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Numerical Radius Of The Powers Of Jordan Block And Its Application For Eigenvalue Of Nonnegative Symmetric Toeplitz Matrices*

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Abstract

In this paper, we present a formula for the numerical radius of the powers of a Jordan block. This formula gives us an analytic and simple upper bound for the maximum eigenvalues of the nonnegative symmetric Toeplitz matrices. Numerical examples are provided to evaluate the accuracy level of the obtained upper bound in comparison with some existing bounds.

1 Introduction

A matrix is said to be Toeplitz if its entries are the same along each diagonal. Let $f : [-\pi, \pi] \to \mathbb{C}$ be a function belonging to $L^1([-\pi, \pi])$. The $n \times n$ Toeplitz matrix $T_n(f)$ generated by the function f is defined by $T_n(f) = [a_{i-j}]_{i,j=1}^n$, where a_k is the kth Fourier coefficient of f,

$$a_k := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ki\theta}d\theta, \quad k = 0, \pm 1, \pm 2, \ldots .$$

These matrices appear in a wide range of applications, mainly among them signal processing (e.g. see [3]). When f is real, the matrices $T_n(f)$ are Hermitian and much is known about their spectral properties. The eigenvalue problem of these matrices is studied extensively in the literature. Results on the individual asymptotic formulas for eigenvalues of Hermitian Toeplitz matrices were obtained e.g. in [2], [4] and [13]. Also, many papers give explicit formulas for the eigenvalues of such matrices in terms of the roots of some special functions, see e.g. [6], [11]. However, while these methods are efficient from the numerical point of view and can be implemented in efficient calculational algorithms, they require computing the zeros of those functions which implies that the results can not be applied to analytic studies such as convergence analyses, directly.

The main purpose of the present paper is to give an *analytic and simple* upper bound for the maximum eigenvalues of symmetric Toeplitz matrices with nonnegative entries (NNST matrices). We believe that despite the existence of algorithms and formulas for computing the eigenvalues of such matrices, our formula would be helpful in related analytical studies. Our method is based on using numerical range and numerical radius. We note that our upper bound can be computed explicitly and we do not propose any new algorithm for computing the eigenvalues. There are also some papers in the literature that give efficient numerical algorithms to compute the extreme eigenvalues of such matrices (see e.g. [10] and references therein).

Assume that f is a real cosine trigonometric polynomial:

$$f(\theta) = a_0 + 2\sum_{k=1}^{m} a_k \cos(k\theta), \quad a_k \geq 0, \ k = 1, \ldots, m.$$

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In this case, the nth Toeplitz matrix generated by f is the $n \times n$ real symmetric banded matrix with the nonnegative parameters $a_1, \dots a_m$, given by

$$T := T_n(f) = \begin{pmatrix} a_0 & a_1 & \dots & a_m & & & \\ a_1 & a_0 & a_1 & \dots & a_m & & \\ \vdots & & \ddots & \ddots & & \ddots & \\ a_m & & & & & & \ddots \\ & \ddots & & & & \ddots & a_m \\ & & & \ddots & & & \vdots \\ & & & & a_m & \dots & a_0 \end{pmatrix}. \quad (1)$$

Recall that (e.g., [9]) the *numerical range* and *numerical radius* of a matrix $A \in \mathbb{C}^{n\times n}$ are defined, respectively, by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\},$$
$$w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, \|x\| = 1\},$$

where by $\|.\|$ we mean the usual $2-$norm in $\mathbb{C}^n$. For a matrix $A \in \mathbb{C}^{n\times n}$ the usual operator norm of A which is induced by the vector norm $\|.\|$ is denoted by $\|A\|$. It is well-known that $w(.)$ is a norm on $\mathbb{C}^{n\times n}$ and for any $A \in \mathbb{C}^{n\times n}$

$$\rho(A) \le w(A) \le \|A\|.$$

The numerical range of the Jordan block $J_n(\lambda)$, the $n \times n$ bidiagonal upper triangular Toeplitz matrix with λ on its main diagonal and 1 on its super-diagonal, is a closed circular disk with the center at λ and the radius $r = \cos(\frac{\pi}{n+1})$, that is $W(J_n(\lambda)) = \mathcal{D}(\lambda, \cos(\frac{\pi}{n+1}))$, [9]. Throughout the paper, we use J_n instead of $J_n(0)$. Also, $[x]$ represents the greatest integer less than or equal to the real number x.

