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Uniqueness Of Meromorphic Functions With Their Q-Shifts Differential-Difference Polynomials Sharing A Small Function*

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Abstract

Under several types of sharing assumptions, this article investigates the uniqueness problems of meromorphic functions whose q-shifts differential-difference polynomials sharing a small function. The results obtained could be seen as partial generalizations and extensions of some exsting results.

1 Introduction

In this article, a nonconstant meromorphic function f is meromorphic in the whole complex plane $\mathbb{C}$ unless otherwise stated, and f is called entire if it has no pole. The readers are assumed to be familiar with the elementrary concepts and standard notations of Nevanlinna value distrubution theory (see [5, 7, 24]), such as the proximity function $m(r, f)$, the (integrated) counting function $N(r, f)$, and the Nevanlinna characteristic function $T(r, f)$.

For convenience, denote by E_1 any set of finite logarithmic measure ($lm(E_1) = \int_{E_1} \frac{dt}{t} < \infty$), and denote by $S(r, f)$ any quantity such that $S(r, f) = o(T(r, f))$ as $r \to \infty$, $r \notin E_1$. A meromorphic function α is called a small function with respect to f if it satisfies $T(r, \alpha) = S(r, f)$. Let $S(f)$ be the set of all small functions with respect to f. Undoubtedly, any finite complex constant is a member of $S(f)$.

For $\alpha \in S(f) \cap S(g)$, two meromorphic functions f and g are said to share α CM (counting multiplicities) if the zeros of $f - \alpha$ and $g - \alpha$ coincide in locations and multiplicities, and they are said to share α IM (ignoring multiplicities) if the zeros of $f - \alpha$ and $g - \alpha$ coincide in locations.

Several decades ago, many scholars have studied the value distribution of $f^n f'$ for a transcendental meromorphic function f and a positive integer n. For instance, in 1959 W. K. Hayman [6] found that $f^n f' = 1$ has infinitely many solutions for $n \geq 3$. Afterwards, the cases that $n = 2$ and $n = 1$ were settled separately by E. Mues [18] in 1979 and by W. Bergweiler and A. Eremenko [3] in 1995.

In 2007, I. Laine and C. C. Yang [12] investigated one type of the difference analogue of the results above for entire functions and proved the following theorem.

Theorem 1 *Let f be a transcendental entire function of finite order, and c be a nonzero complex constant. Then, for $n \geq 2$, $f^n(z)f(z + c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Later, a more complex type of difference polynomials was studied by M. R. Chen and Z. X. Chen in 2012 [4].

Theorem 2 *Let f be a transcendental entire function of finite order, $\alpha(\not\equiv 0)$ be a small function with respect to f, $c_j(\neq 0)$ be distinct finite constants, n, m, d and v_j $(j = 1, 2, \ldots, d)$ be positive integers. If $n \geq 2$, then*

$$f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f^{v_j}(z + c_j) - \alpha(z)$$

has infinitely many zeros.

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In the same paper, the two authors also proved a uniqueness result corresponding to the above theorem. In 2017, A. Banerjee and S. Majumder [1] took the kth derivative ($k \geq 1$) of differential-difference polynomials

$$\left[f^n(z)(a_m f^m(z) + \cdots + a_1 f(z) + a_0)\prod_{j=1}^{d} f^{v_j}(z + c_j)\right]^{(k)} \tag{1}$$

into consideration, where $a_0 \neq 0,\ a_1, \ldots, a_m \neq 0$ are finite constants, $c_j (\neq 0)$ are distinct finite constants, $n,\ m,\ d$ and v_j $(j = 1, 2, \ldots, d)$ are positive integers. In the same paper the two authors [1] also corrected some errors of previous results, and presented their improved and generalized forms under the hypothesis of weighted sharing. Introduced by I. Lahiri [10, 11], weighted sharing is a gradation of sharing of values.

Definition 1 *Let k be a nonnegative integer or ∞. For $a \in \mathbb{C} \cup \{\infty\}$, denote by $E_k(a, f)$ the set of all a-points of f where an a-point with multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m \geq k+1$, and denote by $\overline{E}_{k)}(a, f)$ the set of those distinct a-points of f with multiplicities not greater than k.*

Likewise, for $\alpha \in S(f) \cap S(g)$, we could denote by $E_k(\alpha, f)$ the set of all zeros of $f - \alpha$ where a zero of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m \geq k + 1$. If $E_k(\alpha; f) = E_k(\alpha; g)$, we say that $f,\ g$ share small function α with weight k. Clearly, if $f,\ g$ share α with weight k, then z_0 is a zero of $f - \alpha$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $g - \alpha$ with multiplicity $m(\leq k)$, and z_1 is a zero of $f - \alpha$ with multiplicity $m(\geq k + 1)$ if and only if z_1 is a zero of $g - \alpha$ with multiplicity $n(\geq k + 1)$, here m is not necessarily equal to n. Write $f,\ g$ share (α, k) to mean that $f,\ g$ share small function α with weight k. Apparently if $f,\ g$ share (α, k) then $f,\ g$ share (α, p) for any integer p $(0 \leq p \leq k)$. Also, $f,\ g$ share α IM or CM if and only if $f,\ g$ share $(\alpha, 0)$ or (α, ∞) respectively.

Definition 2 ([9]) *Let $\alpha \in S(f)$ and $k \in \mathbb{N}^+ \cup \{\infty\}$. Denote by $N_{k)}\left(r, \dfrac{1}{f-\alpha}\right)$ the counting function of the zeros of $f - \alpha$ (counted with proper multiplicities) whose multiplicities are not greater than k, denote by $N_{(k+1}\left(r, \dfrac{1}{f-\alpha}\right)$ the counting function of the zeros of $f - \alpha$ whose multiplicities are not less than $k+1$. And let $\overline{N}_{k)}\left(r, \dfrac{1}{f-\alpha}\right)$, $\overline{N}_{(k+1}\left(r, \dfrac{1}{f-\alpha}\right)$ be their corresponding reduced counting functions (ignoring multiplicities), respectively.*

