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k-Fractional Integral Inequalities Of Hadamard Type For Strongly Exponentially $(\alpha, h-m)$-Convex Functions*

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Abstract

In this paper, we present Hadamard type fractional integral inequalities by using k-analogue of Riemann Liouville (RL) fractional integrals. These inequalities are obtained by using a general class of functions called strongly exponentially $(\alpha, h-m)$-convex functions. Error estimates are also established for differentiable functions applying some well known identities.

1 Introduction

A convex function is usually defined as follows:

Definition 1 *A function $f: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be convex function if*

$$f(rx + (1-r)y) \leq rf(x) + (1-r)f(y) \tag{1}$$

holds for all $x, y \in I$ and $r \in [0, 1]$.

The Hadamard inequality is the geometric visualization of convex function which is stated in the following theorem:

Theorem 1 ([25]) *If $f: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

The above inequality is studied very frequently. For some recent versions of this inequality, we refer the researchers to [3, 4, 7, 8, 11, 12, 13, 14, 15, 25, 17].

Convex functions are very useful for the establishment of very known and vital inequalities. An important and significant generalization of convex function is strongly exponentially $(\alpha, h-m)$-convex function. This generalized form of convexity is given in the following definition:

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Definition 2 ([29]) *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $\psi : [0,b] \to \mathbb{R}$ is called strongly exponentially $(\alpha, h-m)$-convex function, if ψ is non-negative and for all $x, y \in [0,b]$, $z \in (0,1)$, $\eta \in \mathbb{R}$ and $(\alpha, m) \in [0,1]^2$, $C \geq 0$ one has*

$$\psi(zx + m(1-z)y) \leq h(z^\alpha)\frac{\psi(x)}{e^{\eta x}} + mh(1-z^\alpha)\frac{\psi(y)}{e^{\eta y}} - \frac{mC}{e^{\eta(x+y)}}h(z^\alpha)h(1-z^\alpha)|y-x|^2. \quad (3)$$

Remark 1 *By selecting suitable function h and particular value of parameters m and η, the above definition produces the functions comprise in the following remarks:*

(i) By setting $C = 0$, $\eta = 0$, $(\alpha, h-m)$-convex function [7] can be obtained.

(ii) By setting $C = 0$, $\eta = 0$ and $\alpha = 1$, $(h-m)$-convex function [20] can be obtained.

(iii) By setting $C = 0$, $\eta = 0$ and $h(t^\alpha) = t^\alpha$, (α, m)-convex function [18] can be obtained.

(iv) By setting $C = 0$, $\eta = 0$, $\alpha = 1$ and $m = 1$, h-convex function [28] can be obtained.

(v) By setting $C = 0$, $\eta = 0$, $\alpha = 1$ and $h(t) = t$, m-convex function [26] can be obtained.

(vi) By setting $C = 0$, $\eta = 0$, $\alpha = 1$, $m = 1$ and $h(t) = t$, convex function [24] can be obtained.

(vii) By setting $C = 0$, $\eta = 0$, $m = 1$, $\alpha = 1$ and $h(t) = 1$, p-function [6] can be obtained.

(viii) By setting $C = 0$, $\alpha = 1$ and $h(t) = t^s$, exponentially (s, m)-convex function [22] can be obtained.

(ix) By setting $C = 0$, $\alpha = 1$, $m = 1$ and $h(t) = t^s$, exponentially s-convex function [19] can be obtained.

(x) By setting $C = 0$, $\alpha = 1$, and $h(t) = t$, exponentially m-convex function [23] can be obtained.

(xi) By setting $C = 0$, $\alpha = 1$, $m = 1$ and $h(t) = t$, exponentially convex function [2] can be obtained.

(xii) By setting $C = 0$, $\eta = 0$, $\alpha = 1$ and $h(t) = t^s$, (s, m)-convex function [1] can be obtained.

(xiii) By setting $C = 0$, $\alpha = 1$, $\eta = 0$, $m = 1$ and $h(t) = t^s$, s-convex function [19] can be obtained.

(xiv) By setting $C = 0$, $\alpha = 1$, $\eta = 0$, $m = 1$ and $h(t) = \frac{1}{t}$, Godunova-Levin function [10] can be obtained.

(xv) By setting $C = 0$, $\alpha = 1$, $\eta = 0$, $m = 1$ and $h(t) = \frac{1}{t^s}$, s-Godunova-Levin function [5] of second kind can be obtained.

The aim of this paper is to give a version of the Hadamard inequality for generalized class of convex function namely strongly exponentially $(\alpha, h-m)$-convex functions by using k-fractional integral operators. Fractional integral operators are very useful in mathematical inequalities. The following definition gives the left as well as right Riemann Liouville (RL) fractional integral operators:

Definition 3 *Let $f \in L_1[a,b]$. The left and right RL fractional integrals $I_{a^+}^\alpha f$ and $I_{b^-}^\alpha f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ of f are defined by*

$$I_{a^+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{1-\alpha} f(t)dt, \quad x > a,$$

and

$$I_{b^-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{1-\alpha} f(t)dt, \quad x < b$$

respectively. Here $\Gamma(\alpha)$ is the Euler's gamma function and $I_{a^+}^0 f(x) = I_{b^-}^0 f(x) = f(x)$.

RL fractional integral operators have been generalized in many ways. In [16], Mubeen and Habibullah gave Riemann Liouville k-fractional integrals as follows:

Definition 4 *Let $f \in L_1[a,b]$. The left and right RL k-fractional integrals $I_{a^+}^{\alpha,k} f$ and $I_{b^-}^{\alpha,k} f$ of order $\alpha \in \mathbb{C}$, $(\alpha) > 0$ of f are defined by*

$$I_{a^+}^{\alpha,k} f(x) := \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

and

$$I_{b^-}^{\alpha,k} f(x) := \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b$$

where $\Gamma_k(\alpha) = \int_0^\infty w^{\alpha-1} e^{\frac{-w^k}{k}} dw$, is the k-gamma function and $I_{a^+}^{0,1} f(x) = I_{b^-}^{0,1} f(x) = f(x)$.

The rest of the article is organized in the following manner. In Section 2, we prove k-fractional integral inequality of Hadamard type for strongly exponentially $(\alpha, h-m)$-convex functions and deduce some related results. In Section 3, we prove a version of k-fractional integral inequality of Hadamard type for differentiable functions f such that $|f'|$ is strongly exponentially $(\alpha, h-m)$-convex. In Section 4, we prove k-fractional integral inequality of Hadamard type for product of two strongly exponentially $(\alpha, h-m)$-convex functions.

