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Note On Weighted Version Of The Young Inequality*

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Abstract

In this note we give weighted version of the Young's inequality.

1 Introduction

The well-known Hölder's inequality

$$\sum_{k=1}^{n} p_k a_k b_k \leq \left(\sum_{k=1}^{n} p_k a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} p_k b_k^p\right)^{\frac{1}{q}} \tag{1}$$

is valid for positive numbers p_i, a_i, b_i $(i = 1, 2, \ldots, n)$ $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The equality holds if and only if $a_k^p = cb_k^q$ for $k = 1, 2, \ldots, n$ and c a positive constant.

The following generalization of Hölder inequality is also valid (see for example [5]). Let p_i $(i = 1, 2, \ldots, n)$, a_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ be positive numbers and $r_i > 0$ such that $\delta_m = \sum_{k=1}^{m} \frac{1}{r_i} = 1$. Then

$$\sum_{i=1}^{n} p_i a_{i1} \cdots a_{im} \leq \left(\sum_{i=1}^{n} p_i a_i^{r_1}\right)^{\frac{1}{r_1}} \cdots \left(\sum_{i=1}^{n} p_i a_{im}^{r_m}\right)^{\frac{1}{r_n}}. \tag{2}$$

In 1936 L.C. Young [7] has given the following Hölder type inequality:

Theorem 1 *Let a_i, b_i $(i = 1, 2, \ldots, n)$ be complex numbers such that*

$$|a_n b_n| \leq |a_{n-1} b_{n-1}| \leq \cdots |a_2 b_2| \leq |a_1 b_1| \tag{3}$$

and $\frac{1}{p} + \frac{1}{q} > 1$. Then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|\right)^{\frac{1}{q}} \tag{4}$$

where

$$\zeta(t) = \sum_{k=1}^{\infty} \frac{1}{k^t}, \quad t > 1.$$

Improvement and generalizations of this inequality was given in [1]. G. H. Hardy, J. E. Littlewood and G. Pólya in their book Inequalities, in the Preface to the first edition of 1934, say:

Historical and bibliographical questions are particularly troublesome in a subject like this, which has applications in every part of mathematics but has never been developed systematically. It is often really

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difficult to trace the origin of a familiar inequality. It is quite likely to occur first as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered, many years later, by half a dozen different authors; and no accessible statement of it may be quite complete We have done our best to be accurate and have given all references we can, but we have never undertaken systematic bibliographical research.

G. H. Hardy in his essay Prolegomena to a Chapter on Inequalities (J. London Math. Soc. 4 (1929), 61–78) displayed an enthusiasm for the subject of inequalities. He gives a masterly account of various types of problems and of the method of proof in the field of elementary inequalities. "Prolegomena" can be considered as the start of the creation of a particular discipline in Analysis, Number Theory and Geometry, namely "Inequalities". The difficulties encountered by renowned analysts such as Hermite, Picard, Hardy, Littlewood and Pólya in the study of inequalities have increased significantly over time. The number of researchers proving refinements, generalizations and variants of a given inequality multiplied, publishing results in a multitude of journals. The question of priorities and historical development is therefore specifically onerous. Recently the authors in [6, 4, 2, 3] have made some contributions in the inequalities field.

In this paper we will give weighted generalization of Young's inequality.

2 Main Results

Theorem 2 *Let p_i, q_j $(i = 1, 2, \ldots, n)$ be positive numbers, a_i, b_i $(i = 1, 2, \ldots, n)$ be complex numbers such that (3) is valid and $p, q > 0$, $p_k^* = \sum_{i=1}^{k} p_i$. Then*

$$\sum_{i=1}^{n} q_i|a_ib_i| \leq \left(\sum_{k=1}^{n} q_k (p_k^*)^{-\frac{1}{p}-\frac{1}{q}}\right) \left(\sum_{i=1}^{n} p_i|a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}}. \tag{5}$$

Proof. Let us start observing that

$$\left(|a_1b_1|^{p_1}\cdots|a_nb_n|^{p_n}\right)^{\frac{1}{p_n^*}} = \left[\left(|a_i|^{pp_1}\cdots|a_n|^{pp_n}\right)^{\frac{1}{p_n^*}}\right]^{\frac{1}{p}} \left[\left(|b_1|^{qp_1}\cdots|b_n|^{qp_n}\right)^{\frac{1}{p_n^*}}\right]^{\frac{1}{q}}. \tag{6}$$