Definition 3 ([10, 11]) *Let $\alpha \in S(f)$ and $k \in \mathbb{N}^+ \cup \{\infty\}$. Denote by $N_k\left(r, \dfrac{1}{f-\alpha}\right)$ the counting function of the zeros of $f - \alpha$ where a zero with multiplicity m is counted m times when $m \leq k$ and k times when $m \geq k + 1$. Put*

$$N_\infty\left(r, \frac{1}{f-\alpha}\right) = N_{\infty)}\left(r, \frac{1}{f-\alpha}\right) = N\left(r, \frac{1}{f-\alpha}\right).$$

Apparently,

$$N_1\left(r, \frac{1}{f-\alpha}\right) = \overline{N}\left(r, \frac{1}{f-\alpha}\right),$$

and

$$N_k\left(r, \frac{1}{f-\alpha}\right) = \overline{N}\left(r, \frac{1}{f-\alpha}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-\alpha}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-\alpha}\right).$$

Recently, the topic of the uniqueness of meromorphic functions related to their differential-difference polynomials, especially the type (1), has captured the attention of a lot of scholars [8, 17, 21, 19]. New types of value sharing or small function sharing (which is essentially value sharing since it could be treated as zeros sharing of the difference between two functions) are also taken into consideration, which contributes to the occurrence of some interesting results [20, 22].

Now we are at a stage to give an introduction of the notion of weakly weighted sharing, a type of sharing introduced by S. H. Lin and W. C. Lin in 2006 [13] and weaker than weighted sharing.

Let $N_E(r,\alpha;f,g)$ ($\overline{N}_E(r,\alpha;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-\alpha$ and $g-\alpha$ with the same multiplicities and $\overline{N}_O(r,\alpha;f,g)$ be the reduced counting function of all common zeros of $f-\alpha$ and $g-\alpha$ ignoring multiplicities. If

$$\overline{N}\left(r,\frac{1}{f-\alpha}\right)+\overline{N}\left(r,\frac{1}{g-\alpha}\right)-2\overline{N}_E(r,\alpha;f,g)=S(r,f)+S(r,g),$$

then we say $f,\ g$ share α "CM"; and if

$$\overline{N}\left(r,\frac{1}{f-\alpha}\right)+\overline{N}\left(r,\frac{1}{g-\alpha}\right)-2\overline{N}_O(r,\alpha;f,g)=S(r,f)+S(r,g),$$

then we say $f,\ g$ share α "IM".

Definition 4 *Let $\alpha\in S(f)\cap S(g)$ and $k\in\mathbb{N}^+\cup\{\infty\}$. Suppose that f and g share α "IM". Denote by $\overline{N}^E_{k)}(r,\alpha;f,g)$ the reduced counting function of zeros of $f-\alpha$ whose multiplicities are equal to the corresponding zeros of $g-\alpha$, both of their multiplicities are not greater than k. And denote by $\overline{N}^O_{(k}(r,\alpha;f,g)$ the reduced counting function of zeros of $f-\alpha$ that are also zeros of $g-\alpha$, both of their multiplicities are not less than k.*

Definition 5 *For $\alpha\in S(f)\cap S(g)$, if $k\in\mathbb{N}^+\cup\{\infty\}$ and*

$$\overline{N}_{k)}\left(r,\frac{1}{f-\alpha}\right)-\overline{N}^E_{k)}(r,\alpha;f,g)=S(r,f);$$

$$\overline{N}_{k)}\left(r,\frac{1}{g-\alpha}\right)-\overline{N}^E_{k)}(r,\alpha;f,g)=S(r,g);$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{f-\alpha}\right)-\overline{N}^O_{(k+1}(r,\alpha;f,g)=S(r,f);$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{g-\alpha}\right)-\overline{N}^O_{(k+1}(r,\alpha;f,g)=S(r,g),$$

or if $k=0$ and

$$\overline{N}\left(r,\frac{1}{f-\alpha}\right)-\overline{N}_O(r,a;f,g)=S(r,f);\ \overline{N}\left(r,\frac{1}{g-\alpha}\right)-\overline{N}_O(r,a;f,g)=S(r,g),$$

then we say f and g weakly share α with weight k, and in such a case we write $f,\ g$ share "(α,k)".

It could be seen easily by Definitions 1 and 5 that weighted sharing is a scaling between IM and CM, while weakly weighted sharing is a scaling between "IM" and "CM".

We would also like to make known to the readers the concept of relaxed weighted sharing, a type of sharing weaker than weakly weighted sharing and is introduced by A. Banerjee and S. Mukherjee in 2007 [2].

Definition 6 *Suppose that $\alpha\in S(f)\cap S(g)$. Denote by $\overline{N}(r,\alpha;f|=p;g|=q)$ the reduced counting function of common zeros of $f-\alpha$ and $g-\alpha$ with multiplicities p and q respectively.*

Definition 7 *Suppose that $\alpha\in S(f)\cap S(g)$ and $k\in\mathbb{N}^+\cup\{\infty\}$. If $f,\ g$ share α "IM" and*

$$\sum_{\substack{p,q\le k\\ p\ne q}}\overline{N}(r,\alpha;f|=p;g|=q)=S(r,f)+S(r,g),$$

then we say $f,\ g$ share α with weight k in a relaxed manner, and in such a case we write $f,\ g$ share $(\alpha,k)^*$.

Using the idea of weakly weighted sharing and relaxed weighted sharing, not so long ago B. Saha, S. Pal and T. Biswas [20] focused attention on shift polynomials

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f^{v_j}(z+c_j)\right]^{(k)}$$

in entire functions f of finite order, where $c_j(\neq 0)$ $(j=1,2,\ldots,d)$ are finite complex constants and $n,\ m,\ d,\ k$ are all positive integers satisfying certain conditions. Actually they acquired the following uniqueness theorems.

Theorem 3 *Let f and g be two transcendental entire functions of finite order, $\alpha(\not\equiv 0)\in S(f)\cap S(g)$ and α has only finitely many zeros. Suppose that $c_j(\neq 0)$ $(j=1,2,\ldots,d)$ are finite complex constants, $n,\ m,\ d,\ v_j,\ k$ are positive integers satisfying $n\geq\max\{2k+m+\sigma+5,\sigma+2d+3\}$, here $\sigma=\sum_{j=1}^{d}v_j$. If*

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f^{v_j}(z+c_j)\right]^{(k)},\ \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g^{v_j}(z+c_j)\right]^{(k)}$$

share "$(\alpha,2)$", then $f\equiv tg$ for some constant t such that $t^{n+\sigma}=t^m=1$.

