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Sums Of Powers Of Integers And Generalized Stirling Numbers Of The Second Kind*

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Abstract

By applying the Newton-Gregory expansion to the polynomial associated with the sum of powers of integers $S_k(n) = 1^k + 2^k + \cdots + n^k$, we derive a couple of infinite families of explicit formulas for $S_k(n)$. One of the families involves the r-Stirling numbers of the second kind $\left\{ {k \atop j} \right\}_r$, $j = 0, 1, \ldots, k$, while the other involves their duals $\left\{ {k \atop j} \right\}_{-r}$, with both families of formulas being indexed by the non-negative integer r. As a by-product, we obtain three additional formulas for $S_k(n)$ involving the numbers $\left\{ {k \atop j} \right\}_{n+m}$, $\left\{ {k \atop j} \right\}_{n-m}$, and $\left\{ {k \atop j} \right\}_{k-j}$, where m is any given non-negative integer. Furthermore, we provide several formulas for the Bernoulli polynomials in terms of the generalized Stirling numbers of the second kind, the harmonic numbers, and the so-called harmonic polynomials.

1 Introduction

Following Broder [4, Equation 57] (see also Carlitz [6, Equation (3.2)]) we define the generalized (or weighted) Stirling numbers of the second kind by

$$\left\{ {k \atop j} \right\}_x = \sum_{i=0}^{k-j} \binom{k}{i} \left\{ {k-i \atop j} \right\} x^i, \quad \text{integers } 0 \leq j \leq k,$$

where x stands for any arbitrary real or complex value, and where the $\left\{ {k \atop j} \right\}$'s are the ordinary Stirling numbers of the second kind. Note that $\left\{ {k \atop j} \right\}_x$ is a polynomial in x of degree $k - j$ with leading coefficient $\binom{k}{j}$ and constant term $\left\{ {k \atop j} \right\}$. Furthermore, we have that $\left\{ {k \atop j} \right\}_1 = \left\{ {k+1 \atop j+1} \right\}$. In general, when x is the non-negative integer r, $\left\{ {k \atop j} \right\}_r$ becomes the r-Stirling number of the second kind $\left\{ {k+r \atop j+r} \right\}_r$ [4]. A combinatorial interpretation of the polynomial $\left\{ {k \atop j} \right\}_x$ is given in [4, Theorem 27] (see also the definition provided by Bényi and Matsusaka in [1, Definition 2.13]).

For convenience and notational simplicity, in this paper we employ the notation $\left\{ {k \atop j} \right\}_r$ to refer to Broder's r-Stirling numbers of the second kind $\left\{ {k+r \atop j+r} \right\}_r$. The former notation has been used recently by Ma and Wang in [21] (see also [1] and [24]). The numbers $\left\{ {k \atop j} \right\}_r$ are then given by

$$\left\{ {k \atop j} \right\}_r = \sum_{i=0}^{k-j} \binom{k}{i} \left\{ {k-i \atop j} \right\} r^i, \quad \text{integer } r \geq 0.$$

Likewise, adopting the notation in [21], we define the counterpart or dual of $\left\{ {k \atop j} \right\}_r$ for negative integer r as

$$\left\{ {k \atop j} \right\}_{-r} = \sum_{i=0}^{k-j} (-1)^i \binom{k}{i} \left\{ {k-i \atop j} \right\} r^i, \quad \text{integer } r \geq 0.$$

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Alternatively, $\left\{ {k \atop j} \right\}_r$ and $\left\{ {k \atop j} \right\}_{-r}$ can equivalently be expressed in the form

$$\left\{ {k \atop j} \right\}_r = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i+r)^k, \quad \text{integer}\, r \geq 0, \tag{1}$$

$$\left\{ {k \atop j} \right\}_{-r} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i-r)^k, \quad \text{integer}\, r \geq 0, \tag{2}$$

respectively. Clearly, both $\left\{ {k \atop j} \right\}_r$ and $\left\{ {k \atop j} \right\}_{-r}$ reduce to $\left\{ {k \atop j} \right\}$ when $r = 0$. It is to be noted that the numbers $\left\{ {k \atop j} \right\}_{-r}$ were introduced and studied by Koutras under the name of non-central Stirling numbers of the second kind and denoted by $S_r(k, j)$ (see [20, Equations (2.5) and (2.6)]).

