

# Oscillation Of Noncanonical Third-Order Delay Differential Equations Via Canonical Transform\*

Radhakrishnan Srinivasan<sup>†</sup>, Sivagandhi Saravanan<sup>‡</sup>, Ethiraju Thandapani<sup>§</sup>, Ercan Tunç<sup>¶</sup>

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## Abstract

The authors examine the oscillatory behavior of solutions of the third-order nonlinear delay differential equation

$$\left(p_2(t) (p_1(t)x'(t))'\right)' + q(t)x^\beta(\sigma(t)) = 0$$

when it is in noncanonical form. The main idea is to transform the noncanonical operator into canonical form and then obtain criteria for the oscillation of all solutions of the studied equation. Particular examples are provided to show the importance and novelty of the main results.

## 1 Introduction

Our goal is to examine the oscillatory behavior of all solutions to the third-order nonlinear delay differential equation

$$\left(p_2(t) (p_1(t)x'(t))'\right)' + q(t)x^\beta(\sigma(t)) = 0, \quad (1)$$

where  $t \geq t_0 > 0$  and  $\beta$  is the ratio of odd positive integers. In the remainder of the paper we assume that:

(i)  $p_1, p_2, q \in C([t_0, \infty), \mathbb{R})$  are positive, and equation (1) is in noncanonical form, i.e.,

$$\int_{t_0}^{\infty} \frac{1}{p_1(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{p_2(t)} dt < \infty; \quad (2)$$

(ii)  $\sigma \in C^1([t_0, \infty), \mathbb{R})$  is strictly increasing,  $\sigma(t) \leq t$ , and  $\lim_{t \rightarrow \infty} \sigma(t) = \infty$ .

By a *solution* of (1), we mean a function  $x \in C([t_x, \infty), \mathbb{R})$  for some  $t_x \geq t_0$  such that  $x \in C^1([t_x, \infty), \mathbb{R})$ ,  $p_1x' \in C^1([t_x, \infty), \mathbb{R})$ ,  $p_2(p_1x')' \in C^1([t_x, \infty), \mathbb{R})$  and  $x$  satisfies (1) on  $[t_x, \infty)$ . We only consider those solutions  $x(t)$  of (1) that exist on some half-line  $[t_x, \infty)$  and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \quad \text{for any } T_1 \geq t_x;$$

in addition, we tacitly assume that (1) possesses such solutions. Such a solution  $x(t)$  of (1) is said to be *oscillatory* if it has arbitrarily large zeros on  $[t_x, \infty)$ , i.e., for any  $t_1 \in [t_x, \infty)$  there exists  $t_2 \geq t_1$  such that  $x(t_2) = 0$ ; otherwise it is called *nonoscillatory*, i.e., it is eventually of one sign. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

The investigation of qualitative properties of equation (1) is important for many real world applications, since these equations are considered as valuable tools in the modeling of many phenomena in different areas of applied mathematics and physics; for example, see Hale's monograph [19] for some applications in science and

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<sup>†</sup>Department of Mathematics, SRM Institute of Science and Technology, Ramapuram Campus, Chennai-600 089, India

<sup>‡</sup>Madras School of Economics, Chennai-600 025, India

<sup>§</sup>Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai-600 005, India

<sup>¶</sup>Department of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpasa University, 60240, Tokat, Turkey

technology. In the last several years, the oscillation theory and asymptotic behavior of third-order differential equations and their applications have received more and more attention in the literature; for some typical results, we refer the reader to the book [24] and the papers [2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 23] and the references contained therein. See also [1, 4, 5, 17, 21] for some interesting work on recent developments in oscillation theory.