Theorem 4 *Under the same conditions as in Theorem 3, if $n\geq\max\{3k+2m+2\sigma+6,\sigma+2d+3\}$ and*

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f^{v_j}(z+c_j)\right]^{(k)},\ \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g^{v_j}(z+c_j)\right]^{(k)}$$

share $(\alpha,2)^*$, then the conclusion of Theorem 3 holds.

Theorem 5 *Under the same conditions as in Theorem 3, if $n\geq\max\{5k+4m+4\sigma+8,\sigma+2d+3\}$ and*

$$\overline{E}_{2)}\left(\alpha(z),\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f^{v_j}(z+c_j)\right]^{(k)}\right)=\overline{E}_{2)}\left(\alpha(z),\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g^{v_j}(z+c_j)\right]^{(k)}\right),$$

then the conclusion of Theorem 3 holds.

Seeing the form of the differential-difference polynomials (1), one may ask: will there be similar uniqueness results if the structure of (1) is changed? Motivated by this, in this paper we are about to investigate a kind of q-shift differential-difference polynomials in meromorphic function of order zero,

$$\left[f^n(z)P(f(z))\prod_{j=1}^{d} f^{v_j}(q_jz+c_j)\right]^{(k)}, \tag{2}$$

where $q_j\in\mathbb{C}\setminus\{0\},\ c_j\in\mathbb{C}\setminus\{0\},\ v_j\in\mathbb{N}^+$ $(j=1,2,\ldots,d),\ k\in\mathbb{N}^+$ are constants, $P(\omega)=a_m\omega^m+a_{m-1}\omega^{m-1}+\cdots+a_1\omega+a_0$ is a nonzero polynomial in ω of degree $m(\geq 0)$ with coefficients $a_l\in S(f)$ $(l=0,1,\ldots,m)$ and $a_m\not\equiv 0$.

The following definition and notation will also be used later.

Definition 8 ([13]) *Let f and g be two nonconstant meromorphic functions that share 1 "IM". Denote by $\overline{N}_L\left(r,\dfrac{1}{f-1}\right)$ the counting function of those 1-points of f with multiplicities greater than that of the corresponding 1-points of g, where each 1-point is counted only once.*

The rest of this paper is organized in this way: in Section 2, the main results of our study are listed first; in Section 3, auxiliary lemmas used in the proof of the theorems are given; in Section 4, the proof of the main results are exhibited in details.

2 Main Results

The main results of this paper are stated as follows. The first one reveals the value distribution of (2).

Theorem 6 *Let f be a transcendental meromorphic function of order zero, $q_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{C}$, $v_j \in \mathbb{N}^+$ $(j = 1, 2, \ldots, d)$, $k \in \mathbb{N}^+$ be constants, $\alpha \in S(f) \cap S(g)$. Let $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \cdots + a_1\omega + a_0$ be a nonzero polynomial in ω of degree $m(\geq 0)$ with coefficients $a_l \in S(f) \cap S(g)$ $(l = 0, 1, \ldots, m)$ and $a_m \not\equiv 0$. Denote $\sigma = \sum_{j=1}^{d} v_j$. If $n \geq 2\sigma + d + k + 3$, then*

$$\left[f^n(z)P(f(z))\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)\right]^{(k)} - \alpha(z)$$

has infinitely many zeros.

The rest four deal with the uniqueness problems on meromorphic functions f related to (2) under divergent types of function sharing hypotheses.

Theorem 7 *Let f, g be two transcendental meromorphic functions of order zero, $q_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{C}$, $v_j \in \mathbb{N}^+$ $(j = 1, 2, \ldots, d)$, $k \in \mathbb{N}^+$ be constants, $\alpha \in S(f) \cap S(g)$. Let*

$$P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \cdots + a_1\omega + a_0$$

be a nonzero polynomial in ω of degree $m(\geq 0)$ with coefficients $a_l \in S(f) \cap S(g)$ $(l = 0, 1, \ldots, m)$ and $a_m \not\equiv 0$. Denote

$$\sigma = \sum_{j=1}^{d} v_j, F(z) := f^n(z)P(f(z))\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)$$

and

$$G(z) := g^n(z)P(g(z))\prod_{j=1}^{d} g^{v_j}(q_j z + c_j).$$

If $F^{(k)}$ and $G^{(k)}$ share "(α, p)", here $p \in \mathbb{N} \cup \{\infty\}$, and the conditions of n are as below:

(i) $n \geq 3\sigma + m + kd + 4d + 3k + 9$ *when* $2 \leq p \leq \infty$;

(ii) $n \geq \dfrac{7\sigma + 3m + 3kd + 9d + 8k + 19}{2}$ *when* $p = 1$;

(iii) $n \geq 6\sigma + 4m + 4kd + 7d + 9k + 15$ *when* $p = 0$,

then one of the following three statements holds:

(i) $F^{(k)}G^{(k)} \equiv \alpha^2$;

(ii) $f = tg$ *for some constant* t *such that* $t^\tau = 1$, *where*

$$\tau = GCD\{n + \sigma + m, n + \sigma + \eta_{m-1}, \ldots, n + \sigma + \eta_1, n + \sigma + \eta_0\}$$

with $\eta_l = l$ *when* $a_l \not\equiv 0$ *and* $\eta_l = m$ *when* $a_l \equiv 0$ $(l = 0, 1, \ldots, m - 1)$

(iii) f, g *satisfy the algebraic equation* $R(f, g) = 0$,

where

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1)\prod_{j=1}^{d} \omega_1^{v_j}(q_j z + c_j) - \omega_2^n P(\omega_2)\prod_{j=1}^{d} \omega_2^{v_j}(q_j z + c_j).$$

Theorem 8 *Under the same conditions as in Theorem 7, if further $n > 4\sigma + 2m + 2kd + 5d + 5k + 10$ and $F^{(k)}$ and $G^{(k)}$ share $(\alpha, 2)^*$, then the conclusions of Theorem 7 hold.*

