For $p > 1$, let E be a p-uniformly convex real Banach space and let S be a bounded subset of E. Then, there exists $\alpha > 0$ such that*

$$\alpha\|x - y\|^p \leq \phi(x, y) \ \ \forall\, x, y \in S.$$

The following definition contains the nonlinear mappings we are working with and that will appear throughout the entire paper.

Definition 1 *Let E be a smooth real Banach space and $T : (T) \subset E \to E$, then T is said to be*

(1) *a contraction if there exists $b \in [0,1)$ such that*

$$\|Tx - Ty\| \leq b\|x - y\|, \;\; x, y \in D(T),$$

where if $b = 1$, T is also called nonexpansive;

(2) *quasi-nonexpansive if $Fix(T) \neq \emptyset$ and*

$$\|Tx - p\| \leq \|x - p\|, \;\; x \in D(T), \;\; p \in Fix(T);$$

(3) *firmly nonexpansive if for all $x, y \in D(T)$, we have*

$$\|Tx - Ty\|^2 \leq \langle x - y, j(Tx - Ty)\rangle.$$

The resolvent operator has the following properties:

Lemma 2 (**[13]**) *For any $r > 0$,*

(i) A is accretive if and only if the resolvent J_r^A of A is single-valued and firmly nonexpansive;

(ii) A is m-accretive if and only if J_r^A of A is single-valued and firmly nonexpansive and its domain is the entire E;

(iii) $0 \in A(x^*)$ if and only if $x^* \in F(J_r^A)$, where $F(J_r^A)$ denotes the fixed-point set of J_r^A.

3 Main Results

We now prove the following result.

Theorem 1 *For $p > 1$, let E be a p-uniformly convex smooth real Banach space and K be a closed, bounded, convex set in E. Let $A : D(A) \subset K \to 2^E$ and $B : D(B) \subset K \to 2^E$ be accretive operators such that*

$$\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I + rA) \quad \text{and} \quad \overline{D(B)} \subset K \subset \bigcap_{r>0} R(I + rB).$$

Then, $A^{-1}0 \cap B^{-1}0 = Fix(J_r^A \circ J_r^B)$ and $J_r^A \circ J_r^B$ is a quasi-nonexpansive mapping on K.

Proof. We split the proof into two steps.

Step 1: First, we show that $Fix(J_r^A) \cap Fix(J_r^B) = Fix(J_r^A \circ J_r^B)$. We note that $Fix(J_r^A) \cap Fix(J_r^B) \subseteq Fix(J_r^A \circ J_r^B)$. Thus, we only need to show that $Fix(J_r^A \circ J_r^B) \subseteq Fix(J_r^A) \cap Fix(J_r^B)$. Let $p \in Fix(J_r^A) \cap Fix(J_r^B)$ and $q \in Fix(J_r^A \circ J_r^B)$. By using properties of J_r^A and J_r^B, we have

$$\|q - p\|^2 = \|J_r^A \circ J_r^B q - J_r^A p\|^2 \leq \|J_r^B q - p\|^2. \tag{3}$$

Using the fact that J_r^B is firmly nonexpansive, we have

$$\|J_r^B q - p\|^2 \leq \langle q - p, j(J_r^B q - p)\rangle. \tag{4}$$

Furthermore, using properties of Lyapunov function, we have

$$\phi(q - p, J_r^B q - p) = \|q - p\|^2 - 2\langle q - p, j(J_r^B q - p)\rangle + \|J_r^B q - p\|^2.$$

Hence,

$$\langle q-p, j(J_r^B q-p)\rangle = \frac{1}{2}\Big(\|q-p\|^2 + \|J_r^B q-p\|^2 - \phi(q-p, J_r^B q-p)\Big). \tag{5}$$

Using (4) and (5), we obtain

$$\|J_r^B q-p\|^2 \le \|q-p\|^2 - \phi(q-p, J_r^B q-p). \tag{6}$$

From (3), we have

$$\phi(q-p, J_r^B q-p) \le 0.$$

By Lemma 1, we have $\|J_r^B q-q\| = 0$ which implies that

$$q = J_r^B q.$$

Keeping in mind that $J_r^A \circ J_r^B q = q$, we have

$$q = J_r^A \circ J_r^B q = J_r^A q.$$

Thus, $q \in Fix(J_r^A) \cap Fix(J_r^B)$. Hence, $Fix(J_r^A) \cap Fix(J_r^B) = Fix(J_r^A \circ J_r^B)$. Since $A^{-1}0 \cap B^{-1}0 = Fix(J_r^A) \cap Fix(J_r^B)$, we come to the conclusion that

$$A^{-1}0 \cap B^{-1}0 = Fix(J_r^A \circ J_r^B).$$

Step 2: We show $J_r^A \circ J_r^B$ is a quasi-nonexpansive mapping on K. Let $x \in K$ and $p \in Fix(J_r^A \circ J_r^B)$. Then, $p \in Fix(J_r^A) \cap Fix(J_r^B)$ by step 1. We observe that,

$$\|J_r^A \circ J_r^B x - p\| = \|J_r^A \circ J_r^B x - J_r^A p\| \le \|J_r^B x - p\| \le \|x-p\|.$$