On the other hand, most of the research work dealing with the oscillatory properties of solutions of (1) are for the cases

$$\int_{t_0}^{\infty} \frac{1}{p_1(t)} dt = \int_{t_0}^{\infty} \frac{1}{p_2(t)} dt = \infty, \quad (3)$$

$$\int_{t_0}^{\infty} \frac{1}{p_1(t)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{p_2(t)} dt < \infty, \quad (4)$$

or

$$\int_{t_0}^{\infty} \frac{1}{p_1(t)} dt < \infty, \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{p_2(t)} dt = \infty. \quad (5)$$

This is due to the fact that the investigation of (1) with (3) or (4) (or (5) holds) is much easier than noncanonical type equations. Further to the best of our observations, the oscillation criteria for (1) when the equation is in noncanonical form were obtained in the literature without changing the form of the equation. In view of the above observation, in this paper, we first convert the noncanonical equation (1) into canonical form and then we obtain some new oscillation criteria for (1). Our second task is to test the strength of our criteria via some particular examples. It should be noted that the results in the present paper improve, extend and simplify the results in [14, 16] in particular (see Example 1 below) as well as many known results on nonlinear oscillation in general. For these reasons, it is our hope that the present paper will stimulate additional interest in research on third and higher odd-order nonlinear noncanonical functional differential equations. We also note that equation (1) is in canonical form if (3) holds, it is in semi-canonical form if (4) or (5) holds, and it is in noncanonical form if (2) holds, as it is here.

## 2 Preliminary Results

In view of (1), while considering nonoscillatory solutions, we can restrict our attention to positive ones. The following lemma is rather standard when studying the oscillatory behavior of solutions of third order equations; for example, see [2, 14]. Therefore, it follows from [2, 14] that the set of positive solutions of (1) has the following structure:

**Lemma 1** *Let (i)–(ii) hold. If  $x$  is an eventually positive solution of (1), then there exists  $t_1 \in [t_0, \infty)$  such that  $x$  satisfies one of following four cases:*

$$(I) \quad x(t) > 0, \quad p_1(t)x'(t) > 0, \quad p_2(t)(p_1(t)x'(t))' > 0, \quad (p_2(t)(p_1(t)x'(t)))' < 0,$$

$$(II) \quad x(t) > 0, \quad p_1(t)x'(t) < 0, \quad p_2(t)(p_1(t)x'(t))' > 0, \quad (p_2(t)(p_1(t)x'(t)))' < 0,$$

$$(III) \quad x(t) > 0, \quad p_1(t)x'(t) < 0, \quad p_2(t)(p_1(t)x'(t))' < 0, \quad (p_2(t)(p_1(t)x'(t)))' < 0,$$

$$(IV) \quad x(t) > 0, \quad p_1(t)x'(t) > 0, \quad p_2(t)(p_1(t)x'(t))' < 0, \quad (p_2(t)(p_1(t)x'(t)))' < 0.$$

for all  $t \geq t_1$ .

So, if we want to derive oscillation criteria for the noncanonical equation (1), we have to eliminate the above mentioned four cases. However, if we transform equation (1) into canonical form, then the number of cases is reduced to only two. Therefore, this significantly simplifies the investigation of oscillation of (1).

In view of (2), we can adopt the following notation:

$$\begin{aligned} \Omega_1(t) &= \int_t^\infty \frac{ds}{p_1(s)}, \quad \Omega_2(t) = \int_t^\infty \frac{ds}{p_2(s)}, \quad \Omega(t) = \int_t^\infty \frac{\Omega_2(s)}{p_1(s)} ds, \\ \Omega_*(t) &= \int_t^\infty \frac{\Omega_1(s)}{p_2(s)} ds, \quad r_1(t) = \frac{p_1(t)\Omega^2(t)}{\Omega_*(t)}, \quad r_2(t) = \frac{p_2(t)\Omega_*^2(t)}{\Omega(t)}. \end{aligned}$$

Instead of using the result of Trench [27], we employ [7, Theorem 2.1] to transform equation (1) in the equivalent canonical form as

$$\left( r_2(t) \left( r_1(t) \left( \frac{x(t)}{\Omega(t)} \right)' \right)' \right)' + \Omega_*(t)q(t)x^\beta(\sigma(t)) = 0. \tag{6}$$

Now, setting  $\mu(t) = x(t)/\Omega(t)$  in (6) and using the notation

$$Q(t) = \Omega_*(t)\Omega^\beta(\sigma(t))q(t),$$

the following results in [7] are immediate.

**Theorem 1** *Noncanonical nonlinear differential equation (1) possesses a solution  $x(t)$  if and only if the canonical equation*

$$\left( r_2(t) (r_1(t)\mu'(t))' \right)' + Q(t)\mu^\beta(\sigma(t)) = 0 \tag{7}$$

has the solution  $\mu(t) = x(t)/\Omega(t)$ .