2 Main Results

In the following we give k-fractional integral inequalities of Hadamard type for strongly exponentially $(\alpha, h-m)$-convex functions.

Theorem 2 *Let $f : [0,\infty) \to \mathbb{R}$ be a strongly exponentially $(\alpha, h-m)$-convex function with modulus $C \geq 0$, $(\alpha, m) \in [0,1]^2$ and $\eta \in \mathbb{R}$. Also let $f \in L_1[a,b]$, $a, b \in [0,\infty)$ and if $h(x+y) \leq h(x)h(y)$, then we will have*

$$\begin{aligned}
& \frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}} [I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a) \\
\leq \quad & \left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}} \right) \int_0^1 t^{\frac{\alpha}{k}-1} h(t^\alpha) dt + m \left(\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}} + \frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}} \right) \int_0^1 t^{\frac{\alpha}{k}-1} h(1-t^\alpha) dt \\
& - \frac{2mCkh(1)|b-a|^2}{\alpha e^{\eta(a+b)}} \\
\leq \quad & \frac{1}{(\frac{\alpha}{k}p - p + 1)^{\frac{1}{p}}} \left[\left(\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}} \right) \left(\int_0^1 (h(t^\alpha))^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + m \left(\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}} + \frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}} \right) \left(\int_0^1 (h(1-t^\alpha))^q dt \right)^{\frac{1}{q}} \right] - \frac{2mCkh(1)|b-a|^2}{\alpha e^{\eta(a+b)}}, \qquad (4)
\end{aligned}$$

where $p^{-1} + q^{-1} = 1$ and $p > 1$.

Proof. Since f is strongly exponentially $(\alpha, h-m)$-convex on $[a,b]$, then for $(\alpha, m) \in [0,1]^2$ and $t \in [0,1]$, we have

$$\begin{aligned}
& f(ta + (1-t)b) + f((1-t)a + tb) \\
\leq \quad & h(t^\alpha) \left[\frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}} \right] + mh(1-t^\alpha) \left[\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}} + \frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}} \right] - \frac{2mCh(t^\alpha)h(1-t^\alpha)|b-a|^2}{e^{\eta(a+b)}}
\end{aligned}$$

from which by multiplying both sides with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we will have

$$\int_0^1 t^{\frac{\alpha}{k}-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt$$

$$\leq \left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)dt+m\left[\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^\alpha)dt$$
$$-\int_0^1 \frac{2mC|b-a|^2}{e^{\eta(a+b)}}h(t^\alpha)h(1-t^\alpha)t^{\frac{\alpha}{k}-1}dt.$$

By changing variables and putting the condition $h(x+y)\leq h(x)h(y)$, we will have

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]$$
$$\leq \left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)dt+m\left[\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^\alpha)dt$$
$$-\frac{2mCkh(1)|b-a|^2}{\alpha e^{\eta(a+b)}}. \tag{5}$$

Which completes the proof of first inequality in (4). The second inequality in (4) follows by using the following inequalities via Holder's inequality

$$\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)dt\leq\frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t^\alpha))^qdt\right)^{\frac{1}{q}},$$

$$\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^\alpha)dt\leq\frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(1-t^\alpha))^qdt\right)^{\frac{1}{q}}.$$

Thus from (5) we will get (4). ■

Corollary 1 *The following inequality holds for $(\alpha,h-m)$-convex function via RL k-fractional integral*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)$$
$$\leq (f(a)+f(b))\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)dt+m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^\alpha)dt$$
$$\leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left[(f(a)+f(b))\left(\int_0^1(h(t^\alpha))^qdt\right)^{\frac{1}{q}}\right.$$
$$\left.+m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\left(\int_0^1(h(1-t^\alpha))^qdt\right)^{\frac{1}{q}}\right]. \tag{6}$$

Proof. By setting $C=0$ and $\eta=0$ in inequality (4) of Theorem 2, we get above inequality (6). ■

Corollary 2 *The following inequality holds for $(h-m)$-convex functions via RL k-fractional integrals*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)$$
$$\leq (f(a)+f(b))\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt+m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt$$
$$\leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t))^qdt\right)^{\frac{1}{q}}\left[(f(a)+f(b))+m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\right]. \tag{7}$$

Proof. By setting $C=0$, $\eta=0$ and $\alpha=1$ in inequality (4) of Theorem 2, we get above inequality (7) which is given in [7, Theorem 2.4] ■

Corollary 3 *The following inequality holds for h-convex function via RL fractional integrals*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[I_{a^+}^{\alpha}f(b)+I_{b^-}^{\alpha}f(a)\right] \\
\leq\ & [f(a)+f(b)]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt \leq \frac{2[f(a)+f(b)]}{(\alpha p-p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}}. \qquad (8)
\end{aligned}$$

Proof. By setting $C=0$, $\eta=0$, $k=1$, $m=1$ and $\alpha=1$ in inequality (4) of Theorem 2, we get the above inequality (8) which is given in [27, Theorem 2.1]. ■

Corollary 4 *The following inequality holds for exponentially (s,m)-convex function via RL k-fractional integral*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)\right] \\
\leq\ & \left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]\frac{k}{\alpha+ks}+m\left[\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right]\beta\left(\frac{\alpha}{k},s+1\right) \\
\leq\ & \left[\frac{f(a)}{e^{\eta a}}+\frac{f(b)}{e^{\eta b}}\right]\frac{k}{\alpha+ks}+\left[\frac{f\left(\frac{b}{m}\right)}{e^{\frac{\eta b}{m}}}+\frac{f\left(\frac{a}{m}\right)}{e^{\frac{\eta a}{m}}}\right]\frac{m}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}. \qquad (9)
\end{aligned}$$

Proof. By setting $C=0$, $\alpha=1$ and $h\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^s$ in inequality (4) of Theorem 2, we get above inequality (9) which is given in [9, Theorem 1]. ■

Corollary 5 *The following inequality holds for (s,m)-convex function via RL k-fractional integral*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)\right] \\
\leq\ & [f(a)+f(b)]\frac{k}{\alpha+ks}+m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\beta\left(\frac{\alpha}{k},s+1\right) \\
\leq\ & [f(a)+f(b)]\frac{k}{\alpha+ks}+\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\frac{m}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}. \qquad (10)
\end{aligned}$$

Proof. By setting $C=0$, $\eta=0$, $\alpha=1$ and $h\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^s$ in inequality (4) of Theorem 2, we get above inequality (10) which is given in [9, Theorem 4]. ■

3 Error Estimations

In the following error estimates of k-fractional Hadamard type inequalities for strongly exponentially $(\alpha, h-m)$-convex functions in terms of their first derivatives are obtained. For next result we will use the following lemma.

