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A Note On Perpendicularities And Inner Products*

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Abstract

A perpendicularity $\perp$ in a module M is a binary relation that is irreflexive (but $0 \perp 0$), symmetric, serial, and preserves addition and scalar multiplication. If $\perp$ is not a subset of another perpendicularity in M, then $\perp$ is maximal. If M is a finite-dimensional vector space and $\perp$ is induced by an inner product, then it is well known that $\perp$ is maximal. We disprove the converse and an analogous result in the context of Abelian groups.

1 Introduction

Perpendicularity is a geometric notion but can be investigated also algebraically. Then the most natural setting is an inner product space and, more generally, a normed space [1], but also certain other structures have been considered [3, 4, 5, 7]. In this note, we disprove certain conjectures [7] on perpendicularities in a vector space and in an Abelian group.

Let M be a module over a ring R. A *perpendicularity in* M is a binary relation $\perp$ satisfying, for all $x, y, y_1, y_2 \in M$, $\gamma \in R$,

(A1) $x \neq 0 \implies x \not\perp x$;

(A2) $x \perp y \implies y \perp x$;

(A3) $x \perp z$ for some $z \in M$;

(A4) $x \perp y_1, y_2 \implies x \perp (y_1 + y_2)$;

(A5) $x \perp y \implies x \perp \gamma y$.

We let $\operatorname{perp} M$ denote the set of all perpendicularities in M. We assumed previously [5, 7] that $M \neq \{0\}$, but we can include the trivial case $M = \{0\}$. Its only binary relation $(0, 0)$ is a perpendicularity. In particular, (A1) is satisfied, since its left-hand side is identically false.

Because an Abelian group is a $\mathbb{Z}$-module, perpendicularity is defined also there. Then (A5) reads simply

$$x \perp y \implies x \perp -y,$$

cf. [5]. The *trivial perpendicularity*

$$x \perp_{\text{triv}} y \iff x = 0 \vee y = 0$$

always exists. A perpendicularity in M is *maximal* if it is not a subset of another perpendicularity in M.

An inner product f in a vector space V *induces* the perpendicularity

$$x \perp y \iff f(x, y) = 0. \tag{1}$$

Then we write $\perp = \perp_f$. The converse does not hold: all perpendicularities in V are not of this form ($\perp_{\text{triv}}$ is a counterexample). But does it hold if $\perp$ is maximal?

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Theorem 1 *A perpendicularity in $V = \mathbb{R}^n$ induced by an inner product is maximal.*

Proof. See [7, pp. 245–246]. Actually this theorem extends to $\mathbb{C}^n$. ■

We conjectured [7, p. 246] the converse: if $\perp \in \operatorname{perp} V$ is maximal, then $\perp = \perp_f$ for some inner product f. We disprove it in Section 2.

We can study this question also in the additive Abelian group $G = \mathbb{Z}^n$, by defining an inner product there. Take $e_1, \ldots, e_n \in G$ such that

$$G = \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle,$$

where $\langle \cdot \rangle$ stands for the spanned subgroup. Express $x, y \in G$ as

$$x = \xi_1 e_1 + \cdots + \xi_n e_n,\ y = \eta_1 e_1 + \cdots + \eta_n e_n, \quad \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathbb{Z},$$

and define the *inner product with respect to the basis* $E = \{e_1, \ldots, e_n\}$ by

$$f_E(x, y) = \xi_1 \eta_1 + \cdots + \xi_n \eta_n.$$

We conjectured [7, Conjecture 1] that $\perp \in \operatorname{perp} G$ is maximal if and only if $\perp = \perp_f$ for some inner product f in G. (In this reference, E is actually of a certain type, but any E applies.) We disprove the "only if" part in Section 3, but the "if" part remains to be conjectured. Finally, we complete our paper with conclusions in Section 4.