**Corollary 1** *Noncanonical nonlinear differential equation (1) has an eventually positive solution if and only if canonical equation (7) has an eventually positive solution.*

Corollary 1 essentially simplifies investigation of equation (1). Because, due to equation (7), we deal with only two classes of an eventually positive solution, namely either

$$N_0 : \mu(t) > 0, \quad r_1(t)\mu'(t) < 0, \quad r_2(t)(r_1(t)\mu'(t))' > 0, \quad (r_2(t)(r_1(t)\mu'(t))')' < 0, \text{ or}$$

$$N_2 : \mu(t) > 0, \quad r_1(t)\mu'(t) > 0, \quad r_2(t)(r_1(t)\mu'(t))' > 0, \quad (r_2(t)(r_1(t)\mu'(t))')' < 0,$$

for sufficiently large  $t$ . Hence the results obtained here are new and different from the existing results.

### 3 Main Results

In this section, we provide criteria for the oscillation of all solutions of (1). We begin with the following lemma.

**Lemma 2** *Assume that  $\mu(t)$  is a positive solutions of (7) which belong to  $N_2$ . Define*

$$R_1(t, t_*) = \int_{t_*}^t \frac{1}{r_1(s)} \int_{t_*}^s \frac{1}{r_2(u)} duds,$$

$$R_{m+1}(t, t_*) = \int_{t_*}^t \frac{1}{r_1(s)} \int_{t_*}^s \frac{1}{r_2(u)} \exp \left( \int_u^t Q(v)R_m(\sigma(v), t_*)dv \right) duds, \quad m \in \mathbb{N},$$

for  $t \geq t_*$  for some  $t_* \geq t_0$ . Then

$$\mu(\sigma(t)) \geq R_m(\sigma(t), t_*)L_2\mu(\sigma(t)) \quad \text{if } \beta = 1, \tag{8}$$

$$\mu(\sigma(t)) \geq R_1(\sigma(t), t_*)L_2\mu(\sigma(t)) \quad \text{if } \beta \neq 1, \tag{9}$$

where  $L_2\mu(t) = r_2(t)(r_1(t)\mu'(t))'$ .

**Proof.** The proof is similar to Lemma 2 of [11] and so it is omitted. ■

**Theorem 2** Assume that there exists a function  $\tau \in C^1([t_0, \infty), \mathbb{R})$  such that

$$\tau'(t) \geq 0, \quad \tau(t) \geq t, \quad \text{and} \quad g(t) = \sigma(\tau(\tau(t))) < t. \tag{10}$$

If both first order delay differential equations

$$z'(t) + Q_1(t)z^\beta(\sigma(t)) = 0 \tag{11}$$

and

$$w'(t) + Q_2(t)w^\beta(g(t)) = 0 \tag{12}$$

are oscillatory, where

$$Q_1(t) = \begin{cases} Q(t)R_1^\beta(\sigma(t), t_*) & \text{if } \beta \neq 1, \\ Q(t)R_m(\sigma(t), t_*) & \text{if } \beta = 1, \end{cases}$$

for some  $m \in \mathbb{N}$  and  $t_* \geq t_0$ , and

$$Q_2(t) = \frac{1}{r_1(t)} \int_t^{\tau(t)} \frac{1}{r_2(s)} \int_s^{\tau(s)} Q(u)du ds,$$

then equation (1) is oscillatory.

**Proof.** Let  $x(t)$  be an eventually positive solution of equation (1), say  $x(t) > 0$  and  $x(\sigma(t)) > 0$  for  $t \geq t_1$  for some  $t_1 \geq t_0$ . Then, by Corollary 1, the corresponding function  $\mu(t) = x(t)/\Omega(t)$  is a positive solution of (7), and so either  $\mu \in N_0$  or  $\mu \in N_2$  for  $t \geq t_2$  for some  $t_2 \geq t_1$ .