Lemma 1 ([7]) *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on interval (a,mb) with $a<mb$. If $f'\in L_1[a,mb]$, then for k-fractional integrals we will have*

$$\begin{aligned}
&\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)\right] \\
=\ & \frac{mb-a}{2}\int_0^1[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}]f'(m(1-t)b+ta)dt.
\end{aligned}$$

Theorem 3 *Let $f:[a,mb]\to\mathbb{R}$ be a function such that $[a,mb]\subseteq[0,\infty)$ and $f\in L_1[a,mb]$. If $|f'|$ is a strongly exponentially $(\alpha,h-m)$-convex function with modulus $C\geq 0$, $(\alpha,m)\in[0,1]^2$, $\eta\in\mathbb{R}$ and $h^q\in[0,1]$, $q>1$. If $h(x+y)\leq h(x)h(y)$, then for RL k-fractional integrals we have*

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{a^+}f(mb)+I^{\alpha,k}_{mb^-}f(a)]\right| \\
\leq\ & \frac{mb-a}{2}\left\{\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\right. \\
&\times\left[\frac{|f'(a)|}{e^{\eta a}}\left(\left[\int_0^{\frac{1}{2}}(h(t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(t^\alpha))^q dt\right]^{\frac{1}{q}}\right)+\frac{m|f'(b)|}{e^{\eta b}}\right. \\
&\left.\left.\left(\left[\int_0^{\frac{1}{2}}(h(1-t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(1-t^\alpha))^q dt\right]^{\frac{1}{q}}\right)\right]-\frac{2mC|b-a|^2}{e^{\eta(a+b)}}\left[\frac{2^{\frac{\alpha}{k}}-1}{2^{\frac{\alpha}{k}}(\frac{\alpha}{k}+1)}\right]\right\} \qquad (11)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the property of modulus from Lemma 1, we will get

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{a^+}f(mb)+I^{\alpha,k}_{mb^-}f(a)]\right| \\
\leq\ & \frac{mb-a}{2}\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right||f'(m(1-t)b+ta)|dt.
\end{aligned}$$

By strongly exponentially $(\alpha,h-m)$-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{a^+}f(mb)+I^{\alpha,k}_{mb^-}f(a)]\right| \\
\leq\ & \frac{mb-a}{2}\int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}]\left[mh(1-t^\alpha)\frac{|f'(b)|}{e^{\eta b}}+h(t^\alpha)\frac{|f'(a)|}{e^{\eta a}}-\frac{mCh(t^\alpha)h(1-t^\alpha)|b-a|^2}{e^{\eta(a+b)}}\right]dt \\
&+\int_{\frac{1}{2}}^1[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}]\left[mh(1-t^\alpha)\frac{|f'(b)|}{e^{\eta b}}+h(t^\alpha)\frac{|f'(a)|}{e^{\eta a}}-\frac{mCh(t^\alpha)h(1-t^\alpha)|b-a|^2}{e^{\eta(a+b)}}\right]dt \\
=\ & \frac{mb-a}{2}\left\{\frac{|f'(a)|}{e^{\eta a}}\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(t^\alpha)dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(t^\alpha)dt\right]\right. \\
&+\frac{m|f'(b)|}{e^{\eta b}}\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(1-t^\alpha)dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(1-t^\alpha)dt\right] \\
&-\frac{mC|b-a|^2}{e^{\eta(a+b)}}\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt\right] \\
&+\frac{|f'(a)|}{e^{\eta a}}\left[\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(t^\alpha)dt-\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}h(t^\alpha)dt\right] \\
&+\frac{m|f'(b)|}{e^{\eta b}}\left[\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(1-t^\alpha)dt-\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}h(1-t^\alpha)dt\right] \\
&\left.-\frac{mC|b-a|^2}{e^{\eta(a+b)}}\left[\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt-\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt\right]\right\}. \qquad (12)
\end{aligned}$$

Now by using Holder inequality in the right hand side of (12), we will get

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)]\right|\\
\leq\ &\frac{mb-a}{2}\left\{\frac{|f'(a)|}{e^{\eta a}}\left[\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left[\int_0^{\frac{1}{2}}[h(t^\alpha)]^q dt\right]^{\frac{1}{q}}\right.\right.\\
&+\left.\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left[\int_{\frac{1}{2}}^1[h(t^\alpha)]^q dt\right]^{\frac{1}{q}}\right]\\
&-\frac{mC|b-a|^2}{e^{\eta(a+b)}}\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt-t^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt\right]\\
&+m\frac{|f'(b)|}{e^{\eta b}}\left[\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left[\int_0^{\frac{1}{2}}[h(1-t^\alpha)]^q dt\right]^{\frac{1}{q}}\right.\\
&+\left.\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left[\int_{\frac{1}{2}}^1[h(1-t^\alpha)]^q dt\right]^{\frac{1}{q}}\right]\\
&\left.-\frac{mC|b-a|^2}{e^{\eta(a+b)}}\left[\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt-(1-t)^{\frac{\alpha}{k}}h(t^\alpha)h(1-t^\alpha)dt\right]\right\}.
\end{aligned}$$

By putting the condition $h(x+y)\leq h(x)h(y)$ and after some calculation one can get inequality (11). ■

Corollary 6 *The following inequality holds for exponentially $(\alpha,h-m)$-convex function via RL fractional integrals*

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[I_{a^+}^{\alpha}f(mb)+I_{mb^-}^{\alpha}f(a)]\right|\\
\leq\ &\frac{mb-a}{2}\left(\left[\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\right)\\
&\times\left[\frac{|f'(a)|}{e^{\eta a}}\left(\left[\int_0^{\frac{1}{2}}(h(t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(t^\alpha))^q dt\right]^{\frac{1}{q}}\right)\right.\\
&\left.+\frac{m|f'(b)|}{e^{\eta b}}\left(\left[\int_0^{\frac{1}{2}}(h(1-t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(1-t^\alpha))^q dt\right]^{\frac{1}{q}}\right)\right].
\end{aligned}\tag{13}$$