2 The Case $V = \mathbb{R}^2$

We consider the vector space $V = \mathbb{R}^2$ in this section. Let

$$S = \{(\cos\theta, \sin\theta) : 0 \le \theta < \pi\}.$$

It is the set of unit vectors in the upper half-plane of $\mathbb{R}^2$, including $(1, 0)$. If $0 \ne x \in V$, then uniquely

$$x = \sigma(x) s(x), \quad 0 \ne \sigma(x) \in \mathbb{R}, \quad s(x) \in S.$$

Define in S the relation

$$\begin{aligned}\rho = \bigcup_{0<\theta<\frac{\pi}{2}} & \big\{((\cos\theta, \sin\theta), (-\cos\theta, \sin\theta))\big\} \cup \big\{((-\cos\theta, \sin\theta), (\cos\theta, \sin\theta))\big\} \\ & \cup \big\{((1,0),(0,1)), ((0,1),(1,0))\big\}.\end{aligned}$$

So, $u, v \in S$ satisfy $u\,\rho\,v$ if and only if they either locate symmetrically to the y-axis or lie on different coordinate axes. Further, define in V

$$x \perp_0 y \iff x = 0 \vee y = 0 \vee s(x)\,\rho\,s(y).$$

Lemma 1 *The relation $\perp_0 \in \operatorname{perp} V$.*

Proof. Clearly, (A1)–(A3) are satisfied. Since $s(\gamma x) = s(x)$ for all $\gamma \ne 0$, (A5) follows. To prove (A4), assume that

$$x \perp_0 y_1, y_2. \tag{2}$$

If $0 \in \{y_1, y_2, y_1 + y_2\}$, then clearly

$$x \perp_0 (y_1 + y_2). \tag{3}$$

So, let $y_1, y_2, y_1 + y_2 \ne 0$. If y_1 and y_2 are linearly dependent, then $s(y_1) = s(y_2) = s(y_1 + y_2)$, and (3) follows. If they are linearly independent, then $s(y_1) \ne s(y_2)$. But (2) implies that $s(y_1) = s(y_2)$, a contradiction. Therefore (3) again follows. ■

Lemma 2 *The perpendicularity $\perp_0$ is maximal.*

Proof. Suppose that there is $\perp \in \operatorname{perp} V$ satisfying

$$\perp \supset \perp_0 \tag{4}$$

(strictly). Then there are $x, y \in V$ such that

$$x \perp y, \quad x \not\perp_0 y. \tag{5}$$

Moreover, $x, y \neq 0$ (otherwise $x \perp_0 y$). Let $z \in V$ satisfy

$$x \perp_0 z, \quad z \neq 0. \tag{6}$$

(It exists by the definition of $\perp_0$.)

If y and z are linearly dependent, then (since they are nonzero) $z = \gamma y$ for some $\gamma \neq 0$. So, (6) reads

$$x \perp_0 \gamma y,$$

which implies by (A5) that $x \perp_0 y$, contradicting (5).

If y and z are linearly independent, then $x = \lambda y + \mu z$ for some $\lambda, \mu \in \mathbb{R}$. But $x \perp y$ by (5), and $x \perp z$ by (6) and (4). Now (A4) and (A5) imply $x \perp x$, contradicting (A1). ■

We let $\mathbb{R}^{n\times n}$ (respectively, $\mathbb{R}_+^{n\times n}$) denote the set of real (real and positive definite) $n \times n$ matrices.

Lemma 3 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is an inner product if and only if there is $Q \in \mathbb{R}_+^{n\times n}$ such that*

$$f(x, y) = x^T Q y.$$

Here x^T is the transpose of x, and x and y are considered as column vectors.

Proof. See [6, Problem 7.2.P32]. ■

Lemma 4 *The perpendicularity $\perp_0$ is not induced by an inner product.*

Proof. Suppose that

$$x \perp_0 y \iff y^T Q x = 0,$$

where

$$Q = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} \in \mathbb{R}_+^{2\times 2}.$$

Let $x = (\cos\theta, \sin\theta)$ and $y = (-\cos\theta, \sin\theta)$, where $0 < \theta < \frac{\pi}{2}$. Since $x \perp_0 y$, we have

$$x^T Q y = -\gamma_{11}\cos^2\theta + \gamma_{22}\sin^2\theta = 0,$$

which cannot hold for all possible θ. ■

Theorem 2 *There is a maximal perpendicularity in $V = \mathbb{R}^2$ that is not induced by an inner product.*

Proof. Apply Lemmas 2 and 4. ■

3 The Case $G = \mathbb{Z}^2$

We consider the additive Abelian group (in other words, the $\mathbb{Z}$-module) $G = \mathbb{Z}^2$ in this section. Define

$$\Gamma = \{\langle (i,j)\rangle : i,j \in \mathbb{Z},\ j \geq 0,\ \gcd(i,j) = 1\}.$$

It is convenient to write

$$G(i,j) = \langle (i,j)\rangle.$$

Lemma 5 *Let $0 \neq x \in G$. There is exactly one $G(i,j) \in \Gamma$ such that $x \in G(i,j)$.*

