Applied Mathematics E-Notes, 23(2023), 360-369 © ISSN 1607-2510
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Global Existence And General Decay Of Solutions For A Quasi-Linear Parabolic System With A Weak-Viscoelastic Term*

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Received 16 October 2022

Abstract

In this work, we study a quasi-linear parabolic equation with a weak-viscoelastic term of the form

$$A(t)|u_t|^{m-2}u_t + \Delta^2 u - \alpha(t)(g * \Delta^2 u) = |u|^{p-2}u,$$

in a bounded domain Ω in R^n where $m, p \geq 2$ and $A(t)$ is a bounded and positive definite matrix. By using the perturbed energy functional technique, we establish global existence and a general decay result, which depends on the behavior of both α and g.

1 Introduction

In this paper, we deal with the following quasi-linear parabolic system with a weak-viscoelastic term

$$\begin{cases} A(t)|u_t|^{m-2}u_t + \Delta^2 u - \alpha(t)(g * \Delta^2 u) = |u|^{p-2}u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with sufficiently smooth boundary $\partial\Omega$, ν represents the unit outer normal to $\partial\Omega$. The values of u are taken in $\mathbb{R}^n (n \geq 1)$ and $A \in C(\mathbb{R}^+)$ is a bounded square matrix satisfying

$$(A(t)v, v) \geq c_0|v|^2, \quad \forall t \in \mathbb{R}^+, \ v \in \mathbb{R}^n, \tag{2}$$

where $(.,.)$ is the inner product in $\mathbb{R}^n$ and $c_0 > 0$. The parameter $m \geq 2$ and p satisfies

$$2 \leq p < \frac{2(n-2)}{n-4} \quad \text{if } n \geq 5; \ 2 < p < \infty \quad \text{if } n \leq 4. \tag{3}$$

The term $(g * \Delta^2 u)$ is defined by

$$(g * \Delta^2 u)(x,t) = \int_0^t g(t-s)\Delta^2 u(x,s)ds.$$

On the mathematical analysis of equations which model the motions of materials with memory, we refer to [3, 16, 19] and references therein. Yin in [19] considered a general equation of the form

$$u_t = \mathrm{div}A(x,t,u,u_x) + a(x,t,u,u_x) + \int_0^t \mathrm{div}B(x,t,\tau,u,u_x)d\tau.$$

Under some conditions on A, B, and a, similarly to the case of parabolic, the existence of a unique weak solution is established.

*Mathematics Subject Classifications: 35K05, 35K65, 35A01, 35B35.

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Regarding the heat equations without the memory term, study of global existence and finite time blow-up of solutions for the following initial boundary value problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0, (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), x \in \Omega, \end{cases}$$

has attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on p, the degree of nonlinearity in f, the dimension n and the size of the initial data. See for example, the works of Levine et al. [10, 11], Kalantarov and Ladyzhenskaya [8] and Messaoudi [14]. Also, concerning the asymptotic behavior of the solution, see [1, 17]. Pucci and Serrin [17] studied the following equation with homogenous Dirichlet boundary condition

$$A(t)|u_t|^{m-2}u_t - \Delta u + f(x,u) = 0. \tag{4}$$

They proved that the strong solution tends to zero when $t \to \infty$ under the condition $(f(x,u),u) > 0$, but did not give the decay rate. Berrimi and Messaoudi [1] proved that if bounded square matrix $A(t)$ in equation (4) satisfying (2), then the solution with small energy decays exponentially for $m = 2$ and polynomially for $m > 2$.

In the presence of the memory term in the heat equations, Messaoudi and Tellab [15] considered the following quasi-linear parabolic system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s) = 0, \tag{5}$$

with Dirichlet boundary condition and proved a general decay result which depends on the behavior of the function g. Ferhat and Hakem [4] considered the quasi-linear parabolic system

$$A(t)|u_t|^{m-2}u_t - Lu + \int_0^t g(t-s)Lu(s) = 0,$$

where

$$Lu = -\operatorname{div}(M\nabla u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right).$$