This completes the proof. ■

4 Application to System of Equilibrium Problems

In this section, we apply our main results and a modified Mann algorithm for approximating a common solution of system of equilibrium problems in real Banach space. Let E be a real Banach space and K be nonempty closed and bounded convex subset of E.

Let $F : K \times K \to \mathbb{R}$ be an equilibrium bifunction. The equilibrium problem is to find $x \in K$ such that

$$F(x,y) \ge 0, \tag{7}$$

for all $y \in K$. We shall denote the set of solutions of this equilibrium problem by $EP(F)$. Thus

$$EP(F) := \Big\{x^* \in K : \ F(x^*, y) \ge 0, \ \forall\, y \in K\Big\}.$$

The equilibrium problem include fixed point problems, optimization problems and variational inequality problems as special cases (see, for example, [3]). Some methods have been proposed to solve the equilibrium problem, see for example, [7]. For solving the equilibrium problem we assume that the bifunction F satisfies the following conditions:

$(A1)$ $F(x,x) = 0$ for all $x \in K$;

$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for all $x, y \in K$;

$(A3)$ for each $x, y, z \in K$,

$$\lim_{t\to 0} F(tz + (1-t)x, y) \le F(x,y);$$

$(A4)$ for each $x \in K, \quad y \to F(x,y)$ is convex and lower semicontinuous.

For solving (7), many authors introduce the following lemma.

Lemma 3 ([16]) *Let K be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : K \times K \to \mathbb{R}$ satisfies $(A1)$–$(A4)$. For $r > 0$ and $x \in E$, define a mapping ${T_r}^F : E \to K$ as follows:*

$$T_r^F(x) = \{z \in K,\ F(z,y) + \frac{1}{r}\langle y - z, jz - jx\rangle \geq 0, \quad \forall y \in K\},$$

for all $z \in E$. Then, the following hold:

1. *T_r^F is single-valued;*
2. *T_r^F is firmly nonexpansive;*
3. *$Fix(T_r^F) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Recently, Sow [15] motivated by the fact that Mann algorithm method is remarkably useful for finding fixed points of nonexpansive mapping, proved the following theorem.

Theorem 2 *Let E be a uniformly convex real Banach space having a weakly continuous duality map J_ϕ and K be a nonempty, closed and convex cone of E. Let $T : K \to K$ be a quasi-nonexpansive mapping such that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n T x_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)T\big(s_n x_n + (1 - s_n)\bar{x}_{n+1}\big), \end{cases}$$

where $\{\beta_n\}, \{\lambda_n\}$, $\{s_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\lim_{n\to\infty} \inf(1 - s_n)\beta_n(1 - \beta_n) > 0$,

(iii) $\lim_{n\to\infty} \lambda_n = 1$,

(iv) $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.

Assume that $I - T$ is demiclosed at the origin. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(T)$.

We now prove the following result.

Corollary 1 *For $p > 1$, let E be a p-uniformly convex smooth real Banach space having a weakly continuous duality map J_ϕ and K be a closed, bounded, convex cone set in E. Let $F : K \times K \to \mathbb{R}$ and $G : K \times K \to \mathbb{R}$ be bifunctions satisfying $(A1)$-$(A4)$ such that $Fix({T_r}^F \circ {T_r}^G) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n){T_r}^F \circ {T_r}^G x_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n){T_r}^F \circ {T_r}^G\big(s_n x_n + (1 - s_n)\bar{x}_{n+1}\big), \end{cases}$$

where $\{\beta_n\}, \{\lambda_n\}$, $\{s_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\lim_{n\to\infty} \inf(1 - s_n)\beta_n(1 - \beta_n) > 0$,

(iii) $\lim_{n\to\infty} \lambda_n = 1$,

(iv) $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

Assume that $I - {T_r}^F \circ {T_r}^G$ *is demiclosed at the origin. Then, the sequence* $\{x_n\}$ *converges strongly to* $x^* \in EP(F) \cap EP(G)$.

Proof. By a similar argument as in Theorem 1, we can show that $Fix({T_r}^F \circ {T_r}^G) = EP(F) \cap EP(G)$ and ${T_r}^F \circ {T_r}^G$ is quasi-nonexpansive mapping. Then, the proof follows Theorem 2. ■

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