Proof. Let $x = (\xi,\eta) \neq (0,0)$ and $\delta = \gcd(\xi,\eta)$. Clearly, $x \in G(\xi/\delta, \eta/\delta)$. If $x \in G(i,j)$, then $\xi = \gamma i$ and $\eta = \gamma j$ for some $\gamma (\neq 0)$. Since $\gcd(i,j) = 1$, we have $\gamma = \pm\delta$. Therefore $(\xi,\eta) = \pm\delta(i,j)$, implying that $G(\xi/\delta, \eta/\delta) = G(i,j)$. ■

We define in G the relation

$$\begin{aligned} x \perp_0 y \iff & x = 0 \vee y = 0 \\ & \vee \exists G(a,b) \in \Gamma : (a,b) \neq (1,1) \wedge x \in G(a,b) \wedge y \in G(b,a). \end{aligned}$$

Lemma 6 *The relation $\perp_0 \in \operatorname{perp} G$.*

Proof. Easy and omitted. ■

Lemma 7 *The perpendicularity $\perp_0$ is maximal.*

Proof. Suppose that there is $\perp \in \operatorname{perp} G$ satisfying

$$\perp \supset \perp_0 \tag{7}$$

(strictly). Then there are $x, y \in G$ such that

$$x \perp y, \quad x \not\perp_0 y. \tag{8}$$

Moreover, $x, y \neq 0$ (otherwise $x \perp_0 y$). Let $z \in G$ satisfy

$$x \perp_0 z, \quad z \neq 0. \tag{9}$$

(It exists by the definition of $\perp_0$.) Then

$$x \perp z \tag{10}$$

by (7).

By (9) and Lemma 5, there is a unique $G(a,b) \in \Gamma$ such that $a \neq b$, $x \in G(a,b)$, and $z \in G(b,a)$. Also there is a unique $G(i,j) \in \Gamma$ such that $y \in G(i,j)$. Since $(i,j) \neq (a,b), (b,a)$ by (8), it follows from Lemma 5 that

$$G(a,b) \cap \langle y \rangle = G(b,a) \cap \langle y \rangle = \{0\}. \tag{11}$$

Let $\mu, \nu, \kappa \in \mathbb{Z}$ satisfy

$$\mu x + \nu y + \kappa z = 0.$$

(At least $\mu = \nu = \kappa = 0$ applies.) If $\mu = 0$, then $\nu y = -\kappa z$, which implies $\nu = \kappa = 0$ by (11). But now the set $\{x, y, z\}$ is linearly independent, which is impossible in $\mathbb{Z}^2$. If $\mu \neq 0$, then $\mu x = -\nu y - \kappa z \perp x$ by (8) and (10). Now $\mu x \perp \mu x$, contradicting (A1). ■

Lemma 8 *The perpendicularity $\perp_0$ is not induced by an inner product.*

Proof. Suppose that G has a basis

$$B = \{g_1, g_2\}, \quad g_1 = (\gamma_{11}, \gamma_{12}), \quad g_2 = (\gamma_{21}, \gamma_{22}), \quad \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22} \in \mathbb{Z},$$

such that

$$x \perp_0 y \iff \xi_1\eta_1 + \xi_2\eta_2 = 0, \tag{12}$$

where

$$x = \xi_1 g_1 + \xi_2 g_2, \quad y = \eta_1 g_1 + \eta_2 g_2, \tag{13}$$

and $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{Z}$.

Let $x \perp_0 y$, $x, y \neq 0$, and let $G(a,b) \in \Gamma$ satisfy $a \neq b$, $x \in G(a,b)$, $y \in G(b,a)$. Because

$$x = (sa, sb), \quad y = (tb, ta), \quad 0 \neq s, t \in \mathbb{Z},$$

(13) holds if and only if

$$\gamma_{11}\xi_1 + \gamma_{21}\xi_2 = sa, \quad \gamma_{12}\xi_1 + \gamma_{22}\xi_2 = sb,$$

$$\gamma_{11}\eta_1 + \gamma_{21}\eta_2 = tb, \quad \gamma_{12}\eta_1 + \gamma_{22}\eta_2 = ta.$$

Hence,

$$\xi_1 = \frac{s(\gamma_{22}a - \gamma_{21}b)}{d}, \quad \xi_2 = \frac{s(\gamma_{11}b - \gamma_{12}a)}{d}, \tag{14}$$

$$\eta_1 = \frac{t(\gamma_{22}b - \gamma_{21}a)}{d}, \quad \eta_2 = \frac{t(\gamma_{11}a - \gamma_{12}b)}{d}, \tag{15}$$

where

$$d = \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \neq 0.$$

(If $d = 0$, then B is linearly dependent and therefore is not a basis. Because we know that $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{Z}$, the denominator d cancels.)