They improved the result obtained by Messaoudi and Tellab [15] and proved a general decay result. Later, Youkana et al. [20] studied the equation (5) where the relaxation function satisfies $g^{'}(t) \le -\xi(t)g^p(t)$, for all $t \ge 0$, $1 \le p < \frac{3}{2}$ and established a general and optimal decay result. Recently, Youkana and Messaoudi [21] considered the equation (5) under a general assumption on the relaxation functions satisfying $g^{'}(t) \le -\xi(t)H(g(t))$, where H is an increasing convex function and ξ is a nonincreasing function. In the case of viscoelastic heat equations with source term, Liu and Chen [12] considered the following quasilinear parabolic system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u,$$

and they proved a general decay of the energy function for the global solution and a blow-up result for the solution with both positive and negative initial energy under suitable conditions on g and p. In another study, Di et al. [7] investigated a nonlinear pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^p u.$$

They obtained finite-time blow-up results for the solutions with initial data at non-positive energy level as well as arbitrary positive energy level and give some upper bounds for the blow-up time T^* depending on the sign and size of initial energy $E(0)$. For more information in this regards we refer to [2, 6, 9, 18, 21] and references therein.

Motivated by the above mentioned papers, our purpose in this research is to investigate the global existence and general decay of solutions to the initial boundary value problem (1).

The rest of this article is organized as follows. In Section 2, we give some materials to be used for the main results. In Section 3, we prove the global existence of weak solutions by introducing a suitable functional to obtain the potential well. Finally, general decay result for the global solutions of the problem (1) has been proved in Section 4.

2 Preliminaries

We give some materials that will be needed in the proof of our results. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms.

We introduce the Sobolev's embedding inequality: assume p is a constant which satisfies (3), then $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$ continuously, and

$$\|u\|_p \le C_p \|\Delta u\|_2, \quad \text{for } u \in H_0^2(\Omega), \tag{6}$$

where C_p is the optimal embedding constant.

For the relaxation function g and the potential α, we assume

(G_1) $g,\, \alpha : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying

$$g(0) > 0, \quad \alpha(t) > 0, \quad 1 - \alpha(t)\int_0^t g(s)ds \ge \ell > 0, \quad \int_0^{+\infty} g(s)ds < \frac{(p-2)\ell}{2p\alpha(0)}.$$

In addition, we assume that there exists a positive constant α_0 such that $\alpha(t) \ge \alpha_0$.

(G_2) There exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\xi(t) > 0, \quad g'(t) \le -\xi(t)g(t) \text{ for } t \ge 0, \quad \lim_{t\to+\infty} \frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0.$$

Remark 1 *Note that (G_1) and (G_2) imply that* $\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$.

We introduce the functional

$$J(u(t)) = \frac{1}{2}(1 - \alpha(t)\int_0^t g(s)ds)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \Delta u)(t) - \frac{1}{p}\|u\|_p^p, \tag{7}$$

as the energy functional associated to problem (1) where

$$(g \circ \Delta u)(t) = \int_0^t g(t-s)\|\Delta u(t) - \Delta u(s)\|_2^2 ds.$$

Lemma 1 *Suppose (2) and assumption (G_1) hold. Let u be the solution of (1). Then, the energy satisfies*

$$\begin{aligned} J(u(t)) &= \int_0^t \Big(-\int_\Omega A(\tau)|u_\tau|^m dx + \frac{1}{2}\alpha'(\tau)(g\circ\Delta u)(\tau) + \frac{1}{2}\alpha(\tau)(g'\circ\Delta u)(\tau) \\ &\quad -\frac{1}{2}\alpha(\tau)g(\tau)\|\Delta u\|_2^2 + \frac{1}{2}(-\alpha'(\tau)\int_0^\tau g(s)ds)\|\Delta u\|_2^2\Big)d\tau + J(u(0)) \\ &\le -\int_0^t \Big(c_0\|u_\tau\|_m^m - \frac{1}{2}\alpha(\tau)(g'\circ\Delta u)(\tau) + \frac{1}{2}(\alpha'(\tau)\int_0^\tau g(s)ds)\|\Delta u\|_2^2\Big)d\tau \\ &\quad +J(u(0)). \end{aligned} \tag{8}$$

Proof. By multiplying the equation (1) by u_t, integrating over $\Omega\times(0,t)$ we get (8), after some manipulations. ■

To illustrate the main results of this paper, we introduce the definition of weak solutions.