The paper is organized as follows. In Section 2, a formula is given for computing the numerical radius of the powers of the matrix J_n. Section 3 includes the main result of the paper in which we give an upper bound for the maximum eigenvalue of the NNST matrices. Finally, in Section 4 some examples are provided to compare the level of accuracy of the given upper bound with the existing ones in the literature.

2 Numerical Range of J_n^k

In this section, using an orthogonal transformation, we determine the Jordan canonical form of the matrix J_n^k. Actually, for any $2 \le k \le n-1$, we use a permutation to gather Jordan sub-blocks of J_n^k. This permutation is determined in terms of the remainder of the division of $n-1$ by k. Note that, for any $k \ge n$, $J_n^k = 0$, and for any $1 \le k \le n-1$, the matrix J_n^k is an upper triangular matrix with 1's on its k^{th} super-diagonal and 0's elsewhere.

Lemma 1 *Let $n \in \mathbb{N}$. Then for any $1 \le k \le n-1$, the matrix J_n^k is orthogonally similar to the matrix*

$$\underbrace{J_{m_1} \oplus \dots \oplus J_{m_1}}_{r_1 times} \oplus \underbrace{J_{m_2} \oplus \dots \oplus J_{m_2}}_{r_2 times},$$

where $m_1 = [\frac{n-1}{k}] + 1$, $m_2 = [\frac{n-1}{k}]$, $r_1 = ((n-1) \mod k) + 1$ and $r_2 = k - r_1$.

Proof. For any $1 \le i \le n$, let $e_{\{i\}}$ be the ith column of the identity matrix I_n. Consider the permutation matrix,

$$P = [e_{\{1\}}e_{\{k+1\}} \dots e_{\{kn_1+1\}}e_{\{2\}}e_{\{k+2\}} \dots e_{\{kn_2+2\}} \dots e_{\{k\}}e_{\{k+k\}} \dots e_{\{kn_k+k\}}].$$

Then, one can check that $P^TP = I_n$ and $P^TJ_n^kP = J_{n_1+1} \oplus J_{n_2+1} \oplus \dots \oplus J_{n_k+1}$, where $n_i + 1$ is the number of columns of the matrix $[e_{\{i\}}e_{\{k+i\}} \dots e_{\{kn_i+i\}}]$ and we have $n_i = [\frac{n-i}{k}]$, $i = 1, \dots, k$. Since

$$\left[\frac{n-1}{k}\right] = \dots = \left[\frac{(n-1) - ((n-1) \mod k)}{k}\right] = \left[\frac{n-r_1}{k}\right],$$

we get $n_1 = \ldots = n_{r_1}$. Also, since

$$\left[\frac{n-(r_1+1)}{k}\right] = \left[\frac{(n-1)-r_1}{k}\right] = \ldots = \left[\frac{(n-1)-(k-1)}{k}\right] = \left[\frac{n-k}{k}\right],$$

we get $n_{r_1+1} = \ldots = n_k$. Therefore the result holds. ■

Example 1 *The matrix J_{16}^3 is orthogonally similar to the matrix $J_6 \oplus J_5 \oplus J_5$ and the matrix J_{16}^{14} to $J_2 \oplus J_2 \oplus \mathbf{0}_{12}$. Also, the matrix J_{21}^8 is orthogonally similar to the matrix $J_3 \oplus J_3 \oplus J_3 \oplus J_3 \oplus J_3 \oplus J_2 \oplus J_2 \oplus J_2$.*

Now, we state our main result in this section.

Theorem 1 *Let $n \in \mathbb{N}$. Then $W(J_n^k) = \mathcal{D}(0, r_{n,k})$, $k = 1, 2, \ldots, n$, where $r_{n,k} = \cos(\frac{\pi}{[\frac{n-1}{k}]+2})$.*

Proof. For any two matrices A and B, $W(A \oplus B) = \mathrm{Conv}(W(A) \cup W(B))$ [9]. Since $m_1 > m_2$, we get $W(J_{m_1}) \supseteq W(J_{m_2})$. Therefore

$$W(J_n^k) = W(J_{m_1}) = \mathcal{D}(0, \cos(\frac{\pi}{m_1+1})) = \mathcal{D}(0, \cos(\frac{\pi}{[\frac{n-1}{k}]+2})).$$