For non-negative integer k, let $S_k(n)$ denote the sum of k-th powers of the first n positive integers

$$S_k(n) = 1^k + 2^k + \cdots + n^k,$$

with $S_k(0) = 0$ for all k. In [27], Orosi derived the classical formula for $S_k(n)$ in terms of the Bernoulli numbers (the so-called Faulhaber formula). Additionally, as is well known, $S_k(n)$ can be expressed in terms of the Stirling numbers of the second kind as (see, e.g., [29])

$$S_k(n) = -\delta_{k,0} + \sum_{j=0}^{k} j! \binom{n+1}{j+1} \left\{ {k \atop j} \right\}, \tag{3}$$

where $\delta_{k,0}$ is the Kronecker delta, which ensures that $S_0(n) = n$. Furthermore, $S_k(n)$ admits the following variant of (3):

$$S_k(n) = \sum_{j=1}^{k+1} (j-1)! \binom{n}{j} \left\{ {k+1 \atop j} \right\} = \sum_{j=0}^{k} j! \binom{n}{j+1} \left\{ {k+1 \atop j+1} \right\}, \tag{4}$$

(see, e.g., [7], [11, Theorem 5] and [30, Equation (9)]). We note that the first equality in (4) can be deduced from the exponential generating function [3, Equation (11)]

$$\sum_{n=1}^{\infty} (1^k + 2^k + \cdots + n^k) \frac{x^n}{n!} = e^x \sum_{j=1}^{k+1} \frac{1}{j} \left\{ {k+1 \atop j} \right\} x^j.$$

Of course, (3) and (4) are equivalent formulas. Indeed, it is a simple exercise to convert (3) into (4), and vice versa, by means of the recursion $\left\{ {k \atop j} \right\} = j \left\{ {k-1 \atop j} \right\} + \left\{ {k-1 \atop j-1} \right\}$ and the well-known combinatorial identity $\binom{n}{j+1} + \binom{n}{j} = \binom{n+1}{j+1}$.

Incidentally, it is worthwhile to mention that, in their 1928 Monthly article [15], Ginsburg wrote down explicitly the first few instances of (4) for $k = 2, 3, 4, 5$ in terms of the binomial coefficients $\binom{n}{j+1}$, where $j = 0, 1, \ldots, k$, namely

$$S_2(n) = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3},$$

$$S_3(n) = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4},$$

$$S_4(n) = \binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5},$$

$$S_5(n) = \binom{n}{1} + 31\binom{n}{2} + 180\binom{n}{3} + 390\binom{n}{4} + 360\binom{n}{5} + 120\binom{n}{6}.$$

As noted by Ginsburg, the above formulas appeared on page 88 of the book by Schwatt, *Introduction to Operations with Series* (Philadelphia, The Press of the University of Pennsylvania, 1924).

In this paper, we obtain a unifying formula for $S_k(n)$ giving (3) and (4) as particular cases. Indeed, we derive a couple of infinite families of explicit formulas for $S_k(n)$, one of them involving the numbers $\left\{ {k \atop j} \right\}_r$ and the other the numbers $\left\{ {k \atop j} \right\}_{-r}$, with $j = 0, 1, \ldots, k$. Specifically, in Section 2, we prove the following theorem which constitutes the main result of this paper.

Theorem 1 *Let k and n be any non-negative integers and let $\left\{ {k \atop j} \right\}_r$ and $\left\{ {k \atop j} \right\}_{-r}$ be the numbers defined in (1) and (2), respectively, where r stands for any arbitrary but fixed non-negative integer. Then*

$$S_k(n) = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r, \tag{5}$$

$$S_k(n) = \sum_{j=0}^{k} j! \left[\binom{n+1+r}{j+1} - \binom{r+1}{j+1} \right] \left\{ {k \atop j} \right\}_{-r}. \tag{6}$$

As a consequence of Theorem 1, we obtain three additional formulas for $S_k(n)$ as a sum over $j = 0, 1, \ldots, k$ involving the numbers $\left\{ {k \atop j} \right\}_{n+m}$, $\left\{ {k \atop j} \right\}_{n-m}$, and $\left\{ {k \atop j} \right\}_{k-j}$ (equations (19), (20), and (21), respectively), with m being any given non-negative integer. Furthermore, in Section 3, we provide several formulas for the Bernoulli polynomials involving the generalized Stirling numbers of the second kind, the harmonic numbers, and the so-called harmonic polynomials, which are defined in [10, Equation (28)]. This will allow us to derive a formula for $S_{k-1}(n)$ in terms of $\left\{ {k \atop j} \right\}_2$, $\left\{ {k \atop j} \right\}_{n+2}$, and the harmonic numbers (equation (29)), and another one in terms of $\left\{ {k \atop j} \right\}_2$ and the above-mentioned harmonic polynomials (equation (30)). We conclude in Section 4 with some final remarks.