Proof. By setting $C=0$, $k=1$ in inequality (11) of Theorem 3, we get the above inequality (13). ■

Corollary 7 *The following inequality holds for $(\alpha,h-m)$-convex function via RL k-fractional integrals*

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)]\right|\\
\leq\ &\frac{mb-a}{2}\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\\
&\times\left[|f'(a)|\left(\left[\int_0^{\frac{1}{2}}(h(t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(t^\alpha))^q dt\right]^{\frac{1}{q}}\right)\right.
\end{aligned}$$

$$+ m|f'(b)|\left(\left[\int_0^{\frac{1}{2}}(h(1-t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^{1}(h(1-t^\alpha))^q dt\right]^{\frac{1}{q}}\right)\Bigg]. \tag{14}$$

Proof. By setting $C=0$, $\eta=0$ in inequality (11) of Theorem 3, we get the above inequality (14). ■

Corollary 8 *The following inequality holds for $(h-m)$-convex function via RL k-fractional integrals*

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)]\right|\\ \le\ &\frac{(mb-a)\left[|f'(a)|+m|f'(b)|\right]}{2}\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\\ &\times\left[\left(\left[\int_0^{\frac{1}{2}}(h(t^\alpha))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^{1}(h(t^\alpha))^q dt\right]^{\frac{1}{q}}\right)\right].\end{aligned} \tag{15}$$

Proof. By setting $C=0$, $\eta=0$ and $\alpha=1$ in inequality (11) of Theorem 3, we get the above inequality (15). ■

Corollary 9 *The following inequality holds for exponentially (s,m)-convex function via RL k-fractional integrals*

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)]\right|\\ \le\ &\frac{(mb-a)\left(\frac{m|f'(b)|}{e^{\eta b}}+\frac{|f'(a)|}{e^{\eta a}}\right)}{2}\left\{\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right.\\ &\left.\left.+\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\right\}.\end{aligned} \tag{16}$$

Proof. By setting $C=0$, $\alpha=1$ and $h(t)=t^s$ in inequality (11) of Theorem 3, we get the above inequality (16) which is given in [9, Theorem 2]. ■

Corollary 10 *The following inequality holds for exponentially (s,m)-convex function via RL k-fractional integrals*

$$\begin{aligned}&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)]\right|\\ \le\ &\frac{(mb-a)\left(m|f'(b)|-|f'(a)|\right)}{2}\left\{\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right.\\ &\left.\left.+\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\right\}.\end{aligned} \tag{17}$$

Proof. By setting $C=0$, $\eta=0$, $\alpha=1$ and $h(t)=t^s$ in inequality (11) of Theorem 3, we get the above inequality (17) which is given in [9, Theorem 5]. ■

4 RL k-Fractional Integral Inequalities for Product of Two Functions

Now we obtain some Hadamard type inequalities for product of two exponentially $(\alpha, h-m)$-convex functions via RL k-fractional integrals.

Theorem 4 *Let $f, g : [0,\infty) \to \mathbb{R}$ be functions such that $fg \in L_1[a,b]$, $a, b \in [0,\infty)$, $a < b$. Also, let function f is strongly exponentially $(\alpha, h-m_1)$-convex and function g is strongly exponentially $(\alpha, h-m_2)$-convex on $[0,\infty)$ with modulus $C \geq 0$ and $(\alpha, m_1), (\alpha, m_2) \in (0,1]^2$. If $h(x+y) \leq h(x)h(y)$, then the following inequalities hold for RL k-fractional integrals*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k} f(b)g(b) + I_{b^-}^{\alpha,k} f(a)g(a)\right] \\
\leq\; & \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt \\
&+\left\{m_2\left[\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}} + \frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}\right] + m_1\left[\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}} + \frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]\right\}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt \\
&+m_1m_2\left[\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}} + \frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt \\
&-\frac{m_1m_2C|b-a|^2}{e^{\eta(a+b)}}\left[\frac{f\left(\frac{b}{m_1}\right)+g\left(\frac{b}{m_2}\right)}{e^{\eta\left(\frac{b}{m_1}+\frac{b}{m_2}\right)}} + \frac{f\left(\frac{a}{m_1}\right)+g\left(\frac{a}{m_2}\right)}{e^{\eta\left(\frac{a}{m_1}+\frac{a}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h^2(1-t^\alpha)dt \\
&-C|b-a|^2\left[\frac{m_1g(a)+m_2f(a)}{e^{\eta(2a+b)}} + \frac{m_1g(b)+m_2f(b)}{e^{\eta(2b+a)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h(1-t^\alpha)dt \\
&+\frac{2m_1m_2C^2|b-a|^4}{e^{\eta(2a+2b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h^2(1-t^\alpha)dt \\
\leq\; & \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)dt\right)^{\frac{1}{q}}\left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right]\right. \\
&+\left[\frac{m_2f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}} + \frac{m_1g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}} + \frac{m_2f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}} + \frac{m_1g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 h^{2q}(1-t^\alpha)dt\right)^{\frac{1}{q}}\left[m_1m_2\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}} + \frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\right]\right\} + \frac{2m_1m_2C^2kh^2(1)|b-a|^4}{\alpha e^{\eta(2a+2b)}} \\
&-\frac{m_1m_2Ch^2(1)|b-a|^2}{e^{\eta(a+b)}}\left[\frac{f\left(\frac{b}{m_1}\right)+g\left(\frac{b}{m_2}\right)}{e^{\eta\left(\frac{b}{m_1}+\frac{b}{m_2}\right)}} + \frac{f\left(\frac{a}{m_1}\right)+g\left(\frac{a}{m_2}\right)}{e^{\eta\left(\frac{a}{m_1}+\frac{a}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^\alpha)dt \\
&-Ch^2(1)|b-a|^2\left[\frac{m_1g(a)+m_2f(a)}{e^{\eta(2a+b)}} + \frac{m_1g(b)+m_2f(b)}{e^{\eta(2b+a)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1+t^\alpha)dt, \qquad (18)
\end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since function f is strongly exponentially $(\alpha, h-m_1)$-convex and function g is exponentially $(\alpha, h-m_2)$-convex, for $t \in [0,1]$, we have