■

Example 2 *For $n = 21$, we have $r_{21,1} = \cos(\frac{\pi}{22})$, $r_{21,2} = \cos(\frac{\pi}{12})$, $r_{21,3} = \cos(\frac{\pi}{8})$, $r_{21,4} = \cos(\frac{\pi}{7})$, $r_{21,5} = \cos(\frac{\pi}{6})$, $r_{21,6} = \cos(\frac{\pi}{5})$, $r_{21,j} = \cos(\frac{\pi}{4})$, $j = 7, 8, 9, 10$ and $r_{21,j} = \cos(\frac{\pi}{3})$, $j = 11, \ldots, 20$. In Figure 1, we plot the sets $W(J_{21}^k), k = 1, 2, \ldots, 20$.*

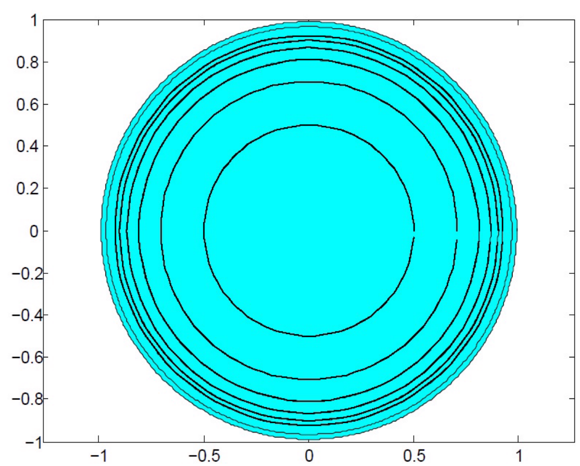

Figure 1: The sets $W(J_{21}^k)$, $k = 1, 2, \ldots, 20$.

Remark 1 *Using Theorem 1, one can obtain a circular disk as an inclusion region for the numerical range of the matrix $J_n(\lambda)^k$:*

$$W(J_n(\lambda)^k) = W\left(\sum_{j=0}^{m} \binom{k}{j} \lambda^{k-j} J_n^j\right) \subseteq \mathcal{D}\left(\lambda^k, \sum_{j=1}^{m} \binom{k}{j} \lambda^{k-j} \cos(\frac{\pi}{[\frac{n-1}{j}]+2})\right),$$

where $m = \min\{k, n\}$.

3 Upper Bound for the Maximum Eigenvalue of NNST Matrices

In this section, we give an upper bound for the maximum eigenvalue of NNST matrices. Let T be an $n \times n$ NNST matrix with bandwidth m and parameters $a_0 = 0$ and $a_1, \ldots, a_m$, where $a_k \geq 0, k = 1, \ldots, m$. The Perron-Frobenius theorem [8] asserts that the matrix T has a maximum nonnegative eigenvalue $\lambda_{\max}(T)$ with a corresponding eigenvector whose components are also nonnegative. Here, $\lambda_{\max}(T)$ will be greater than or equal, in absolute value, to all other eigenvalues of T, hence $\lambda_{\max}(T) = \rho(T)$. A rough upper bound for $\lambda_{\max}(T)$ can be obtained in the following way:

$$\begin{aligned}\lambda_{\max}(T) \leq \|T\| &= \left\| \sum_{k=1}^{m} a_k \left((J_n)^k + (J_n^T)^k \right) \right\| \\ &\leq \sum_{k=1}^{m} a_k \left(\left\|(J_n^T)^k\right\| + \left\|(J_n)^k\right\| \right) = 2\sum_{k=1}^{m} a_k,\end{aligned}$$

where the last equality holds since $\left\|(J_n^T)^k\right\| = \left\|(J_n)^k\right\| = 1$, for any $1 \leq k \leq n-1$. Thus, for any real parameter a_0 we get

$$\lambda_{\max}(T) \leq a_0 + 2\sum_{k=1}^{m} a_k. \tag{2}$$

In the following, we give an upper bound for $\lambda_{\max}(T)$ which is smaller than the given one in (2).

Theorem 2 *Let T be an $n \times n$ NNST matrix with bandwidth m and parameters $a_0, a_1, \ldots, a_m$, where $a_k \geq 0$, $k = 1, \ldots, m$. Then*

$$\lambda_{\max}(T) \leq a_0 + 2\sum_{k=1}^{m} a_k \cos\left(\frac{\pi}{[\frac{n-1}{k}] + 2}\right). \tag{3}$$