Theorem 9 *Under the same conditions as in Theorem 7, if further $n > 6\sigma + 4m + 4kd + 9k + 7$ and $\overline{E}_{p)}\left(\alpha, F^{(k)}\right) = \overline{E}_{p)}\left(\alpha, G^{(k)}\right)$, here $p \geq 2$, then the conclusions of Theorem 7 hold.*

Theorem 10 *Under the same conditions as in Theorem 7, if further $n > \sigma + 2m + d + 4$ and $F^{(k)}$ and $G^{(k)}$ share $(1, \infty)$ and (∞, ∞), then* (ii) *or* (iii) *in the conclusions of Theorem 7 holds.*

Remark 1 *As we can see, Theorems 7–9 are partial generalizations of Theorems 3, 4 and 5 to some extent.*

3 Preliminaries

This section presents some lemmas which are of great significance in the sequel. Let F and G be two nonconstant meromorphic functions, we first denote by H the following function.

$$H = \left(\frac{F''}{F'} - 2\frac{F'}{F-1}\right) - \left(\frac{G''}{G'} - 2\frac{G'}{G-1}\right).$$

Lemma 1 ([23]) *Let f be a nonconstant meromorphic function, $a_k \in S(f)$ $(k = 0, 1, \ldots, n)$, $a_n \not\equiv 0$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 ([15]) *Let f be a nonconstant meromorphic function of order zero, $q \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ are constants. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r, f)$$

on a set E_2 of logarithmic density 1, i.e.,

$$ld(E_2) = \varlimsup_{r\to\infty} \frac{1}{\log r} \int_{E_2 \cap [1,r]} \frac{dt}{t} = 1.$$

Remark 2 *This lemma is a q-shift analogue of the logarithmic derivative lemma.*

Lemma 3 ([16]) *Let f be a nonconstant meromorphic function of order zero, $q \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ are constants. Then*

$$N(r, f(qz+c)) \leq N(r, f(z)) + S(r, f),\ N\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f),$$

$$\overline{N}(r, f(qz+c)) \leq \overline{N}(r, f(z)) + S(r, f),\ \overline{N}\left(r, \frac{1}{f(qz+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Remark 3 *This lemma is a vital tool and will be used frequently in the proof of our theorems. It indicates that the counting function of the q-shift of a meromorphic function f could be controlled by the counting function of f.*

Lemma 4 ([25]) *Let f be a nonconstant meromorphic function and p, k be positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f)$$

and

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 5 ([13]) *Let m be a nonnegative integer or ∞. Let F and G be two nonconstant meromorphic functions, and F, G share "$(1,m)$". If $H \not\equiv 0$, then*

(i) If $2 \le m \le \infty$, then

$$T(r,F) \le N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G).$$

(ii) If $m = 1$, then

$$\begin{aligned} T(r,F) \quad \le \quad & N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) \\ & + S(r,F) + S(r,G). \end{aligned}$$

(iii) If $m = 0$, then

$$\begin{aligned} T(r,F) \quad \le \quad & N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) \\ & + \overline{N}_L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G). \end{aligned}$$

The same inequalities holds for $T(r,G)$.

Lemma 6 ([2]) *Let F and G be two nonconstant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \quad \le \quad & N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) \\ & + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) - m\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G). \end{aligned}$$

The same inequalities holds for $T(r,G)$.

Lemma 7 ([14]) *Let F and G be two nonconstant meromorphic functions and $p \ge 2$ be an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then*

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G).$$

The same inequalities holds for $T(r,G)$.

Lemma 8 *Let f be a nonconstant meromorphic function of order zero, $q_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{C}$ are complex constants, n, v_j $(j = 1, 2, \ldots, d)$ are positive integers. Define a q-shift difference polynomial in f as $F(z) = f^n(z)P(f(z))\prod_{j=1}^d f^{v_j}(q_j z + c_j)$, here $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \cdots + a_1\omega + a_0$ is a nonzero polynomial in ω of degree $m(\ge 0)$ with small coefficients $a_l \in S(f)$ $(l = 0, 1, \ldots, m)$ and $a_m \not\equiv 0$. Then*

$$(n + m - \sigma)T(r,f) \le T(r,F) + S(r,f) \le (n + m + \sigma)T(r,f) + S(r,f),$$

where $\sigma = \sum_{j=1}^d v_j$.

Remark 4 *This lemma implies that both $S(r,F)$ and $S(r,f)$ could be replaced by each other when $n \ge \sigma - m + 1$.*

Remark 5 *The above four lemmas are all estimations of the characteristic function of the q-shift differential-difference polynomials* (2)*, they play significant roles in the proof of our theorems.*

Proof of Lemma 8. It could be deduced by properties of the counting function that

$$N(r, f^n P(f)) \leq N(r, F) + N\left(r, \frac{1}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right), \tag{3}$$

and from Lemmas 2 and 3 we see that

$$\begin{aligned}
m(r, f^n P(f)) &= m\left(r, \frac{F(z)}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right) \\
&\leq m(r, F) + T\left(r, \prod_{j=1}^{d} f^{v_j}(q_j z + c_j)\right) - N\left(r, \frac{1}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right) + S(r, f) \\
&\leq m(r, F) + m(r, f^{\sigma}) + m\left(r, \prod_{j=1}^{d}\left(\frac{f(q_j z + c_j)}{f(z)}\right)^{v_j}\right) \\
&+ N\left(r, \prod_{j=1}^{d} f^{v_j}(q_j z + c_j)\right) - N\left(r, \frac{1}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right) + S(r, f) \\
&\leq m(r, F) + \sigma T(r, f) - N\left(r, \frac{1}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right) + S(r, f).
\end{aligned}$$

Combining Lemma 1 with this inequality and (3) implies

$$\begin{aligned}
(n+m)T(r, f) &= T(r, f^n P(f)) + S(r, f) = m(r, f^n P(f)) + N(r, f^n P(f)) + S(r, f) \\
&\leq m(r, F) + \sigma T(r, f) - N\left(r, \frac{1}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right) + N(r, F) \\
&+ N\left(r, \frac{1}{\prod_{j=1}^{d} f^{v_j}(q_j z + c_j)}\right) + S(r, f) \\
&= \sigma T(r, f) + T(r, F) + S(r, f),
\end{aligned}$$

which means $(n + m - \sigma)T(r, f) \leq T(r, F) + S(r, f)$.