Definition 1 *Let $u_0 \in H_0^2(\Omega)$. A function u is called a weak solution of the problem (1) defined on $[0,T)$ if*

$$u \in C\left([0,T);[H_0^2(\Omega)]^n\right) \cap C^1\left([0,T);[L^m(\Omega)]^n\right)$$

satisfies

$$(A(t)|u_t|^{m-2}u_t,\phi) + (\Delta u, \Delta\phi) - \alpha(t)((g*\Delta^2 u),\phi) = (|u|^{p-2}u,\phi),$$

for all $t\in[0,T)$ and $\phi \in C\left([0,T);[H_0^2(\Omega)]^n\right)$.

Let $T_{max} := \sup\{T>0\}$. The problem (1) admits weak solution on $[0,T)$. If $T_{max}<\infty$, then u is called a local weak solution; if $T_{max}=\infty$, u is called a global weak solution of problem (1) for $0\le t<\infty$.

Remark 2 *Similar to [17], we assume the existence of solution. For the linear case ($m=2$), one can easily establish the existence of a weak solution by the Galerkin method. In the one-dimensional case ($n=1$), the existence is established, in a more general setting, by Yin [19].*

3 Global Existence

We define the following functional in order to obtain the potential well.

$$I(u(t)) = \left(1-\alpha(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 - \|u\|_p^p. \tag{9}$$

From (7) and (9) it tells us that

$$J(u(t)) = \frac{1}{p}I(u(t)) + \frac{p-2}{2p}\left(1-\alpha(t)\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)(g\circ\Delta u)(t). \tag{10}$$

Lemma 2 *Assume p satisfies (3) and assumption (G_1) holds. Let $u_0\in H_0^2(\Omega)$ satisfy*

$$C_p^p\left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{p-2}{2}} < \ell; I(u(0))>0. \tag{11}$$

Then, $I(u(t))>0$ for all $t\in[0,T_{max})$, where C_p is the optimal constant of the embedding $H_0^2(\Omega)\hookrightarrow L^p(\Omega)$, and

$$J(u(0)) = \frac{1}{p}I(u(0)) + \frac{p-2}{2p}\|\Delta u_0\|_2^2, \quad I(u(0)) = \|\Delta u_0\|_2^2 - \|u_0\|_p^p.$$

Proof. Let u be a weak solution of problem (1). By definition of functional $I(u(t))$ in (9) we know $I\in C[0,T_{max})$. Suppose that there exists $t_0\in[0,T_{max})$ such that $I(u(t_0))\ge 0$ if the conclusion is not true, so there exists an interval $[t_0,t_1]\in[0,T_{max})$ such that $I(u(t_0))=0$ and $I(u(t))<0$ for all $t\in(t_0,t_1]$. From first inequality of (11) then we obtain that for $\varepsilon>0$ sufficiently small such that

$$C_p^p\left(\frac{4p}{(p-2)\ell}\left(J(u(0))+\varepsilon\right)\right)^{\frac{p-2}{2}} \le \ell, \tag{12}$$

there exists $\tilde{t}\in(t_0,t_1]$ satisfying

$$I(u(\tilde{t})) = -p\varepsilon < 0. \tag{13}$$

Therefore, from assumption $(G1)$, using (8), (10) and (13), we have

$$\begin{aligned} J(u(\tilde{t})) &\leq J(u(0)) + \frac{\alpha(0)}{2}\left(\int_0^{+\infty} g(s)ds\right)\|\Delta u(\tilde{t})\|_2^2 \\ &\leq J(u(0)) + \frac{(p-2)\ell}{4p}\|\Delta u(\tilde{t})\|_2^2, \end{aligned} \tag{14}$$

and

$$J(u(\tilde{t})) \geq -\varepsilon + \frac{(p-2)\ell}{2p}\|\Delta u(\tilde{t})\|_2^2. \tag{15}$$

Combining (14) with (15), we arrive at

$$\|\Delta u(\tilde{t})\|_2^2 \leq \frac{4p}{(p-2)\ell}\left(J(u(0)) + \varepsilon\right). \tag{16}$$

Therefore, it follows from (6) and (16) that

$$\|u(\tilde{t})\|_p^p \leq \frac{C_p^p}{\ell}\|\Delta u(\tilde{t})\|_2^{p-2}\ell\|\Delta u(\tilde{t})\|_2^2 \leq \frac{C_p^p}{\ell}\left(\frac{4p}{(p-2)\ell}\left(J(u(0)) + \varepsilon\right)\right)^{\frac{p-2}{2}}\ell\|\Delta u(\tilde{t})\|_2^2. \tag{17}$$