Before proceeding further, a few observations are in order.

Remark 1 *It should be stressed that both (5) and (6) hold irrespective of the value taken by the non-negative integer parameter r. This means that, actually, the right-hand side of (5) and (6) provides us with an infinite supply of explicit formulas for $S_k(n)$, one for each choice of r. For example, for $r = 2$, and noting that $S_k(1) = 1$ for all k, we have from (5)*

$$S_k(n) = 1 + \sum_{j=0}^{k} j! \binom{n-1}{j+1} \left\{ {k \atop j} \right\}_2,$$

where

$$\left\{ {k \atop j} \right\}_2 = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i+2)^k.$$

Analogously, for $r = 2$, we have from (6)

$$S_k(n) = -\delta_{k,0} + (-1)^{k+1}(1 + 2^k) + \sum_{j=0}^{k} j! \binom{n+3}{j+1} \left\{ {k \atop j} \right\}_{-2},$$

where

$$\left\{ {k \atop j} \right\}_{-2} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i-2)^k.$$

Remark 2 *It is easily seen that both* (5) *and* (6) *reduce to* (3) *when* $r = 0$*. Furthermore,* (5) *reduces to* (4) *when* $r = 1$*. Moreover, setting* $r = n$ *in* (5) *leads to*

$$S_k(n) = n^{k+1} + \sum_{j=1}^{k} (-1)^j j! \binom{n+j-1}{j+1} \left\{ {k \atop j} \right\}_n . \tag{7}$$

Similarly, setting $r = n+1$ *in* (5) *yields*

$$S_k(n) = \sum_{j=0}^{k} (-1)^j j! \binom{n+j}{j+1} \left\{ {k \atop j} \right\}_{n+1}, \tag{8}$$

retrieving the result obtained in [16, Equation (4.8)]. Interestingly, by performing the Stirling transform of (8)*, we obtain the convolution*

$$\sum_{j=0}^{k} (-1)^j Q_{k-j}(n) S_j(n) = k! \binom{n+k}{k+1},$$

where $Q_{k-j}(n)$ *is the following polynomial in* n *of degree* $k-j$*:*

$$Q_{k-j}(n) = \sum_{i=0}^{k-j} \binom{i+j}{j} \left[{k+1 \atop i+j+1} \right] n^i,$$

and where the $\left[{k \atop j} \right]$*'s are the (unsigned) Stirling numbers of the first kind.*

Remark 3 *By renaming* r *as* n *in equation* (18) *below, we find that*

$$S_k(n) = (-1)^k \left(-\delta_{k,0} + \sum_{j=0}^{k} j! \binom{n+1}{j+1} \left\{ {k \atop j} \right\}_{-n} \right),$$

which may be compared with (3)*.*

2 Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma.

Lemma 1 *For a real or complex variable* x*, let* $S_k(x)$ *denote the unique interpolating polynomial in* x *of degree* $k+1$ *such that* $S_k(x) = 1^k + 2^k + \cdots + x^k$ *whenever* x *is a positive integer (with* $S_k(0) = 0$*). Then,*

$$S_k(x) = S_k(a-1) + \sum_{j=0}^{k} j! \binom{x+1-a}{j+1} \left\{ {k \atop j} \right\}_a, \tag{9}$$

where a *is a parameter taking any arbitrary but fixed real or complex value.*

Proof. As is well known (see, e.g., [14, Equation (15)]), $S_k(x)$ can be expressed in terms of the Bernoulli polynomials $B_k(x)$ as follows:

$$S_k(x) = \frac{1}{k+1} \big[B_{k+1}(x+1) - B_{k+1}(1) \big], \quad k \geq 0. \tag{10}$$

Let us recall further that the forward difference operator Δ acting on the function $f(x)$ is defined by $\Delta f(x) = f(x+1) - f(x)$. Thus, the following elementary result

$$\Delta S_k(x) = (x+1)^k \tag{11}$$

follows immediately from (10) and the difference equation $\Delta B_{k+1}(x) = (k+1)x^k$ [14, Equation (12)].