$$f(ta + (1-t)b)g(ta + (1-t)b)$$

$$\begin{aligned}
\leq\ & h^2(t^\alpha)\frac{f(a)g(a)}{e^{2\eta a}}+m_2h(t^\alpha)h(1-t^\alpha)\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}-m_2Ch^2(t^\alpha)h(1-t^\alpha)\frac{f(a)|b-a|^2}{e^{\eta(2a+b)}}\\
&+m_1h(t^\alpha)h(1-t^\alpha)\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}+m_1m_2h^2(1-t^\alpha)\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\\
&-m_1m_2Ch(t^\alpha)h^2(1-t^\alpha)\frac{f\left(\frac{b}{m_1}\right)|b-a|^2}{e^{\eta(a+b+\frac{b}{m_1})}}-m_1Ch^2(t^\alpha)h(1-t^\alpha)\frac{g(a)|b-a|^2}{e^{\eta(2a+b)}}\\
&-m_1m_2Ch(t^\alpha)h^2(1-t^\alpha)\frac{g\left(\frac{b}{m_2}\right)|b-a|^2}{e^{\eta(a+b+\frac{b}{m_2})}}+m_1m_2C^2h^2(t^\alpha)h^2(1-t^\alpha)\frac{|b-a|^4}{e^{\eta(2a+2b)}}.
\end{aligned}$$

By multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)b)g(ta+(1-t)b)dt\\
\leq\ & \frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt+\frac{m_2f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt\\
&-\frac{m_2Cf(a)|b-a|^2}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h(1-t^\alpha)dt\\
&+\frac{m_1g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt+\frac{m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt\\
&-\frac{m_1m_2Cf\left(\frac{b}{m_1}\right)|b-a|^2}{e^{\eta(a+b+\frac{b}{m_1})}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h^2(1-t^\alpha)dt-\frac{m_1Cg(a)|b-a|^2}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h(1-t^\alpha)dt\\
&-\frac{m_1m_2Cg\left(\frac{b}{m_2}\right)|b-a|^2}{e^{\eta(a+b+\frac{b}{m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h^2(1-t^\alpha)dt+\frac{m_1m_2C^2|b-a|^4}{e^{\eta(2a+2b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h^2(1-t^\alpha)dt.
\end{aligned}$$

By substituting $z=ta+(1-t)b$ in left hand side of above inequality, we get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}I_{a^+}^{\alpha,k}f(b)g(b)\\
\leq\ & \frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt+\frac{m_2f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt\\
&+\frac{m_1g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt+\frac{m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt\\
&-\frac{m_1m_2C|b-a|^2}{e^{\eta(a+b)}}\left[\frac{f\left(\frac{b}{m_1}\right)+g\left(\frac{b}{m_2}\right)}{e^{\eta\left(\frac{b}{m_1}+\frac{b}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h^2(1-t^\alpha)dt\\
&-\frac{C|b-a|^2\left[m_1g(a)+m_2f(a)\right]}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h(1-t^\alpha)dt\\
&+\frac{m_1m_2C^2|b-a|^4}{e^{\eta(2a+2b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h^2(1-t^\alpha)dt.
\end{aligned}$$

By using Holder inequality we will have

$$\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt \leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1 h^{2q}(t^\alpha)dt\right)^{\frac{1}{q}},$$

$$\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt \leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q dt\right)^{\frac{1}{q}},$$

$$\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt \leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1 h^{2q}(1-t^\alpha)dt\right)^{\frac{1}{q}}.$$

Thus we will get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}I_{a^+}^{\alpha,k}f(b)g(b)\\
\leq\;& \frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt + \frac{m_2 f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt\\
&+\frac{m_1 g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt + \frac{m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt\\
&-\frac{m_1m_2C|b-a|^2}{e^{\eta(a+b)}}\left[\frac{f\left(\frac{b}{m_1}\right)+g\left(\frac{b}{m_2}\right)}{e^{\eta\left(\frac{b}{m_1}+\frac{b}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h^2(1-t^\alpha)dt\\
&-\frac{C|b-a|^2\left[m_1g(a)+m_2f(a)\right]}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h(1-t^\alpha)dt\\
&+\frac{m_1m_2C^2|b-a|^4}{e^{\eta(2a+2b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h^2(1-t^\alpha)dt\\
\leq\;& \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)dt\right)^{\frac{1}{q}}\frac{f(a)g(a)}{e^{2\eta a}}+\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q dt\right)^{\frac{1}{q}}\right.\\
&\left.\left[\frac{m_2f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}+m_1\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\right]+\left(\int_0^1 h^{2q}(1-t^\alpha)dt\right)^{\frac{1}{q}}\frac{m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\right\}\\
&+\frac{m_1m_2C^2kh^2(1)|b-a|^4}{\alpha e^{\eta(2a+2b)}}-\frac{m_1m_2Ch^2(1)|b-a|^2}{e^{\eta(a+b)}}\left[\frac{f\left(\frac{b}{m_1}\right)+g\left(\frac{b}{m_2}\right)}{e^{\eta\left(\frac{b}{m_1}+\frac{b}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^\alpha)dt\\
&-\frac{Ch^2(1)|b-a|^2\left[m_1g(a)+m_2f(a)\right]}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(1+t^\alpha)dt. \qquad (19)
\end{aligned}$$

Similarly by changing the roles of a and b, after a little computation one can get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}I_{b^-}^{\alpha,k}f(a)g(a)\\
\leq\;& \frac{f(b)g(b)}{e^{2\eta b}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt + \frac{m_2 f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt\\
&+\frac{m_1 g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt + \frac{m_1m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt
\end{aligned}$$

$$
\begin{aligned}
&-\frac{m_1 m_2 C|a-b|^2}{e^{\eta(b+a)}}\left[\frac{f\left(\frac{a}{m_1}\right)+g\left(\frac{a}{m_2}\right)}{e^{\eta\left(\frac{a}{m_1}+\frac{a}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h^2(1-t^\alpha)dt \\
&-\frac{C|a-b|^2\left[m_1 g(b)+m_2 f(b)\right]}{e^{\eta(2b+a)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h(1-t^\alpha)dt \\
&+\frac{m_1 m_2 C^2|a-b|^4}{e^{\eta(2b+2a)}}\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)h^2(1-t^\alpha)dt \\
\leq\ & \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)dt\right)^{\frac{1}{q}}\frac{f(b)g(b)}{e^{2\eta b}}+\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q dt\right)^{\frac{1}{q}}\right. \\
&\left.\left[m_2\frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}+m_1\frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]+\left(\int_0^1 h^{2q}(1-t^\alpha)dt\right)^{\frac{1}{q}} m_1 m_2\frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1 m_2})}}\right\} \\
&+\frac{m_1 m_2 C^2 k h^2(1)|a-b|^4}{\alpha e^{\eta(2b+2a)}}-\frac{m_1 m_2 C h^2(1)|a-b|^2}{e^{\eta(b+a)}}\left[\frac{f\left(\frac{a}{m_1}\right)+g\left(\frac{a}{m_2}\right)}{e^{\eta\left(\frac{a}{m_1}+\frac{a}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^\alpha)dt \\
&-\frac{Ch^2(1)|a-b|^2\left[m_1 g(a)+m_2 f(b)\right]}{e^{\eta(2b+a)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(1+t^\alpha)dt. \qquad (20)
\end{aligned}
$$