Proof. Without loss of generality, we may assume $a_0 = 0$. Then, since the matrices $J_n^k + (J_n^k)^T$, $k = 1, \cdots, m$ are Hermitian (see [15, Theorem 8.12]), we have:

$$\begin{aligned}\lambda_{\max}(T) = \lambda_{\max}\left(\sum_{k=1}^{m} a_k (J_n^k + (J_n^k)^T) \right) &\leq \sum_{k=1}^{m} a_k \lambda_{\max}\left(J_n^k + (J_n^k)^T\right) \\ \leq \sum_{k=1}^{m} a_k w\left(J_n^k + (J_n^k)^T\right) \leq 2\sum_{k=1}^{m} a_k w(J_n^k) &= 2\sum_{k=1}^{m} a_k \cos\left(\frac{\pi}{[\frac{n-1}{k}] + 2}\right),\end{aligned}$$

in which Theorem 1 is used in the last equality. ■

Remark 2 *If T is a symmetric Toeplitz matrix with real parameters, $a_0, a_1, \ldots, a_m$, such that $a_1 \leq 0$ and $a_i a_{i+1} \leq 0$, $i = 1, 2, \ldots, m-1$, then by considering orthogonal transformation $Q = \mathrm{diag}\{1, -1, 1, \ldots, 1\}$ or $Q = \mathrm{diag}\{1, -1, 1, \ldots, -1\}$, the matrix T is orthogonally similar to the matrix T' whose parameters are a_0, $|a_1|, \ldots, |a_m|$. Therefore T and T' have the same eigenvalues and the upper bound (3) can be applied for the maximum eigenvalue of T as follows:*

$$\lambda_{\max}(T) \leq a_0 + 2\sum_{k=1}^{m} |a_k| \cos\left(\frac{\pi}{[\frac{n-1}{k}] + 2}\right). \tag{4}$$

4 Numerical Examples

In this section, numerical examples are provided to compare the level of the accuracy of the proposed upper bound in (3) with the existing ones in the literature. As the special NNST matrices, we consider the symmetric pentadiagonal (or 5-diagonal) and 7-diagonal NNST matrices. For pentadiagonal matrices we compare our upper bound with the results of [5], while for the 7-diagonal case we compare our results with the one given in [1, relation (1.1)].

Let $T_n(p, q, r)$ be an n-by-n symmetric pentadiagonal Toeplitz matrix:

$$T_n(p,q,r) := \begin{pmatrix} p & q & r & & & & \\ q & p & q & r & & & \\ r & q & p & q & r & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & r \\ & & & & & p & q \\ & & & & r & q & p \end{pmatrix},$$

where $p \in \mathbb{R}$ and q, r are two nonnegative real numbers. If $r = 0$, then $T_n(p, q, 0)$ is a tridiagonal matrix and its eigenvalues are known [7]. So, we assume $r \neq 0$. Based on (3), our upper bound on the maximum eigenvalue of the matrix $T_n(p, q, r)$ becomes:

$$\lambda_{\max}\left(T_n(p,q,r)\right) \leq p + 2q\cos\left(\frac{\pi}{n+1}\right) + 2r\cos\left(\frac{\pi}{[\frac{n-1}{2}]+2}\right). \tag{5}$$

In [5, Theorem 4], Eloufi obtained interlacing upper and lower bounds for the eigenvalues of the pentadiagonal symmetric Toeplitz matrices. Consequently, one can obtain the following bound for the maximum eigenvalue of $T_n(p, q, r)$:

$$\lambda_{\max}(T_n(p,q,r)) \leq p + 2q\cos(\tfrac{\pi}{n+3}) + 2r\cos(\tfrac{2\pi}{n+3}). \tag{6}$$

Let r and q be two nonnegative numbers. Then for any $n \geq 1$, since $\cos(\frac{\pi}{n+1}) < \cos(\frac{\pi}{n+3})$ and $\cos(\frac{\pi}{[\frac{n-1}{2}]+2}) = \cos(\frac{2\pi}{n+3})$, our upper bound in inequality (5) is always smaller than the upper bound in inequality (6), which is derived from [5].

Remark 3 *Based on [5, Theorem 4], a lower bound on the minimum eigenvalue of the matrix $T_n(p, q, r)$ can be obtained as follows:*

$$\lambda_{\min}(T_n(p,q,r)) \geq p + \min_{j=1,2}\left\{2q\cos(\tfrac{k_j\pi}{n+3}) + 2r\cos(\tfrac{2k_j\pi}{n+3})\right\},$$

where $k_1 = [\frac{(n+3)\arccos(\frac{-q}{r})}{\pi}]$ and $k_2 = k_1 + 1$.