On the other hand, the Newton-Gregory expansion of the function $f(x)$ is given by (see, e.g., [28, Equation (A.9), p. 230])

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a),$$

where, for any integer $j \geq 1$, the j-th order difference operator Δ^j is defined by $\Delta^j f(x) = \Delta(\Delta^{j-1} f(x)) = \Delta^{j-1}(\Delta f(x))$ and $\Delta^0 f(x) = f(x)$, and where $\Delta^j f(a) = \Delta^j f(x)|_{x=a}$. Hence, applying the Newton-Gregory expansion to the power sum polynomial $S_k(x)$ and using (11) yields

$$S_k(x) = S_k(a) + \sum_{j=0}^{k} \binom{x-a}{j+1} \Delta^j (a+1)^k, \tag{12}$$

where we have omitted the terms in the sum with index j greater than k because $\Delta^j (x+1)^k = 0$ for all $j \geq k+1$ [28, Equation (6.16), p. 68].

The connection between (12) and the generalized Stirling numbers $\left\{ {k \atop j} \right\}_x$ stems from the fact that (see, e.g., [4, Theorem 29] and [6, Equation (3.8)])

$$\left\{ {k \atop j} \right\}_x = \frac{1}{j!} \Delta^j x^k. \tag{13}$$

Thus, we obtain (9) by combining (12) and (13), and making $a \to a-1$. ■

When x and a are the non-negative integers n and r, respectively, (9) becomes

$$S_k(n) = S_k(r-1) + \sum_{j=0}^{k} j! \binom{n+1-r}{j+1} \left\{ {k \atop j} \right\}_r, \tag{14}$$

where $S_k(-1) = 0$ for all $k \geq 1$, and $S_0(-1) = -1$. Now, by letting $n = 0$ in (14) and using the relation

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k} \tag{15}$$

we obtain

$$S_k(r-1) = \sum_{j=0}^{k} (-1)^j j! \binom{r+j-1}{j+1} \left\{ {k \atop j} \right\}_r. \tag{16}$$

Hence, substituting (16) into (14), we obtain (5).

Moreover, by setting $r \to -r$ in (14) and invoking the symmetry property of the power sum polynomials (see, e.g., [26, Theorem 10])

$$S_k(-r-1) = -\delta_{k,0} + (-1)^{k+1} S_k(r),$$

we obtain

$$S_k(n) = -\delta_{k,0} - (-1)^k S_k(r) + \sum_{j=0}^{k} j! \binom{n+1+r}{j+1} \left\{ {k \atop j} \right\}_{-r}. \tag{17}$$

For $n = 0$, the last expression can be put as

$$(-1)^k S_k(r) = -\delta_{k,0} + \sum_{j=0}^{k} j! \binom{r+1}{j+1} \left\{ {k \atop j} \right\}_{-r}. \tag{18}$$

Hence, substituting (18) into (17), we obtain (6).

We conclude this section with the following implications of Theorem 1.

Remark 4 *By letting $r = n + m$ in (5), where m is any given non-negative integer, and using (15), we obtain*

$$S_k(n) = \sum_{j=0}^{k} (-1)^j j! \left[\binom{n+m+j-1}{j+1} - \binom{m+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_{n+m}. \tag{19}$$

Of course, (19) reduces to (7) and (8) when $m = 0$ and $m = 1$, respectively. Similarly, by putting $r = n - m$ in (5), where m is any given non-negative integer, we obtain

$$S_k(n) = \sum_{j=0}^{k} j! \left[\binom{m+1}{j+1} + (-1)^j \binom{n+j-m-1}{j+1} \right] \left\{ {k \atop j} \right\}_{n-m}. \tag{20}$$

Note that, when $m = n$, (20) reduces to (3).

Remark 5 *Using the relation $\left\{ {k \atop j} \right\}_{-r} = (-1)^{k-j} \left\{ {k \atop j} \right\}_{r-j}$ (see [21, Equation (2.4)]) and taking $r = k$ in (6) yields*

$$S_k(n) = \sum_{j=0}^{k} (-1)^{k-j} j! \left[\binom{n+k+1}{j+1} - \binom{k+1}{j+1} \right] \left\{ {k \atop j} \right\}_{k-j}. \tag{21}$$

Incidentally, setting $n = 1$ in (21) gives the identity

$$\sum_{j=0}^{k} (-1)^{k-j} j! \binom{k+1}{j} \left\{ {k \atop j} \right\}_{k-j} = 1.$$