Adding (19) and (20), we get the required result. ■

Corollary 11 *The following inequality holds for exponentially $(\alpha, h-m)$-convex function via RL fractional integrals*

$$
\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[I_{a^+}^\alpha f(b)g(b)+I_{b^-}^\alpha f(a)g(a)\right] \\
\leq\ & \left[\frac{f(a)g(a)}{e^{2\eta a}}+\frac{f(b)g(b)}{e^{2\eta b}}\right]\int_0^1 t^{\alpha-1}h^2(t^\alpha)dt \\
&+\left\{m_2\left[\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}+\frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}\right]+m_1\left[\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}+\frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]\right\}\int_0^1 t^{\alpha-1}h(t^\alpha)h(1-t^\alpha)dt \\
&+m_1 m_2\left[\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1 m_2})}}+\frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1 m_2})}}\right]\int_0^1 t^{\alpha-1}h^2(1-t^\alpha)dt \\
\leq\ & \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)dt\right)^{\frac{1}{q}}\left[\frac{f(a)g(a)}{e^{2\eta a}}+\frac{f(b)g(b)}{e^{2\eta b}}\right]\right. \\
&+\left[\frac{m_2 f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}+\frac{m_1 g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}+\frac{m_2 f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}+\frac{m_1 g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 h^{2q}(1-t^\alpha)dt\right)^{\frac{1}{q}}\left[m_1 m_2\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1 m_2})}}+\frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1 m_2})}}\right]\right\}. \qquad (21)
\end{aligned}
$$

Proof. By setting $C=0$ and $k=1$ in inequality (18) of Theorem 4, we get the above inequality (21). ■

Corollary 12 *The following inequality holds for $(\alpha, h-m)$-convex function via RL k-fractional integrals*

$$
\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k} f(b)g(b)+I_{b^-}^{\alpha,k} f(a)g(a)\right]
$$

$$
\begin{aligned}
\leq\ & [f(a)g(a)+f(b)g(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t^\alpha)dt \\
&+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]+m_1\left[g(a)f\left(\frac{b}{m_1}\right)+g(b)f\left(\frac{a}{m_1}\right)\right]\right\}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^\alpha)h(1-t^\alpha)dt \\
&+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t^\alpha)dt \\
\leq\ & \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t^\alpha)dt\right)^{\frac{1}{q}}[f(a)g(a)]\right. \\
&+\left[m_2f(a)g\left(\frac{b}{m_2}\right)+m_1g(a)f\left(\frac{b}{m_1}\right)+m_2f(b)g\left(\frac{a}{m_2}\right)+m_1g(b)f\left(\frac{a}{m_1}\right)\right]\left(\int_0^1 (h(t^\alpha)h(1-t^\alpha))^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 h^{2q}(1-t^\alpha)dt\right)^{\frac{1}{q}}\left[m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\right\}. \qquad (22)
\end{aligned}
$$

Proof. By setting $C=0$ and $\eta=0$ in inequality (18) of Theorem 4, we get the above inequality (22). ■

Corollary 13 *The following inequality holds for $(h-m)$-convex function via RL k-fractional integrals*

$$
\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right] \\
\leq\ & [f(a)g(a)+f(b)g(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt \\
&+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]+m_1\left[g(a)f\left(\frac{b}{m_1}\right)+g(b)f\left(\frac{a}{m_1}\right)\right]\right\}\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t)dt \\
\leq\ & \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t)dt\right)^{\frac{1}{q}}\left[f(a)g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right.\right. \\
&\left.+f(b)g(b)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]+\left(\int_0^1 (h(t)h(1-t))^q dt\right)^{\frac{1}{q}}\left[m_2f(a)g\left(\frac{b}{m_2}\right)+m_1g(a)f\left(\frac{b}{m_1}\right)\right. \\
&\left.\left.+\ m_2f(b)g\left(\frac{a}{m_2}\right)+m_1g(b)f\left(\frac{a}{m_1}\right)\right]\right\}. \qquad (23)
\end{aligned}
$$

Proof. By setting $C=0$, $\eta=0$ and $\alpha=1$ in inequality (18) of Theorem 4, we get the above inequality (23) which is given in [[7], Theorem 4.3]. ■

Corollary 14 *The following inequality holds for exponentially (s,m)-convex function via RL k-fractional integrals*

$$
\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right] \\
\leq\ & \left(\frac{k}{\alpha+2ks}\right)\left[\frac{f(a)g(a)}{e^{2\eta a}}+\frac{f(b)g(b)}{e^{2\eta b}}\right] \\
&+\left[m_1\left(\frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}+\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\right)+m_2\left(\frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}+\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\right)\right]\beta\left(\frac{\alpha}{k}+s,\ s+1\right)
\end{aligned}
$$

$$+m_1m_2\left[\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}+\frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\right]\beta\left(\frac{\alpha}{k},\,2s+1\right)$$

$$\leq \left(\frac{k}{\alpha+2ks}\right)\left[\frac{f(a)g(a)}{e^{2\eta a}}+\frac{f(b)g(b)}{e^{2\eta b}}\right]+\left[m_1\left(\frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}+\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\right)\right.$$

$$\left.+m_2\left(\frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}+\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\right)\right]\left(\frac{k}{p(\alpha-k+ks)+k}\right)^{\frac{1}{p}}\left(\frac{1}{qs+1}\right)^{\frac{1}{q}}$$

$$+m_1m_2\left[\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}+\frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\right]\left(\frac{k}{\alpha p-kp+k}\right)^{\frac{1}{p}}\left(\frac{1}{2qs+1}\right)^{\frac{1}{q}}. \tag{24}$$