On the other hand, by Lemmas 1–3 we obtain

$$\begin{aligned}
T(r, F) &\leq T(r, f^n P(f)) + T\left(r, \prod_{j=1}^{d} f^{v_j}(q_j z + c_j)\right) \\
&\leq (n+m)T(r, f) + m\left(r, \prod_{j=1}^{d} f^{v_j}(q_j z + c_j)\right) + N\left(r, \prod_{j=1}^{d} f^{v_j}(q_j z + c_j)\right) + S(r, f) \\
&\leq (n+m)T(r, f) + m\left(r, \prod_{j=1}^{d}\left(\frac{f(q_j z + c_j)}{f(z)}\right)^{v_j}\right) + m(r, f^{\sigma}) + \sigma N(r, f) + S(r, f) \\
&= (n + m + \sigma)T(r, f) + S(r, f).
\end{aligned}$$

This proves the lemma. ∎

Lemma 9 *Let $f,\ g$ be two nonconstant meromorphic functions, $q_j\in\mathbb{C}\setminus\{0\},\ c_j\in\mathbb{C}$ be complex constants, $n,\ k,\ v_j\ (j=1,2,\ldots,d)$ be positive integers, and let*

$$F(z)=f^n(z)P(f(z))\prod_{j=1}^{d}f^{v_j}(q_jz+c_j),\quad G(z)=g^n(z)P(g(z))\prod_{j=1}^{d}g^{v_j}(q_jz+c_j),$$

here $P(\omega)=a_m\omega^m+a_{m-1}\omega^{m-1}+\cdots+a_1\omega+a_0$ is a nonzero polynomial in ω of degree $m(\geq 0)$ with small coefficients $a_l\in S(f)\ (l=0,1,\ldots,m)$ and $a_m\not\equiv 0$. If there exists two distinct small functions $b_1,\ b_2\in S(f)\cap S(g)$ such that

$$\overline{N}\left(r,\frac{1}{F^{(k)}-b_1}\right)=\overline{N}\left(r,\frac{1}{G^{(k)}}\right),\ \overline{N}\left(r,\frac{1}{G^{(k)}-b_2}\right)=\overline{N}\left(r,\frac{1}{F^{(k)}}\right),$$

then $n\leq 3\sigma+m+kd+d+3k+3$.

Proof of Lemma 9. From Lemma 8 we know

$$(n+m-\sigma)T(r,f)\leq T(r,F)+S(r,f).$$

This, together with Lemma 4 and the second fundamental theorem concerning small functions, yields that

$$\begin{aligned}(n+m-\sigma)T(r,f)&\leq T(r,F)+S(r,f)\\&\leq T(r,F^{(k)})-\overline{N}\left(r,\frac{1}{F^{(k)}}\right)+N_{1+k}\left(r,\frac{1}{F}\right)+S(r,f)\\&\leq \overline{N}(r,F^{(k)})+\overline{N}\left(r,\frac{1}{G^{(k)}}\right)+N_{1+k}\left(r,\frac{1}{F}\right)+S(r,f)\\&\leq \overline{N}(r,F^{(k)})+k\overline{N}(r,G)+N_{1+k}\left(r,\frac{1}{G}\right)+N_{1+k}\left(r,\frac{1}{F}\right)+S(r,f)+S(r,g)\\&\leq(\sigma+m+d+k+2)T(r,f)+(\sigma+m+kd+2k+1)T(r,g)+S(r,f)+S(r,g),\end{aligned}$$

i.e.,

$$(n-2\sigma-d-k-2)T(r,f)\leq(\sigma+m+kd+2k+1)T(r,g)+S(r,f)+S(r,g).$$

A similar inequality holds for $T(r,g)$, which means

$$(n-2\sigma-d-k-2)(T(r,f)+T(r,g))\leq(\sigma+m+kd+2k+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

hence $n\leq 3\sigma+m+kd+d+3k+3$. ■

4 Proof of Theorems

Proof of Theorem 6. Denote $F(z)=f^n(z)P(f(z))\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)$, by the second fundamental theorem and Lemma 4 we have

$$\begin{aligned}T(r,F^{(k)})&\leq\overline{N}(r,F^{(k)})+\overline{N}\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-\alpha}\right)+S(r,f)\\&\leq T(r,F^{(k)})-T(r,F)+N_{1+k}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F^{(k)}-\alpha}\right)+S(r,f).\end{aligned}$$

This inequality and Lemma 8 imply that

$$\begin{aligned}(n+m-\sigma)T(r,f) &\le T(r,F)+S(r,f)\\ &\le N_{1+k}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F^{(k)}-\alpha}\right)+S(r,f)\\ &\le (\sigma+m+k+1)T(r,f)+(d+1)T(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-\alpha}\right)+S(r,f)\\ &= (\sigma+m+d+k+2)T(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-\alpha}\right)+S(r,f),\end{aligned}$$

thus

$$(n-2\sigma-d-k-2)T(r,f)\le \overline{N}\left(r,\frac{1}{F^{(k)}-\alpha}\right)+S(r,f).$$

Since $n\ge 2\sigma+d+k+3$, it could be concluded that $F^{(k)}-\alpha$ has infinitely many zeros. ■

Proof of Theorem 7. Set $F_1=\dfrac{F^{(k)}}{\alpha}$ and $G_1=\dfrac{G^{(k)}}{\alpha}$, then F_1 and G_1 share "$(1,p)$" possibly except the zeros and poles of α. Assume that

$$H=\left(\frac{F_1''}{F_1'}-\frac{2F_1'}{F_1-1}\right)-\left(\frac{G_1''}{G_1'}-\frac{2G_1'}{G_1-1}\right)\not\equiv 0,$$

then from Lemmas 4 and 8 we get

$$\begin{aligned}N_2\left(r,\frac{1}{F_1}\right) &= N_2\left(r,\frac{1}{F^{(k)}}\right)+S(r,f)\\ &\le T(r,F^{(k)})-T(r,F)+N_{2+k}\left(r,\frac{1}{F}\right)+S(r,f)\\ &\le T(r,F^{(k)})-(n+m-\sigma)T(r,f)+N_{2+k}\left(r,\frac{1}{F}\right)+S(r,f),\end{aligned}$$

i.e.,

$$(n+m-\sigma)T(r,f)\le T(r,F^{(k)})-N_2\left(r,\frac{1}{F_1}\right)+N_{2+k}\left(r,\frac{1}{F}\right)+S(r,f). \tag{4}$$