First assume that  $\mu \in N_0$ . Integrating (7) from  $t$  ( $t \geq t_2$ ) to  $\tau(t)$ , we get

$$r_2(t)(r_1(t)\mu'(t))' \geq \int_t^{\tau(t)} Q(s)\mu^\beta(\sigma(s))ds.$$

Using the fact that  $\mu$  is decreasing and taking (10) into account, we see that

$$(r_1(t)\mu'(t))' \geq \frac{1}{r_2(t)}\mu^\beta(\sigma(\tau(t))) \int_t^{\tau(t)} Q(s)ds.$$

Integrating the last inequality from  $t$  to  $\tau(t)$ , we obtain

$$-r_1(t)\mu'(t) \geq \int_t^{\tau(t)} \frac{\mu^\beta(\sigma(\tau(s)))}{r_2(s)} \int_s^{\tau(s)} Q(u)du ds,$$

which yields

$$\mu'(t) + Q_2(t)\mu^\beta(g(t)) \leq 0.$$

Therefore, by [25, Theorem 1], we conclude that there exists a positive solution of (12) which tends to zero as  $t \rightarrow \infty$ , which contradicts the fact that (12) is oscillatory.

Next, assume  $\mu \in N_2$ . Combining (8) with (7), we see that  $z(t) = L_2\mu(t)$  is a positive solution of the first order delay differential inequality

$$z'(t) + Q_1(t)z^\beta(\sigma(t)) \leq 0. \tag{13}$$

As in the case  $\mu \in N_0$ , we obtain a contradiction to the fact that (11) is oscillatory. Thus (7) is oscillatory, which in turn implies (1) is oscillatory. This completes the proof of the theorem. ■

The proofs of the following Corollaries 2 and 3 follows from Theorem 2 of [22] and Theorem 2 and hence the details are omitted.

**Corollary 2** Let  $\beta = 1$ . Assume that there exists a function  $\tau \in C^1([t_0, \infty), \mathbb{R})$  such that (10) holds. If

$$\liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t Q(s)R_m(\sigma(s), t_*)ds > \frac{1}{e} \tag{14}$$

for some  $m \in \mathbb{N}$  and  $t_* \geq t_0$ , and

$$\liminf_{t \rightarrow \infty} \int_{g(t)}^t Q_2(s)ds > \frac{1}{e}, \tag{15}$$

then (1) is oscillatory.

**Corollary 3** Let  $\beta \in (0, 1)$ . Assume that there exists a function  $\tau \in C^1([t_0, \infty), \mathbb{R})$  such that (10) holds. If

$$\limsup_{t \rightarrow \infty} \int_{\sigma(t)}^t Q_1(s)ds > 0 \tag{16}$$

and

$$\limsup_{t \rightarrow \infty} \int_{g(t)}^t Q_2(s)ds > 0, \tag{17}$$

then (1) is oscillatory.

To present our next results, we assume that

$$\sigma(t) = \theta_1 t \text{ and } \tau(t) = \theta_2 t \text{ with } \theta_3 = \theta_1 \theta_2^2 < 1, \tag{18}$$

or

$$\sigma(t) = t^{\theta_1} \text{ and } \tau(t) = t^{\theta_2} \text{ with } \theta_3 = \theta_1 \theta_2^2 < 1, \tag{19}$$

where  $\theta_1, \theta_3 \in (0, 1)$  and  $\theta_2 > 1$ .

**Corollary 4** Let  $\beta \in (1, \infty)$  and let (18) holds. Suppose also that there exists a function  $\tau \in C^1([t_0, \infty), \mathbb{R})$  such that (10) holds. If there exists  $\lambda_1 > -\ln(\beta)/\ln(\theta_1)$  such that

$$\liminf_{t \rightarrow \infty} [Q_1(t) \exp(-t^{\lambda_1})] > 0, \tag{20}$$

and  $\lambda_2 > -\ln(\beta)/\ln(\theta_3)$  such that

$$\liminf_{t \rightarrow \infty} [Q_2(t) \exp(-t^{\lambda_2})] > 0 \tag{21}$$

hold, then (1) is oscillatory.

**Corollary 5** Let  $\beta \in (1, \infty)$  and let (19) holds. Suppose also that there exists a function  $\tau \in C^1([t_0, \infty), \mathbb{R})$  such that (10) holds. If there exists  $\lambda_1 > -\ln(\beta)/\ln(\theta_1)$  such that

$$\liminf_{t \rightarrow \infty} [Q_1(t) \exp(-(\ln t)^{\lambda_1})] > 0, \tag{22}$$

and  $\lambda_2 > -\ln(\beta)/\ln(\theta_3)$  such that

$$\liminf_{t \rightarrow \infty} [Q_2(t) \exp(-(\ln t)^{\lambda_2})] > 0 \tag{23}$$

hold, then (1) is oscillatory.