3 Connection with the Bernoulli Polynomials

By using (16) in (10), we readily obtain the following formula for the Bernoulli polynomials evaluated at the non-negative integer r:

$$B_{k+1}(r) = B_{k+1}(1) + (k+1) \sum_{j=0}^{k} (-1)^j j! \binom{r+j-1}{j+1} \left\{ {k \atop j} \right\}_r. \tag{22}$$

Furthermore, making $r \to -r$ in (22) and using (15), we get the following formula for the Bernoulli polynomials evaluated at the negative integer $-r$:

$$B_{k+1}(-r) = B_{k+1}(1) - (k+1) \sum_{j=0}^{k} j! \binom{r+1}{j+1} \left\{ {k \atop j} \right\}_{-r}, \quad r \geq 0.$$

Formula (22) should be compared with the corresponding formula derived by Kargın and Çekim in [17, p. 896], namely (in our notation)

$$B_{k+1}(r) = B_{k+1} + (k+1) \sum_{j=0}^{k} (-1)^j j! \binom{r+j}{j+1} \left\{ {k \atop j} \right\}_r. \tag{23}$$

By equating the right-hand sides of (22) and (23), we further obtain the identity

$$r^k = \sum_{j=1}^{k} (-1)^{j+1} j! \binom{r+j-1}{j} \left\{ {k \atop j} \right\}_r,$$

which holds for any integers $r \geq 0$ and $k \geq 1$.

One can naturally extend the above formulas (22) and (23) in order for $B_{k+1}(r)$ to apply to any real or complex variable x as follows:

$$B_{k+1}(x) = B_{k+1}(1) + (k+1)\sum_{j=0}^{k}(-1)^j j!\binom{x+j-1}{j+1}\left\{ {k \atop j} \right\}_x$$

and

$$B_{k+1}(x) = B_{k+1} + (k+1)\sum_{j=0}^{k}(-1)^j j!\binom{x+j}{j+1}\left\{ {k \atop j} \right\}_x,$$

respectively, where

$$\left\{ {k \atop j} \right\}_x = \frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}(i+x)^k.$$

On the other hand, from [18, Equation (20)] (see also [25, p. 967]), it is known that, for all non-negative integers k, m, r,

$$B_k(m-r) = \sum_{j=0}^{k}(-1)^j j! H_{j+1}^{(r)}\left\{ {k \atop j} \right\}_m, \tag{24}$$

where $H_j^{(r)}$ is the j-th hyperharmonic number of order r defined recursively by (see, e.g., [12, p. 258])

$$H_j^{(r)} = \sum_{i=1}^{j} H_i^{(r-1)}, \ \text{for } r > 1, \quad \text{and } H_j^{(1)} = H_j,$$

where $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$ is the j-th harmonic number. Several generalizations of (24) can be found in [5], where plenty of number theoretic and combinatoric identities involving generalized Bernoulli polynomials and Stirling numbers of both kinds are established. Thus, taking $r = 1$ and letting $m = x$ in (24) gives rise to the following formula expressing the Bernoulli polynomials $B_k(x-1)$ in terms of $\left\{ {k \atop j} \right\}_x$ and the harmonic numbers:

$$B_k(x-1) = \sum_{j=0}^{k}(-1)^j j! H_{j+1}\left\{ {k \atop j} \right\}_x. \tag{25}$$

Let us note at this point that (25) also arises as a specialization of the formula

$$B_k(x) = \sum_{j=0}^{k}(-1)^j j!\left\{ {k \atop j} \right\}_r H_j(x-r+1), \tag{26}$$

where the so-called harmonic polynomials $H_j(x)$ are defined by the generating function (see [10, Equation (28)])

$$\frac{-\ln(1-t)}{t(1-t)^{1-x}} = \sum_{j=0}^{\infty} H_j(x)t^j.$$

The harmonic polynomials admit, among others, the representation (see [10, Equation (33)])

$$H_k(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{x}{k-j}H_{j+1}, \tag{27}$$

from which it follows, in particular, that $H_k(0) = H_{k+1}$. Hence, making $x \to x-1$ and $r \to x$ in (26) gives (25). Note that (26) holds for any choice of r. Specifically, for $r = 2$, we have

$$B_k(x+1) = \sum_{j=0}^{k}(-1)^j j!\left\{ {k \atop j} \right\}_2 H_j(x). \tag{28}$$

As an application of (25), we can use it, in conjunction with (10), to obtain the following formula for the power sum $S_{k-1}(n)$:

$$S_{k-1}(n) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j j! H_{j+1}\left(\left\{ {k \atop j} \right\}_{n+2} - \left\{ {k \atop j} \right\}_{2}\right), \quad k \geq 1. \tag{29}$$