By Lemma 4 and the definition of F_1 and G_1, it could be seen that

$$\overline{N}\left(r,\frac{1}{F_1}\right)\le k\overline{N}(r,F)+N_{1+k}\left(r,\frac{1}{F}\right)+S(r,f), \tag{5}$$

$$\overline{N}\left(r,\frac{1}{G_1}\right)\le k\overline{N}(r,G)+N_{1+k}\left(r,\frac{1}{G}\right)+S(r,g), \tag{6}$$

$$N_2\left(r,\frac{1}{G_1}\right)\le k\overline{N}(r,G)+N_{2+k}\left(r,\frac{1}{G}\right)+S(r,g). \tag{7}$$

(i) When $p \geq 2$, according to (4), (7) and Lemma 5, we have

$$\begin{aligned}
(n+m-\sigma)T(r,f) &\leq T(r,F_1) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f)\\
&\leq N_2\left(r,\frac{1}{F_1}\right) + N_2\left(r,\frac{1}{G_1}\right) + N_2(r,F_1) + N_2(r,G_1)\\
&\quad -N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g)\\
&\leq N_{2+k}\left(r,\frac{1}{G}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + N_2(r,F_1) + N_2(r,G_1)\\
&\quad +k\overline{N}(r,G) + S(r,f) + S(r,g)\\
&\leq (\sigma+m+k+2)(T(r,f)+T(r,g)) + 2(1+d)T(r,f)\\
&\quad +(2+k)(1+d)T(r,g) + S(r,f) + S(r,g)\\
&= (\sigma+m+2d+k+4)T(r,f) + (\sigma+m+kd+2d+2k+4)T(r,g)\\
&\quad +S(r,f) + S(r,g).
\end{aligned}$$

Similary, we have

$$\begin{aligned}
(n+m-\sigma)T(r,g) &\leq (\sigma+m+2d+k+4)T(r,g) + (\sigma+m+kd+2d+2k+4)T(r,f)\\
&\quad +S(r,f) + S(r,g).
\end{aligned}$$

The above two inequalities signify

$$(n-3\sigma-m-kd-4d-3k-8)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which is a contradiction with $n \geq 3\sigma+m+kd+4d+3k+9$. Hence $H \equiv 0$, that is,

$$\frac{F_1''}{F_1'} - \frac{2F_1'}{F_1-1} \equiv \frac{G_1''}{G_1'} - \frac{2G_1'}{G_1-1}.$$

A simple calculation helps us to see that

$$\frac{1}{F_1-1} \equiv \frac{a}{G_1-1} + b, \tag{8}$$

where $a(\neq 0)$, b are two constants.

Case 1. Suppose that $b \neq 0$, $a = b$, then (8) means

$$\frac{1}{F_1-1} \equiv \frac{bG_1}{G_1-1}.$$

Subcase 1.1. If $b = -1$, then $F_1G_1 \equiv 1$, i.e., $F^{(k)}G^{(k)} \equiv \alpha^2$.

Subcase 1.2. If $b \neq -1$, then $\dfrac{1}{F_1} \equiv \dfrac{bG_1}{(1+b)G_1-1}$ and thus

$$\overline{N}\left(r,\frac{1}{G_1-\frac{1}{1+b}}\right) = \overline{N}\left(r,\frac{1}{F_1}\right).$$

Together with Lemmas 4 and 8 and the second fundamental theorem, this equality implies

$$\begin{aligned}
T(r,G_1) &\leq \overline{N}(r,G_1)+\overline{N}\left(r,\frac{1}{G_1}\right)+\overline{N}\left(r,\frac{1}{G_1-\frac{1}{1+b}}\right)+S(r,g)\\
&\leq \overline{N}(r,G_1)+\overline{N}\left(r,\frac{1}{G_1}\right)+\overline{N}\left(r,\frac{1}{F_1}\right)+S(r,g)\\
&\leq \overline{N}(r,G_1)+T(r,G_1)-T(r,G)+N_{1+k}\left(r,\frac{1}{G}\right)+k\overline{N}(r,F)+N_{1+k}\left(r,\frac{1}{F}\right)\\
&\quad +S(r,f)+S(r,g)\\
&\leq T(r,G_1)-(n+m-\sigma)T(r,g)+N_{1+k}\left(r,\frac{1}{G}\right)+k\overline{N}(r,F)+N_{1+k}\left(r,\frac{1}{F}\right)\\
&\quad +\overline{N}(r,G_1)+S(r,f)+S(r,g),
\end{aligned}$$

which means

$$\begin{aligned}
(n+m-\sigma)T(r,g) &\leq k\overline{N}(r,F)+N_{1+k}\left(r,\frac{1}{F}\right)+N_{1+k}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+S(r,f)+S(r,g)\\
&\leq (k+kd)T(r,f)+(\sigma+m+k+1)(T(r,f)+T(r,g))\\
&\quad +(1+d)T(r,g)+S(r,f)+S(r,g)
\end{aligned}$$

and thus

$$(n-2\sigma-d-k-2)T(r,g)\leq(\sigma+m+kd+2k+1)T(r,f)+S(r,f)+S(r,g).$$

In a similar manner, we have

$$(n-2\sigma-d-k-2)T(r,f)\leq(\sigma+m+kd+2k+1)T(r,g)+S(r,f)+S(r,g).$$

Hence $n\leq 3\sigma+m+kd+d+3k+3$, a contradiction with $n\geq 3\sigma+m+kd+4d+3k+9$.