The proofs of Corollaries 4 and 5 follows from Theorem 4 and Theorem 5 of [26], respectively.

**Theorem 3** Let  $\beta = 1$ . Assume that there exists a function  $\sigma_1 \in C^1([t_0, \infty), \mathbb{R})$  such that  $\sigma_1'(t) \geq 0$ ,  $\sigma_1(t) < t$  and  $\lim_{t \rightarrow \infty} \sigma_1(t) = \infty$ . If (14) holds for some  $m \in \mathbb{N}$  and  $t \geq t_* \geq t_0$ , and

$$\limsup_{t \rightarrow \infty} \int_{\sigma(t)}^t \frac{1}{r_1(u)} \int_u^{\sigma_1(t)} \frac{1}{r_2(s)} \int_s^{\sigma_1(t)} Q(v) dv ds du > 1, \tag{24}$$

then (1) is oscillatory.

**Proof.** Let  $x(t)$  be an eventually positive solution of equation (1), say  $x(t) > 0$  and  $x(\sigma(t)) > 0$  for  $t \geq t_1$  for some  $t_1 \geq t_0$ . Then, by Corollary 1, the corresponding function  $\mu(t) = x(t)/\Omega(t)$  is a positive solution of (7), and so either  $\mu \in N_0$  or  $\mu \in N_2$  for  $t \geq t_2$  for some  $t_2 \geq t_1$ .

First assume that  $\mu \in N_0$ . Integrating (7) from  $u$  ( $u \geq t_2$ ) to  $\sigma_1(t)$  and using the fact that  $\mu$  is decreasing, we have

$$r_2(u)(r_1(u)\mu'(u))' \geq \int_u^{\sigma_1(t)} Q(v)\mu(\sigma(v))dv \geq \mu(\sigma(t)) \int_u^{\sigma_1(t)} Q(v)dv,$$

or

$$(r_1(u)\mu'(u))' \geq \frac{\mu(\sigma(t))}{r_2(u)} \int_u^{\sigma_1(t)} Q(v)dv.$$

Integrating the above inequality from  $u$  to  $\sigma_1(t)$  yields

$$-\mu'(u) \geq \frac{\mu(\sigma(t))}{r_1(u)} \int_u^{\sigma_1(t)} \frac{1}{r_2(x)} \int_x^{\sigma_1(t)} Q(v)dv dx.$$

Again integrating the last inequality from  $u$  to  $t$ , we get

$$\mu(u) \geq \mu(\sigma(t)) \int_u^t \frac{1}{r_1(v)} \int_v^{\sigma_1(t)} \frac{1}{r_2(x)} \int_x^{\sigma_1(t)} Q(s)ds dx dv. \tag{25}$$

Letting  $u = \sigma(t)$  in (25), we obtain a contradiction with (24).

Next assume  $\mu \in N_2$ . In view of (14) and Corollary 2, we see that the set  $N_2 = \emptyset$ . Thus, (7) is oscillatory, which in turn implies (1) is oscillatory. The proof of the theorem is complete. ■

### 4 Numerical Examples

Let us illustrate the importance of our newly obtained results on Euler type linear and nonlinear third-order delay differential equations.

**Example 1** Consider the third-order linear delay differential equation

$$\left( t^2 (t^2 x'(t))' \right)' + atx(\lambda t) = 0, \quad t \geq 1, \tag{26}$$

where  $a > 0$  is a constant and  $\lambda \in (0, 1)$ . Using Theorem 1, equation (26) has the canonical representation

$$\mu'''(t) + \frac{a}{\lambda^2 t^3} \mu(\lambda t) = 0.$$

Moreover,  $r_1(t) = r_2(t) = 1$ ,  $Q(t) = \frac{a}{\lambda^2 t^3}$ , and  $\sigma(t) = \lambda t$ .