Example 3 *As an applied example, consider the fourth-order model problem*

$$\begin{aligned} &\alpha_1 u - \alpha_2 u_{xx} + u_{xxxx} = f, \\ &u(0) = u_x(0) = \lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} u_x(x) = 0. \end{aligned}$$

Here the coefficients α_1 and α_2 are constants. When the Laguerre-Galerkin spectral method ([12]) is applied for discretizing this problem, we obtain a linear system whose coefficient matrix is the symmetric pentadiagonal Toeplitz matrix $A = (a_{ij})$ [14], where

$$a_{ij} = \begin{cases} 6\alpha_1 + \frac{1}{2}\alpha_2 + \frac{3}{8}, & i = j, \\ -4\alpha_1 + \frac{1}{4}, & |i-j| = 1, \\ \alpha_1 - \frac{1}{4}\alpha_2 + \frac{1}{16}, & |i-j| = 2, \\ 0, & \textit{otherwise}. \end{cases}$$

For instance for $\alpha_1 = \alpha_2 = 10$, the coefficient matrix can be calculated as

$$A = \begin{pmatrix} 65.375 & -39.75 & 7.562 & & & \\ -39.75 & 65.375 & -39.75 & \ddots & & \\ 7.562 & -39.75 & 65.375 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 7.562 \\ & & \ddots & \ddots & \ddots & -39.75 \\ & & & 7.562 & -39.75 & 65.375 \end{pmatrix}. \quad (7)$$

Note that despite the fact that the second parameter is -39.75, which is negative, according to Remark 2 we can apply (5) by considering $q = 39.75$. In Table 1, we compute the values of the maximum eigenvalue and the upper bounds in (5) and (6) for the matrix A when $n = 10, 100$ and 1000 (all values are rounded to 4 significant digits). It is evident that while the upper bound in (5) provides a better approximation for the $\lambda_{\max}(A)$, as the size of the matrix gets larger both upper bounds in (5) and (6) approach to $\lambda_{\max}(A)$ from above.

n	$\lambda_{\max}(A)$	upper bound in (6)	upper bound in (5)	
10	154.619	155.956	154.753	
100	159.9316	159.9339	159.9319	
1000	159.9983	159.9983	159.9983	

Table 1: The values of the maximum eigenvalue and the upper bounds in (5) and (6) for the matrix A in (7).

In [1], Bini et al. gave some upper and lower bounds for the eigenvalues of 7−diagonal symmetric Toeplitz matrices. For an $n \times n$, 7-diagonal NNST matrix $T = T_n(a_0, a_1, a_2, a_3)$ with the parameters $a_1, a_2, a_3 \geq 0$ and $a_0 \in \mathbb{R}$, when n is even, their upper bound for the maximum eigenvalue is

$$\lambda_{\max}(T) \leq \max\left\{ p\left(2\cos(\frac{j\pi}{n+3})\right), \quad j = 1, 2, \ldots, n+2 \right\}, \quad (8)$$

where $p(\mu) = a_3\mu^3 + a_2\mu^2 + (a_1 - 3a_3)\mu + a_0 - 2a_2$, $\mu \in \mathbb{R}$. In the following example, we compare our upper bound in (3) with (8) for a 7-diagonal NNST matrix.

Example 4 *Let $T = T_n(0, a_1, a_2, a_3)$ be a 7-diagonal NNST matrix, with $a_1 = m^2$, $a_2 = m$, $a_3 = 1$. Then, for $n = 10$ and $m = 5$, the maximum eigenvalue of the matrix T equals $\lambda_{\max}(T) = 57.983$, while the upper bounds in (3) and (8) are 58.253 and 58.899, respectively. Also, for $n = 20$ and $m = 10$ the maximum eigenvalue of the matrix T is equal to $\lambda_{\max}(T) = 218.743$, while the upper bounds in (3) and (8) are 218.803 and 219.230, respectively. Hence our upper bound in (3) seems to be a better upper bound.*

5 Conclusion

Giving a formula for the numerical radius of the powers of a Jordan block, in our main result (Theorem 2), we derived an analytic and simple upper bound for the maximum eigenvalue of nonnegative symmetric Toeplitz matrices. Although there are effective formulas and algorithms for calculating the eigenvalues in the previous research, in its own right, our simple and ready-to-use formula can play a role in analytical studies. The main idea of this paper can be extended to the eigenvalues of symmetric block Toeplitz matrices. However, the inequalities related to the maximum eigenvalue of the sum of Hermitian block matrices are more complex than the numeric case. Therefore, other methods should be developed instead of using the direct inequality in our main theorem's proof.

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