CASE 2. Suppose that $b\neq 0,\ a\neq b$. Then by (8) we obtain

$$F_1\equiv\frac{(b+1)G_1+a-b-1}{a-b+bG_1},$$

and thus

$$\overline{N}\left(r,\frac{1}{G_1-\frac{b-a+1}{b+1}}\right)=\overline{N}\left(r,\frac{1}{F_1}\right).$$

Use the same reasoning as in subcase 1.2, a contradiction is reached.

CASE 3. Suppose that $b=0,\ a\neq 0$. Then (8) implies

$$F_1\equiv\frac{G_1+a-1}{a},\ G_1\equiv aF_1-a+1. \tag{9}$$

SUBCASE 3.1. If $a\neq 1$, then

$$\overline{N}\left(r,\frac{1}{F_1-\frac{a-1}{a}}\right)=\overline{N}\left(r,\frac{1}{G_1}\right),\ \overline{N}\left(r,\frac{1}{G_1+a-1}\right)=\overline{N}\left(r,\frac{1}{F_1}\right).$$

By Lemma 9 we have $n\leq 3\sigma+m+kd+d+3k+3<3\sigma+m+kd+4d+3k+9$, a contradiction.

SUBCASE 3.2. If $a=1$, then (9) turns into $F_1\equiv G_1$, that is,

$$\left(f^n(z)P(f(z))\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)^{(k)}\equiv\left(g^n(z)P(g(z))\prod_{j=1}^{d}g^{v_j}(q_jz+c_j)\right)^{(k)}.$$

Integrating both sides of this equality yeilds

$$\left(f^n(z)P(f(z))\prod_{j=1}^{d} f^{v_j}(q_jz+c_j)\right)^{(k-1)} \equiv \left(g^n(z)P(g(z))\prod_{j=1}^{d} g^{v_j}(q_jz+c_j)\right)^{(k-1)} + \xi_{k-1}.$$

If $\xi_{k-1} \neq 0$, since

$$n \geq 3\sigma + m + kd + 4d + 3k + 9 > 3\sigma + m + (k-1)d + d + 3(k-1) + 3,$$

by Lemma 9 we get a contradiction. Thus $\xi_{k-1} = 0$.

Carrying out the same process $k-1$ times helps us derive

$$f^n(z)P(f(z))\prod_{j=1}^{d} f^{v_j}(q_jz+c_j) \equiv g^n(z)P(g(z))\prod_{j=1}^{d} g^{v_j}(q_jz+c_j). \tag{10}$$

Now let $t = \dfrac{f}{g}$. If t is a constant, then by substituding $f = tg$ into equation (10) we see that

$$\begin{aligned} g^n \prod_{j=1}^{d} g^{v_j}(q_jz+c_j)\Big[a_m g^m(t^{\sigma+m+n}-1) + a_{m-1}g^{m-1}(t^{\sigma+m+n-1}-1) \\ + \cdots + a_1 g(t^{\sigma+n+1}-1) + a_0(t^{\sigma+n}-1)\Big] \equiv 0. \end{aligned} \tag{11}$$

Notice that g is transcendental, clearly $g^n \prod_{j=1}^{d} g^{v_j}(q_jz+c_j) \not\equiv 0$ and it follows from (11) that

$$\begin{aligned} &a_m g^m(t^{\sigma+m+n}-1) + a_{m-1}g^{m-1}(t^{\sigma+m+n-1}-1) \\ &+ \cdots + a_1 g(t^{\sigma+n+1}-1) + a_0(t^{\sigma+n}-1) \equiv 0. \end{aligned}$$

The equality above means that $t^\tau = 1$, where

$$\tau = GCD\{n+\sigma+m, n+\sigma+\eta_{m-1}, \ldots, n+\sigma+\eta_1, n+\sigma+\eta_0\}$$

with $\eta_l = l$ when $a_l \not\equiv 0$ and $\eta_l = m$ when $a_l \equiv 0$ $(l = 0, 1, \ldots, m-1)$. As a result, $f = tg$ with such a constant t.

If t is not a constant, we conclude immediately that f, g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1)\prod_{j=1}^{d} \omega_1^{v_j}(q_jz+c_j) - \omega_2^n P(\omega_2)\prod_{j=1}^{d} \omega_2^{v_j}(q_jz+c_j).$$

(ii) When $p = 1$, F_1 and G_1 share "$(1,1)$". Thus in light of (5) we know

$$\begin{aligned} \overline{N}_L\left(r, \frac{1}{F_1 - 1}\right) &\leq \frac{1}{2}N\left(r, \frac{F_1}{F_1'}\right) + S(r,f) \leq \frac{1}{2}N\left(r, \frac{F_1'}{F_1}\right) + S(r,f) \\ &\leq \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F_1}\right) + S(r,f) \\ &\leq \frac{1+k}{2}\overline{N}(r,F) + \frac{1}{2}N_{1+k}\left(r, \frac{1}{F}\right) + S(r,f). \end{aligned} \tag{12}$$

On account of (4), (7), (12) and Lemma 5, we see that

$$\begin{aligned}
(n+m-\sigma)T(r,f) \leq & T(r,F_1) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) \\
\leq & N_2\left(r,\frac{1}{F_1}\right) + N_2\left(r,\frac{1}{G_1}\right) + N_2(r,F_1) + N_2(r,G_1) \\
& + \overline{N}_L\left(r,\frac{1}{F_1-1}\right) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g) \\
\leq & N_{2+k}\left(r,\frac{1}{G}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + N_2(r,F_1) + N_2(r,G_1) \\
& + k\overline{N}(r,G) + \frac{1+k}{2}\overline{N}(r,F) + \frac{1}{2}N_{1+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g) \\
\leq & \left(\sigma+m+2d+k+4+\frac{(1+k)(1+d)}{2}+\frac{\sigma+m+k+1}{2}\right)T(r,f) \\
& + (2(1+d)+k(1+d)+(\sigma+m+k+2))T(r,g) + S(r,f) + S(r,g) \\
= & \frac{3\sigma+3m+kd+5d+4k+10}{2}T(r,f) \\
& + (\sigma+m+kd+2d+2k+4)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(n+m-\sigma)T(r,g) \leq & \frac{3\sigma+3m+kd+5d+4k+10}{2}T(r,g) \\
& + (\sigma+m+kd+2d+2k+4)T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

The above two inequalities indicate

$$\left(n-\frac{7\sigma+3m+3kd+9d+8k+18}{2}\right)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which contradicts with $n \geq \dfrac{7\sigma+3m+3kd+9d+8k+19}{2}$. Therefore $H \equiv 0$.