Likewise, using (28) together with (10) yields

$$S_{k-1}(n) = \frac{1}{k}\sum_{j=0}^{k}(-1)^j j! \left\{ {k \atop j} \right\}_{2} \big(H_j(n) - H_{j+1}\big), \quad k \geq 1, \tag{30}$$

where $H_j(n)$ are the harmonic polynomials given in (27). Formula (30) can equally be expressed as

$$S_{k-1}(n) = \frac{1}{k}\sum_{j=0}^{k}\left\{ {k \atop j} \right\}_{2} \big(D_j(n-1) - D_j(-1)\big), \quad k \geq 1,$$

where the Daehee polynomials $D_k(x)$ are defined by the generating function (see, e.g., [19])

$$\left(\frac{\ln(1+t)}{t}\right)(1+t)^x = \sum_{k=0}^{\infty} D_k(x)\frac{t^k}{k!}.$$

4 Concluding Remarks

Equation (14) above can be written in the equivalent form

$$S_k(n+r) - S_k(r-1) = \sum_{j=0}^{k} j! \binom{n+1}{j+1}\left\{ {k \atop j} \right\}_{r}, \tag{31}$$

which applies to any non-negative integers k, n, r. As it turns out, (31) can be obtained as a particular case of [2, Theorem 2.1]. The object of this theorem concerns the sum of the k-th powers of the first $(n+1)$-terms of the general arithmetic sequence

$$S_{k,(a,d)}(n) = a^k + (a+d)^k + \cdots + (a+nd)^k,$$

where k and n are non-negative integers and a and d are complex numbers with $d \neq 0$. According to [2, Theorem 2.1], $S_{k,(a,d)}(n)$ can be expressed in terms of the generalized Stirling numbers of the second kind as follows (in our notation):

$$S_{k,(a,d)}(n) = d^k \sum_{j=0}^{k} j! \binom{n+1}{j+1}\left\{ {k \atop j} \right\}_{a/d}, \tag{32}$$

where

$$\left\{ {k \atop j} \right\}_{a/d} = \frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}\left(i + \frac{a}{d}\right)^k.$$

In particular, taking $d = 1$ and assuming that a is the non-negative integer r, (32) becomes

$$r^k + (r+1)^k + \cdots + (r+n)^k = \sum_{j=0}^{k} j! \binom{n+1}{j+1}\left\{ {k \atop j} \right\}_{r},$$

which is just (31). For completeness' sake, let us remind that $S_{k,(a,d)}(n)$ can alternatively be expressed in terms of the Bernoulli polynomials as follows (see, e.g., [2, Equation (2)] and [8, Equation (16)]):

$$S_{k,(a,d)}(n) = \frac{d^k}{k+1}\Big[B_{k+1}\Big(n + \frac{a}{d} + 1\Big) - B_{k+1}\Big(\frac{a}{d}\Big)\Big],$$

which reduces to (10) for $a = d = 1$ and $n \to n - 1$.

We conclude by quoting the following formula for $S_k(n)$ involving the (unsigned) Stirling numbers of the first and second kind $\left[{k \atop j}\right]$ and $\left\{{k \atop j}\right\}$:

$$S_k(n) = \sum_{j=1}^{k} (-1)^{j-1} j \left[{n+1 \atop n+1-j}\right] \left\{{n+k-j \atop n}\right\}, \quad k \geq 1. \tag{33}$$

Formula (33) was derived by Merca [22] by manipulating the formal power series for the Stirling numbers. It can also be obtained starting from the Newton-Girard identities ([9, Exercise 2]). Actually, formula (33) is a special case of an identity connecting the power sum symmetric functions $p_m(x_1, x_2, \ldots, x_n)$ with the elementary symmetric functions $\sigma_m(x_1, x_2, \ldots, x_n)$ and the complete homogenous symmetric functions $h_m(x_1, x_2, \ldots, x_n)$, namely

$$p_k(x_1, x_2, \ldots, x_n) = \sum_{m=1}^{k} (-1)^{m-1} m \sigma_m(x_1, x_2, \ldots, x_n) h_{k-m}(x_1, x_2, \ldots, x_n), \tag{34}$$

which holds for all $k \geq 1$ (see, e.g., [13, Proposition 3.2] and [23, Lemma 2.1]). Formula (33) is then obtained from (34) when $x_i = i$ for all $i = 1, 2, \ldots, n$.

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