Then, by the definition of functional (9), using (12) and (17), we get

$$I(u(\tilde{t}))=\left(1-\alpha(\tilde{t})\int_0^{\tilde{t}} g(s)ds\right)\|\Delta u(\tilde{t})\|_2^2 - \|u(\tilde{t})\|_p^p \geq \ell\|\Delta u(\tilde{t})\|_2^2 - \ell\|\Delta u(\tilde{t})\|_2^2 = 0,$$

which contradicts (13). Then the conclusion of Lemma is true. ■

Theorem 1 *Assume p satisfies (3) and assumption (G_1) holds. Let $u_0 \in H_0^2(\Omega)$. Then the local weak solution u exists globally.*

Proof. To show $T_{max} = \infty$, it suffices to show there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T_{max})}\|\Delta u\|_2^2 \leq C. \tag{18}$$

By virtue of Lemma 2 and assumption (G_1), using (8) and (10), we get

$$\begin{aligned} J(u(0)) + \frac{(p-2)\ell}{4p}\|\Delta u\|_2^2 &\geq \frac{1}{p}I(u(t)) + \frac{p-2}{2p}(1-\alpha(t)\int_0^t g(s)ds)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)(g\circ\Delta u)(t) \\ &\geq \frac{(p-2)\ell}{2p}\|\Delta u\|_2^2. \end{aligned}$$

Therefore,

$$\|\Delta u\|_2^2 \leq \frac{4p}{(p-2)\ell}J(u(0)).$$

So let

$$C = \frac{4p}{(p-2)\ell}J(u(0)),$$

then (18) holds and consequently the solution is global. ■

4 General Decay

In this section, we prove a general decay result for the solution energy. Our main result in this section reads in the following theorem.

Theorem 2 *Given $u_0 \in H_0^2(\Omega)$. Assume that (2) and (G_1), (G_2) hold. Then, there exist two positive constants k and K, depending only on the initial data that the solution of (1) satisfies*

$$J(u(t)) \leq Ke^{-k\int_0^t \alpha(s)\xi(s)ds}, \quad \forall t \geq 0. \tag{19}$$

To prove above theorem, we need the following technical Lemmas. First, we state an important Lemma by Martinez [13].

Lemma 3 *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^2-increasing function with $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = +\infty$. Assume that there exist $c > 0$ for which*

$$\int_a^{+\infty} E(t)\psi'(t)dt \leq cE(a)\forall a \geq 0.$$

Then,

$$E(t) \leq \gamma e^{-\omega\psi(t)}, \forall t \geq 0$$

where γ and ω are positive constants.

Lemma 4 *Suppose that (G_1) and (G_2) hold and the initial data $u_0 \in H_0^2(\Omega)$. Then $J(u(t))$ is a non-increasing function satisfying*

$$J(u(t)) \leq \int_0^t \left(-\int_\Omega A(t)|u_\tau|^m dx + \frac{1}{2}\alpha(\tau)(g' \circ \Delta u)(\tau)\right) d\tau + J(u(0)) \leq J(u(0)). \tag{20}$$

Proof. Taking (8) in $(0,\tilde{t})$ for $|t-\tilde{t}| < \delta_0$ and using assumption (G_2), we get for δ_0 small enough

$$J(u(\tilde{t})) \leq \int_0^{\tilde{t}} \alpha(t)\left(-\frac{1}{\alpha(t)}\int_\Omega A(t)|u_t|^m dx + (g' \circ \Delta u)(t) + \frac{1}{2}\left(\frac{-\alpha'(t)}{\alpha(t)}\int_0^t g(s)ds\right)\|\Delta u\|_2^2\right) dt + J(u(0)).$$

From the fact that $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$ for $\tilde{t}$ belonging to some small neighborhood of $t = 0$, that is (20) obtained for $t > \tilde{t} > 0$. ■

Proof of Theorem 2. By multiplying the equation in (1) by $\alpha(t)\xi(t)u$ and integrating over $\Omega \times (a, T_{max})$ and using the boundary data, we get

$$\begin{aligned}
&\int_a^{T_{max}} \alpha(t)\xi(t)\left(\int_\Omega A(t)|u_t|^{m-2}u_t u dx + (1-\alpha(t)\int_0^t g(s)ds)\|\Delta u\|_2^2 - \|u\|_p^p\right) dt \\
= &\int_a^{T_{max}} \alpha(t)\xi(t)\left(\alpha(t)\int_\Omega \Delta u(t)\int_0^t g(t-s)(\Delta u(s) - \Delta u(t))dsdx\right) dt.
\end{aligned} \tag{21}$$