By using mean $[0, p]$ and $[0, q]$ inequalities we have

$$\left(|a_1|^{p_1}\cdots|a_n|^{p_n}\right)^{\frac{1}{p_n^*}} \leq \left(\frac{1}{p_n^*}\sum_{i=1}^{n} p_i|a_i|^p\right)^{\frac{1}{p}} \tag{7}$$

and

$$\left(|b_1|^{p_1}\cdots|b_n|^{p_n}\right)^{\frac{1}{p_n^*}} \leq \left(\frac{1}{p_n^*}\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}}. \tag{8}$$

On the other hand side by $[-\infty, 0]$ mean inequality we have

$$|a_nb_n| \leq \left(|a_1b_1|^{p_1}\cdots|a_nb_n|^{p_n}\right)^{\frac{1}{p_n^*}}. \tag{9}$$

So by (8) and (9) we have

$$|a_nb_n| \leq \left(\frac{1}{p_n^*}\sum_{i=1}^{n} p_i|a_i|^p\right)^{\frac{1}{p}} \left(\frac{1}{p_n^*}\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}} = {p_n^*}^{-\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} p_i|a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}}.$$

Similarly,

$$|a_{n-1}b_{n-1}| \leq {p_{n-1}^*}^{-\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n-1} p_i|a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}} \leq {p_{n-1}^*}^{-\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} p_i|a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}}.$$

Proceeding in this way, we can get

$$\sum_{i=1}^{n} g_i|a_ib_i| \le \left(\sum_{i=1}^{n} q_i {p_i^*}^{-\frac{1}{p}-\frac{1}{q}}\right)\left(\sum_{i=1}^{n} p_i|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} p_i|b_i|^q\right)^{\frac{1}{q}}.$$

■

Similarly, we can prove such results for more sequences.

Theorem 3 *Let a_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ be complex numbers such that*

$$|a_{n1}\cdots a_{nm}| \le \cdots |a_{21}\cdots a_{2m}| \le |a_{11}\cdots a_{1m}|.$$

Let p_i, q_i $(i = 1, \ldots, n)$ be positive numbers $r_1, \ldots, r_n > 0$. Then

$$\sum_{i=1}^{n} q_i|a_{i1}\cdots a_{im}| \le \left(\sum_{k=1}^{n} q_k (p_k^*)^{-\delta_m}\right)\left(\sum_{i=1}^{n} p_i|a_i|^{p_1}\right)^{\frac{1}{q}}\cdots\left(\sum_{i=1}^{n} p_i|a_{im}|^{r_m}\right)^{\frac{1}{r_m}}.$$

Let us note that the following results is also valid.

Theorem 4 *Let a_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ be complex numbers $p_i \ge 1$ $(i = 1, \ldots, n)$ and r_i $(i = 1, 2, \ldots, m)$ are positive numbers such that $\delta_m \ge 1$. Then*

$$\sum_{i=1}^{n} p_i|a_{i1}\cdots a_{im}| \le \left(\sum_{i=1}^{n} p_i|a_{i1}|^{r_1}\right)^{\frac{1}{r_1}}\cdots\left(\sum_{i=1}^{n} p_i|a_i|^{r_n}\right)^{\frac{1}{r_n}}.$$

Proof. Let us note that this theorem was proved in [5, p. 103] for positive real number a_j. So it is obvious that is valid for absolute values of complex numbers. ■

Theorem 5 *Let a_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ be complex numbers p_i $(i = 1, 2, \ldots, n)$ and r_i $(i = 1, 2, \ldots, m)$ are positive numbers such that $\delta_m \le 1$. Then*

$$\sum_{i=1}^{n} p_i|a_{i1}\cdots a_{im}| \le (p_n^*)^{1-\delta_m}\left(\sum_{i=1}^{n} p_i|a_{i1}|^{r_1}\right)^{\frac{1}{r_1}}\cdots\left(\sum_{i=1}^{n} p_i|a_{im}|^{r_m}\right)^{\frac{1}{r_m}}.$$

Proof. Set in (2) $a_{ij} \to |a_{ij}|$, $m \to m+1$, $a_{im+1} = 1$. Then we have

$$\sum_{i=1}^{n} p_i|a_{i1}\cdots a_{im}|\cdot 1 \le \left(\sum_{i=1}^{n} p_i|a_{i1}|^{r_1}\right)^{\frac{1}{r_1}}\cdots\left(\sum_{i=1}^{n} p_i|a_{im}|^{r_m}\right)^{\frac{1}{r_m}}\left(\sum_{i=1}^{n} p_i^{-1}\right)^{1-\delta_n}$$

what is our inequality. ■

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