Now let  $m = 2$ , then condition (14) becomes

$$\liminf_{t \rightarrow \infty} \int_{\lambda t}^t Q(s)R_2(\lambda s, 1)ds = \frac{4a}{(2-a)(4-a)} \ln(1/\lambda) > \frac{1}{e},$$

that is, (14) holds if  $\frac{4a}{(2-a)(4-a)} \ln(1/\lambda) > \frac{1}{e}$ . Let  $\sigma_1(t) = \lambda_1 t$  with  $\lambda_1 \in (0, 1)$ , we see that (24) becomes

$$\limsup_{t \rightarrow \infty} \int_{\lambda t}^t \int_u^{\lambda_1 t} \int_s^{\lambda_1 t} \frac{a}{\lambda^2 v^3} dv ds du = \frac{a}{\lambda^2} \left( \frac{4\lambda}{\lambda_1} - \frac{4}{\lambda_1} + \frac{1}{\lambda_1^2} - \frac{\lambda^2}{\lambda_1^2} + 2 \ln(1/\lambda) \right) > 4,$$

that is, (24) holds if

$$\frac{a}{\lambda^2} \left( \frac{4\lambda}{\lambda_1} - \frac{4}{\lambda_1} + \frac{1}{\lambda_1^2} - \frac{\lambda^2}{\lambda_1^2} + 2 \ln(1/\lambda) \right) > 4.$$

Let  $\lambda = 3/10$  and  $a = 0.45$ . Then (14) is clearly satisfied. With  $\lambda_1 = 1/5$ , we see that condition (24) holds for  $a > 0.03226$ . Hence by Theorem 3, equation (26) is oscillatory if  $a = 0.45$ .

The same equation (26) is found in [16]. They used the expression (see (4.5) in [16])

$$a \left( \frac{3}{\lambda^2} + \frac{4}{\lambda} - 1 - \frac{2}{\lambda^2} \ln \lambda \right) > 4$$

to prove their claim. But this expression is wrong and the correct one is

$$a \left( \frac{4}{\lambda} - \frac{3}{\lambda^2} - 1 - \frac{2}{\lambda^2} \ln \lambda \right) > 4.$$

If we let  $\lambda = 3/10$  in the above expression, we see that  $a > 0.6949$ , and therefore (26) is oscillatory for  $a > 0.6949$ . Thus, our Theorem 3 improves Theorem 3.3 of [16] and also the results in [14].

**Example 2** Consider the third-order sublinear delay differential equation

$$\left( t^2 (t^2 x'(t))' \right)' + at^{5/3} x^{1/3} \left( \frac{t}{3} \right) = 0, \quad t \geq 1, \tag{27}$$

where  $a > 0$ . By simple calculation, we see that (27) has the canonical representation

$$\mu'''(t) + \frac{\sqrt[3]{36a}}{t} \mu^{1/3} \left( \frac{t}{3} \right) = 0.$$

By choosing  $\tau(t) = 3t/2$ , we see that all conditions of Corollary 3 are satisfied and hence equation (27) is oscillatory.

**Example 3** Consider the third-order superlinear delay differential equation

$$\left( t^2 (t^2 x'(t))' \right)' + t^8 e^{t^4} x^3 \left( \frac{t}{3} \right) = 0, \quad t \geq 1. \tag{28}$$

The transformed canonical equation is

$$\mu'''(t) + \frac{729}{4} e^{t^4} \mu^3 \left( \frac{t}{3} \right) = 0.$$

By choosing  $\theta_2 = 4/3$ , we see that  $\theta_3 = 16/27 < 1$  and so condition (18) holds. By choosing  $\lambda_1 = 2$  and  $\lambda_2 = 3$ , we see that condition (20) and (21) are satisfied. Hence, by Corollary 4, equation (28) is oscillatory.

**Remark 1** It is worth mentioning that the oscillation of equations (27) and (28) cannot be commented by previous results reported in the literature.

## 5 Conclusion

In this paper, we have established new type of oscillation criteria for third order noncanonical delay differential equations by transforming it to canonical type equations. The canonical transformation is based on the recent result obtained in [7]. Also our criteria applicable to linear and nonlinear equations and they improve, extend and simplify the results in [2, 6, 9, 11, 12, 14, 16, 20]. This is illustrated via examples. Establishing oscillation criteria for fractional order differential equations using this technique could be a promising topic for future work.

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