Adding $\alpha(t)(g \circ \Delta u)(t)$ to both sides of (21) and taking (7) into account, we get

$$\begin{aligned}
&2\int_a^{T_{max}} \alpha(t)\xi(t)J(u(t))dt \\
\leq &\int_a^{T_{max}} \alpha(t)\xi(t)\alpha(t)(g \circ \Delta u)(t)dt + \frac{p-2}{p}\int_a^{T_{max}} \alpha(t)\xi(t)\|u\|_p^p dt \\
&-\int_a^{T_{max}} \alpha(t)\xi(t)\int_\Omega A(t)|u_t|^{m-2}u_t u dx dt \\
&+\int_a^{T_{max}} \alpha(t)\xi(t)\left(\alpha(t)\int_\Omega \Delta u(t)\int_0^t g(t-s)(\Delta u(s) - \Delta u(t))dsdx\right) dt.
\end{aligned} \tag{22}$$

Thanks to the Young's inequality, (2), assumptions (G_1), (G_2), the boundedness of matrix A and using the fact that, $\alpha'(t) \leq 0$ for all $t > 0$, we get for $\delta > 0$

$$\left| -\int_\Omega A(t)|u_t|^{m-2}u_t u dx \right| \leq \delta \|u\|_m^m + C_\delta \|u_t\|_m^m, \tag{23}$$

and

$$\begin{aligned} \alpha(t)\int_\Omega \Delta u(t)\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))dsdx &\leq \delta\alpha(0)\|\Delta u\|_2^2 + \frac{\int_0^{+\infty} g(s)ds}{4\delta}\alpha(t)(g\circ\Delta u)(t) \\ &\leq \delta\alpha(0)\|\Delta u\|_2^2 + \frac{(p-2)\ell}{8p\delta\alpha(0)}\alpha(t)(g\circ\Delta u)(t). \end{aligned} \tag{24}$$

By combining (22)–(24) and using

$$\xi(t)\alpha(t)(g\circ\Delta u)(t) \leq -\alpha(t)(g'\circ\Delta u)(t),$$

we arrive at

$$\begin{aligned} 2\int_a^{T_{max}} \alpha(t)\xi(t)J(u(t))dt \;\leq\; & \alpha(0)\xi(0)C_\delta \int_a^{T_{max}} \|u_t\|_m^m dt \\ & -\alpha(0)\left(1+\frac{(p-2)\ell}{8p\delta\alpha(0)}\right)\int_a^{T_{max}} \alpha(t)(g'\circ\Delta u)(t)dt \\ & +\delta\int_a^{T_{max}} \alpha(t)\xi(t)\|u\|_m^m dt + \frac{p-2}{p}\int_a^{T_{max}} \alpha(t)\xi(t)\|u\|_p^p dt \\ & +\delta\alpha(0)\int_a^{T_{max}} \alpha(t)\xi(t)\|\Delta u\|_2^2 dt. \end{aligned} \tag{25}$$

By recalling (8), we deduce from the first two integrals on the right-hand side of (25) that

$$\begin{aligned} & \alpha(0)\xi(0)C_\delta\int_a^{T_{max}} \|u_t\|_m^m dt - \alpha(0)\left(1+\frac{(p-2)\ell}{8p\delta\alpha(0)}\right)\int_a^{T_{max}} \alpha(t)(g'\circ\Delta u)(t)dt \\ \leq\; & \left(\frac{C_\delta}{2c_0} + \alpha(0)(1+\frac{(p-2)\ell}{8p\delta\alpha(0)})\alpha(t)\xi(t)\right)\int_a^{T_{max}} \left(\frac{\alpha'(t)}{\xi(t)\alpha(t)}\int_0^t g(s)ds\right)dt \\ & -\gamma_0\int_a^{T_{max}} J'(u(t))dt, \end{aligned} \tag{26}$$

where

$$\gamma_0 := \frac{\alpha(0)\xi(0)C_\delta}{c_0} + 2\alpha(0)\left(1+\frac{(p-2)\ell}{8p\delta\alpha(0)}\right).$$