Proof. By setting $C=0$, $\alpha=1$ and $h(t)=t^s$ in inequality (18) of Theorem 4, we get the above inequality (24) which is given in [9, Theorem 3]. ■

Corollary 15 *The following inequality holds for (s,m)-convex function via RL k-fractional integrals*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right]$$

$$\leq \left(\frac{k}{\alpha+2ks}\right)[f(a)g(a)+f(b)g(b)]$$

$$+\left[m_1\left(g(b)f\left(\frac{a}{m_1}\right)+g(a)f\left(\frac{b}{m_1}\right)\right)+m_2\left(f(b)g\left(\frac{a}{m_2}\right)+f(a)g\left(\frac{b}{m_2}\right)\right)\right]\beta\left(\frac{\alpha}{k}+s,\,s+1\right)$$

$$+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\beta\left(\frac{\alpha}{k},\,2s+1\right)$$

$$\leq \left(\frac{k}{\alpha+2ks}\right)[f(a)g(a)+f(b)g(b)]+\left[m_1\left(g(b)f\left(\frac{a}{m_1}\right)+g(a)f\left(\frac{b}{m_1}\right)\right)\right.$$

$$\left.+m_2\left(f(b)g\left(\frac{a}{m_2}\right)+f(a)g\left(\frac{b}{m_2}\right)\right)\right]\left(\frac{k}{p(\alpha-k+ks)+k}\right)^{\frac{1}{p}}\left(\frac{1}{qs+1}\right)^{\frac{1}{q}}$$

$$+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\left(\frac{k}{\alpha p-kp+k}\right)^{\frac{1}{p}}\left(\frac{1}{2qs+1}\right)^{\frac{1}{q}}. \tag{25}$$

Proof. By setting $C=0$, $\eta=0$, $\alpha=1$ and $h(t)=t^s$ in inequality (18) of Theorem 4, we get the above inequality (25) which is given in [9, Theorem 6]. ■

Theorem 5 *Let $f,g:[0,\infty)\to\mathbb{R}$ be functions such that $fg\in L_1[a,b]$, $a,b\in[0,\infty)$, $a<b$. Also, let function f is strongly exponentially $(\alpha_1,h-m_1)$-convex and function g is strongly exponentially $(\alpha_2,h-m_2)$-convex on $[0,\infty)$ with modulus $C\geq 0$ and $(\alpha,m_1),(\alpha,m_2)\in(0,1]^2$. If $h(x+y)\leq h(x)h(y)$, then the following inequalities hold for RL k-fractional integrals*

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right]$$

$$\leq \left[\frac{f(a)g(a)}{e^{2\eta a}}+\frac{f(b)g(b)}{e^{2\eta b}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt$$

$$+m_1m_2\left[\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}+\frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt$$

$$+\left\{m_2\left[\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}+\frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}\right]+m_1\left[\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}+\frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]\right.$$
$$\left.\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\right\}-m_2C|b-a|^2\left[\frac{f(a)}{e^{\eta(2a+b)}}+\frac{f(b)}{e^{\eta(2b+a)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1}+1)dt$$
$$m_1C|b-a|^2\left[\frac{g(a)}{e^{\eta(2a+b)}}+\frac{g(b)}{e^{\eta(2b+a)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2}+1)dt-m_1m_2C|b-a|^2$$
$$\times\left[\frac{f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+b+\frac{b}{m_1}\right)}}+\frac{f\left(\frac{a}{m_1}\right)}{e^{\eta\left(a+b+\frac{a}{m_1}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^{\alpha_1})dt-m_1m_2C|b-a|^2$$
$$\left[\frac{g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+b+\frac{b}{m_2}\right)}}+\frac{g\left(\frac{a}{m_2}\right)}{e^{\eta\left(a+b+\frac{a}{m_2}\right)}}\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^{\alpha_2})dt+\frac{2m_1m_2kh(1)C^2|b-a|^4}{\alpha e^{\eta(2a+2b)}}. \tag{26}$$

Proof. Since function f is strongly exponentially $(\alpha_1, h-m_1)$-convex and function g is strongly exponentially $(\alpha_2, h-m_2)$-convex, then for $t \in [0,1]$, we have

$$\begin{aligned}
& f(ta+(1-t)b)g(ta+(1-t)b) \\
\leq\; & h(t^{\alpha_1})h(t^{\alpha_2})\frac{f(a)g(a)}{e^{2\eta a}}+m_2h(t^{\alpha_1})h(1-t^{\alpha_2})\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}} \\
& +m_1h(t^{\alpha_2})h(1-t^{\alpha_1})\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}+m_1m_2h(1-t^{\alpha_1})h(1-t^{\alpha_2})\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}} \\
& -m_2C|b-a|^2h(t^{\alpha_1})h(t^{\alpha_2})h(1-t^{\alpha_2})\frac{f(a)}{e^{\eta(2a+b)}}-m_1C|b-a|^2h(t^{\alpha_2})h(t^{\alpha_1})h(1-t^{\alpha_1})\frac{g(a)}{e^{\eta(2a+b)}} \\
& -m_1m_2C|b-a|^2h(t^{\alpha_2})h(1-t^{\alpha_1})h(1-t^{\alpha_2})\frac{f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+b+\frac{b}{m_1}\right)}} \\
& -m_1m_2C|b-a|^2h(t^{\alpha_1})h(1-t^{\alpha_1})h(1-t^{\alpha_2})\frac{g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+b+\frac{b}{m_2}\right)}} \\
& +\frac{m_1m_2}{e^{\eta(2a+2b)}}C^2|b-a|^4h(t^{\alpha_1})h(t^{\alpha_2})h(1-t^{\alpha_1})h(1-t^{\alpha_2}).
\end{aligned}$$

By multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
& \int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)b)g(ta+(1-t)b)dt \\
\leq\; & \frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt \\
& +\frac{m_2f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1-1})h(1-t^{\alpha_2})dt+\frac{m_1g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt \\
& +\frac{m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt \\
& -m_2C|b-a|^2\frac{f(a)}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})h(1-t^{\alpha_2})dt
\end{aligned}$$

$$-m_1C|b-a|^2\frac{g(a)}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(t^{\alpha_1})h(1-t^{\alpha_1})dt$$
$$-m_1m_2C|b-a|^2\frac{f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+b+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt$$
$$-m_1m_2C|b-a|^2\frac{g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+b+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt$$
$$+\frac{m_1m_2C^2|b-a|^4}{e^{\eta(2a+2b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt.$$