By (14) and (15),

$$\begin{aligned}
&\frac{(d\xi_1)(d\eta_1) + (d\xi_2)(d\eta_2)}{st} \\
= \;& (\gamma_{22}a - \gamma_{21}b)(\gamma_{22}b - \gamma_{21}a) + (\gamma_{11}b - \gamma_{12}a)(\gamma_{11}a - \gamma_{12}b) \\
= \;& (\gamma_{11}^2 + \gamma_{12}^2 + \gamma_{21}^2 + \gamma_{22}^2)ab - (\gamma_{11}\gamma_{12} + \gamma_{21}\gamma_{22})(a^2 + b^2).
\end{aligned}$$

On the other hand, $x \perp_0 y$ implies $dx \perp_0 dy$. Hence, by (12),

$$(d\xi_1)(d\eta_1) + (d\xi_2)(d\eta_2) = 0.$$

Consequently,

$$(\gamma_{11}^2 + \gamma_{12}^2 + \gamma_{21}^2 + \gamma_{22}^2)ab - (\gamma_{11}\gamma_{12} + \gamma_{21}\gamma_{22})(a^2 + b^2) = 0,$$

which cannot hold for all possible a and b. ■

Theorem 3 *There is a maximal perpendicularity in $G = \mathbb{Z}^2$ that is not induced by an inner product.*

Proof. Apply Lemmas 7 and 8. ■

4 Conclusions

The following conjectures were the starting point for the present article: A perpendicularity $\perp$ in the vector space $V = \mathbb{R}^n$ and, respectively, in the additive Abelian group $G = \mathbb{Z}^n$, is maximal if and only if $\perp$ is induced by an inner product. For V, the "if" part had already been proved in [7]. Theorems 2 and 3 above disprove the "only if" parts whenever $n = 2$, but the "if" part remains open for G.

Can Theorem 2 be extended to $V = \mathbb{R}^n$ and further to $V = \mathbb{C}^n$, and can Theorem 3 be extended to $G = \mathbb{Z}^n$? The answer to the first question is positive, but we do not give the proof here for the following reason. We have not yet been able to prove (as we conjectured above) that the answer also to the second question is positive. As these questions are closely related, it is reasonable to solve both issues in one paper.

The proof of Lemma 4 raises a few new questions. Namely, it is actually enough that Q is symmetric and has, at least, one nonzero diagonal entry. This motivates us to study (1) in $V = \mathbb{R}^n$ assuming only that f is bilinear. That is, $f(x, y) = x^T Qy$, where $Q \in \mathbb{R}^{n\times n}$ is symmetric. Clearly, $\perp_f \in \operatorname{perp} V$ if and only if f is nonsingular (i.e., Q is invertible).

Theorem 2 states that there is a maximal perpendicularity in $V = \mathbb{R}^2$ that is not induced by an inner product. In Lemma 4, $\perp_0$ is induced (although not by an inner product) by the bilinear mapping

$$f(x, y) = x^T Qy, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence it is reasonable to ask if a maximal perpendicularity in $V = \mathbb{R}^n$ is always induced by a bilinear mapping. Further, if f is an inner product, then $\perp_f$ is maximal by Theorem 1. This raises another question: Is $\perp_f$ maximal even if f is only assumed to be a nonsingular bilinear mapping? We conjecture that the answer to both questions is positive.

For a maximal perpendicularity, a nonsingular bilinear mapping is actually a better counterpart than an inner product. Namely, the positivity condition ($x \neq 0 \Rightarrow f(x, x) > 0$), which enables f to induce the norm $\sqrt{f(x, x)}$, is useless in inducing the perpendicularity (1). The above conjecture is therefore more interesting than Theorems 1 and 2.

More generally, let $V = F^n$, where F is a field. Because all fields cannot be ordered [2, p. 268, Prop. 6], we cannot define an inner product in the ordinary way. However, as remarked above, nonsingular bilinear mappings are here more interesting. (In this context, Cohn [2, Section 8.1] defines an inner product as a bilinear mapping.) More generally, we conjecture that a perpendicularity in $V = F^n$ is maximal if and only if it is induced by a nonsingular bilinear mapping.

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