By exploiting Lemma 4 and using $\lim_{t\to+\infty}\frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0$ to choose $t > \tilde{t} > 0$, (26) takes the form

$$\alpha(0)\xi(0)C_\delta\int_a^{T_{max}} \|u_t\|_m^m dt - \alpha(0)\left(1+\frac{(p-2)\ell}{8p\delta\alpha(0)}\right)\int_a^{T_{max}} \alpha(t)(g'\circ\Delta u)(t)dt \leq \gamma_0 J(u(a)).$$

Therefore, (25) yields that

$$\begin{aligned} & 2\int_a^{T_{max}} \alpha(t)\xi(t)J(u(t))dt - \gamma_0 J(u(a)) \\ \leq\; & \delta\int_a^{T_{max}} \alpha(t)\xi(t)\|u\|_m^m dt + \frac{p-2}{p}\int_a^{T_{max}} \alpha(t)\xi(t)\|u\|_p^p dt \\ & +\delta\alpha(0)\int_a^{T_{max}} \alpha(t)\xi(t)\|\Delta u\|_2^2 dt. \end{aligned} \tag{27}$$

Since g is positive, we have, for any $t_0 > 0$

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 > 0, \quad \forall t \geq t_0. \tag{28}$$

To estimate the last integrals in the right-hand side of (27), we use (28), assumption (G_1), (6), (10), (11) and repeatedly embedding inequalities as follows

$$\begin{aligned} J(u(t)) &= \frac{1}{p}I(u(t)) + \frac{p-2}{2p}(1-\alpha(t)\int_0^t g(s)ds)\|\Delta u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \Delta u)(t) \\ &\geq \frac{p-2}{2p}(1-\alpha(t)\int_0^t g(s)ds)\|\Delta u\|_2^2 \\ &\geq \frac{(p-2)\ell}{2p}\|\Delta u\|_2^2, \end{aligned} \tag{29}$$

$$\begin{aligned} \|u\|_p^p &\leq C_p^p \|\Delta u\|_2^{p-2}\|\Delta u\|_2^2 \\ &\leq \frac{C_p^p}{\ell}\left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{p-2}{2}} \ell\|\Delta u\|_2^2 \\ &\leq (1-\alpha(t)\int_0^t g(s)ds)\|\Delta u\|_2^2 \\ &\leq (1-\alpha_0 g_0)\|\Delta u\|_2^2 \\ &\leq \frac{2p(1-\alpha_0 g_0)}{(p-2)\ell}J(u(t)), \end{aligned} \tag{30}$$

where C_p is the optimal constant of the embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$, and

$$\begin{aligned} \|u\|_m^m &\leq C_m^m \|\Delta u\|_2^{m-2}\|\Delta u\|_2^2 \\ &\leq C_m^m \left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{m-2}{2}} \|\Delta u\|_2^2 \\ &\leq \gamma_1 J(u(t)), \end{aligned} \tag{31}$$

where C_m is the optimal constant of the embedding $H_0^2(\Omega) \hookrightarrow L^m(\Omega)$, and

$$\gamma_1 = \frac{2pC_m^m}{(p-2)\ell}\left(\frac{4p}{(p-2)\ell}J(u(0))\right)^{\frac{m-2}{2}}.$$

Inserting (29)–(31) in (27) the estimate

$$(1 - \frac{\delta\gamma_1}{2} - \frac{\delta p\alpha(0)}{(p-2)\ell} - \frac{1-\alpha_0 g_0}{\ell})\int_a^{T_{max}} \alpha(t)\xi(t)J(u(t))dt \leq \frac{\gamma_0}{2}J(u(a)),$$

is established.

At this point, we pick $\alpha_0 > \frac{1-\ell}{g_0}$ and choose δ small enough, to have

$$\lambda := 1 - \frac{\delta\gamma_1}{2} - \frac{\delta p\alpha(0)}{(p-2)\ell} - \frac{1-\alpha_0 g_0}{\ell} > 0.$$

Thanks to the Lemma 3, by taking $\psi(t) = \int_0^t \alpha(s)\xi(s)ds$ and letting T_{max} goes to infinity, we obtain the desired result in (19) and the proof is completed. ■

Acknowledgment. The authors would like to thank the referee for his/her valuable comments and corrections which improved this work.

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