By substituting $w = ta + (1-t)b$ in left hand side of above inequality, we get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}I_{a^+}^{\alpha,k}f(b)g(b)\\
\leq\ &\frac{f(a)g(a)}{e^{2\eta a}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt\\
&+\frac{m_2f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(1-t^{\alpha_2})dt+\frac{m_1g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\\
&+m_1m_2\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt\\
&-m_2C|b-a|^2\frac{f(a)}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1}+1)dt\\
&-\frac{m_1C|b-a|^2g(a)}{e^{\eta(2a+b)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2}+1)dt-\frac{m_1m_2C|b-a|^2f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+b+\frac{b}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^{\alpha_1})dt\\
&-\frac{m_1m_2C|b-a|^2g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+b+\frac{b}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^{\alpha_2})dt+\frac{m_1m_2kh(1)C^2|b-a|^4}{\alpha e^{\eta(2a+2b)}},
\end{aligned}\tag{27}$$

and similarly by changing the rules of a and b, after a little computation one can get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}I_{b^-}^{\alpha,k}f(a)g(a)\\
\leq\ &\frac{f(b)g(b)}{e^{2\eta b}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt+\frac{m_2f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(1-t^{\alpha_2})dt\\
&+\frac{m_1g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt+\frac{m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt\\
&-\frac{m_2C|b-a|^2f(b)}{e^{\eta(2b+a)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1}+1)dt-m_1C|b-a|^2\frac{g(b)}{e^{\eta(2b+a)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2}+1)dt\\
&-m_1m_2C|b-a|^2\frac{f\left(\frac{a}{m_1}\right)}{e^{\eta\left(a+b+\frac{a}{m_1}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^{\alpha_1})dt\\
&-m_1m_2C|b-a|^2\frac{g\left(\frac{a}{m_2}\right)}{e^{\eta\left(a+b+\frac{a}{m_2}\right)}}\int_0^1 t^{\frac{\alpha}{k}-1}h(2-t^{\alpha_2})dt+\frac{m_1m_2kh(1)C^2|b-a|^4}{\alpha e^{\eta(2a+2b)}}.
\end{aligned}\tag{28}$$

Adding (27) and (28), we get the required result. ■

Corollary 16 *The following inequality holds for exponentially $(\alpha, h-m)$-convex function via RL fractional integrals*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[I_{a^+}^{\alpha} f(b)g(b) + I_{b^-}^{\alpha} f(a)g(a)\right] \\
\leq\ & \left[\frac{f(a)g(a)}{e^{2\eta a}} + \frac{f(b)g(b)}{e^{2\eta b}}\right]\int_0^1 t^{\alpha-1}h(t^{\alpha_1})h(t^{\alpha_2})dt \\
&+m_1m_2\left[\frac{f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)}{e^{\eta b(\frac{m_1+m_2}{m_1m_2})}} + \frac{f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)}{e^{\eta a(\frac{m_1+m_2}{m_1m_2})}}\right]\int_0^1 t^{\alpha-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt \\
&+\left\{m_2\left[\frac{f(a)g\left(\frac{b}{m_2}\right)}{e^{\eta\left(a+\frac{b}{m_2}\right)}} + \frac{f(b)g\left(\frac{a}{m_2}\right)}{e^{\eta\left(b+\frac{a}{m_2}\right)}}\right]\right. \\
&\left.+\ m_1\left[\frac{g(a)f\left(\frac{b}{m_1}\right)}{e^{\eta\left(a+\frac{b}{m_1}\right)}} + \frac{g(b)f\left(\frac{a}{m_1}\right)}{e^{\eta\left(b+\frac{a}{m_1}\right)}}\right]\int_0^1 t^{\alpha-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\right\}. \qquad (29)
\end{aligned}$$

Proof. By setting $C = 0$ and $k = 1$ in inequality (26) of Theorem 5, we get the above inequality (29). ■

Corollary 17 *The following inequality holds for $(\alpha, h-m)$-convex function via RL k-fractional integrals*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k} f(b)g(b) + I_{b^-}^{\alpha,k} f(a)g(a)\right] \\
\leq\ & [f(a)g(a) + f(b)g(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt \\
&+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) + f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt \\
&+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right) + f(b)g\left(\frac{a}{m_2}\right)\right]\right. \\
&\left.+\ m_1\left[g(a)f\left(\frac{b}{m_1}\right) + g(b)f\left(\frac{a}{m_1}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\right\}. \qquad (30)
\end{aligned}$$

Proof. By setting $C = 0$ and $\eta = 0$ in inequality (26) of Theorem 5, we get the above inequality (30) which is given in [7, Theorem 4.6]. ■

Corollary 18 *The following inequality holds for (α, m)-convex function via RL fractional integrals*

$$\begin{aligned}
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}I_{a^+}^{\alpha} f(b)g(b) \leq\ & \frac{1}{\alpha_1+\alpha_2+\alpha}f(a)g(a) + \frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_2f(a)g\left(\frac{b}{m_2}\right) \\
&+\frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_1g(a)f\left(\frac{b}{m_1}\right) \\
&+\left(\frac{1}{\alpha} - \frac{1}{\alpha+\alpha_1} - \frac{1}{\alpha+\alpha_2} + \frac{1}{\alpha_1+\alpha_2+\alpha}\right)m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right), \qquad (31)
\end{aligned}$$

and

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}I_{b^-}^{\alpha} f(a)g(a) \leq \frac{1}{\alpha_1+\alpha_2+\alpha}f(b)g(b) + \frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_2f(b)g\left(\frac{b}{m_2}\right)$$

$$+\frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)}m_1 g(b) f\left(\frac{b}{m_1}\right)$$
$$+\left(\frac{1}{\alpha}-\frac{1}{\alpha+\alpha_1}-\frac{1}{\alpha+\alpha_2}+\frac{1}{\alpha_1+\alpha_2+\alpha}\right)m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right). \quad (32)$$

Proof. From (27) by setting $C=0$, $\eta=0$, $k=1$ and $h(t)=t$, we get (31). Similarly, using $C=0$, $\eta=0$, $k=1$ and $h(t)=t$ in (28), we get (32) which is given in [21, Theorem 12]. ■

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