The rest of the proof of case (ii) could be finished along a similar argument as shown in case (i).

(iii) When $p=0$, F_1 and G_1 share "$(1,0)$", thus from (5) we have

$$\begin{aligned}
\overline{N}_L\left(r,\frac{1}{F_1-1}\right) \leq & N\left(r,\frac{F_1}{F_1'}\right) + S(r,f) \leq N\left(r,\frac{F_1'}{F_1}\right) + S(r,f) \\
\leq & \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F_1}\right) + S(r,f) \\
\leq & (1+k)\overline{N}(r,F) + N_{1+k}\left(r,\frac{1}{F}\right) + S(r,f) \qquad (13)
\end{aligned}$$

and

$$\overline{N}_L\left(r,\frac{1}{G_1-1}\right) \leq (1+k)\overline{N}(r,G) + N_{1+k}\left(r,\frac{1}{G}\right) + S(r,f). \qquad (14)$$

By virtue of (4), (7), (13), (14) and Lemma 5, we obtain

$$\begin{aligned}(n+m-\sigma)T(r,f) \leq & T(r,F_1) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f)\\ \leq & N_2\left(r,\frac{1}{F_1}\right) + N_2\left(r,\frac{1}{G_1}\right) + N_2(r,F_1) + N_2(r,G_1)\\ & + 2\overline{N}_L\left(r,\frac{1}{F_1-1}\right) + \overline{N}_L\left(r,\frac{1}{G_1-1}\right) - N_2\left(r,\frac{1}{F_1}\right)\\ & + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g)\\ \leq & N_{2+k}\left(r,\frac{1}{G}\right) + k\overline{N}(r,G) + N_2(r,F_1) + N_2(r,G_1)\\ & + 2(1+k)\overline{N}(r,F) + 2N_{1+k}\left(r,\frac{1}{F}\right) + (1+k)\overline{N}(r,G)\\ & + N_{1+k}\left(r,\frac{1}{G}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g)\\ \leq & \big[2(1+k+m+\sigma) + 2(1+d) + 2(1+k)(1+d)\\ & + (2+k+m+\sigma)\big]T(r,f) + \big[(2+k+m+\sigma) + 2(1+d) + k(1+d)\\ & + (1+k)(1+d) + (1+k+m+\sigma)\big]T(r,g) + S(r,f) + S(r,g)\\ = & (3\sigma+3m+2kd+4d+5k+8)T(r,f)\\ & + (2\sigma+2m+2kd+3d+4k+6)T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(n+m-\sigma)T(r,g) \leq & (3\sigma+3m+2kd+4d+5k+8)T(r,g)\\ & + (2\sigma+2m+2kd+3d+4k+6)T(r,f) + S(r,f) + S(r,g).\end{aligned}$$

The above two inequalities mean

$$(n-6\sigma-4m-4kd-7d-9k-14)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which contradicts with $n \geq 6\sigma+4m+4kd+7d+9k+15$. Hence $H \equiv 0$.

The rest of the proof of case (iii) could also be finished along a similar argument as shown in case (i). ■

Proof of Theorem 8. Set $F_1 = \frac{F^{(k)}}{\alpha}$ and $G_1 = \frac{G^{(k)}}{\alpha}$, then F_1 and G_1 share $(1,2)^*$ possibly except the zeros and poles of α. We also have (4)–(7). Assume that

$$H = \left(\frac{F_1''}{F_1'} - \frac{2F_1'}{F_1-1}\right) - \left(\frac{G_1''}{G_1'} - \frac{2G_1'}{G_1-1}\right) \not\equiv 0,$$

then by (4), (5), (7) and Lemma 6, we obtain

$$\begin{aligned}(n+m-\sigma)T(r,f) \leq & T(r,F_1) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) \\ \leq & N_2\left(r,\frac{1}{F_1}\right) + N_2\left(r,\frac{1}{G_1}\right) + N_2(r,F_1) + N_2(r,G_1) + \overline{N}(r,F_1) \\ & + \overline{N}\left(r,\frac{1}{F_1}\right) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g) \\ \leq & N_{2+k}\left(r,\frac{1}{G}\right) + k\overline{N}(r,G) + N_2(r,F_1) + N_2(r,G_1) + \overline{N}(r,F_1) \\ & + k\overline{N}(r,F) + N_{1+k}\left(r,\frac{1}{F}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g) \\ \leq & \big[(2+k+m+\sigma) + k(1+d) + 2(1+d)\big]T(r,g) + \big[3(1+d) + k(1+d) \\ & + (1+k+m+\sigma) + (2+k+m+\sigma)\big]T(r,f) + S(r,f) + S(r,g) \\ = & (2\sigma + 2m + kd + 3d + 3k + 6)T(r,f) \\ & + (\sigma + m + kd + 2d + 2k + 4)T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Similary, we have

$$\begin{aligned}(n+m-\sigma)T(r,g) \leq & (2\sigma + 2m + kd + 3d + 3k + 6)T(r,g) \\ & + (\sigma + m + kd + 2d + 2k + 4)T(r,f) + S(r,f) + S(r,g).\end{aligned}$$

The above two inequalities signify

$$(n - 4\sigma - 2m - 2kd - 5d - 5k - 10)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which contradicts with $n \geq 4\sigma + 2m + 2kd + 5d + 5k + 11$. Hence $H \equiv 0$.

To complete the proof, we just need a similar discussion as done in the proof of Theorem 7. Hence we omit the details. ■

Proof of Theorem 9. Set $F_1 = \dfrac{F^{(k)}}{\alpha}$ and $G_1 = \dfrac{G^{(k)}}{\alpha}$, then $\overline{E}_{p)}(1,F_1) = \overline{E}_{p)}(1,G_1)$ possibly except the zeros and poles of α. And we also have (4)–(7). Assume that

$$H = \left(\frac{F_1''}{F_1'} - \frac{2F_1'}{F_1 - 1}\right) - \left(\frac{G_1''}{G_1'} - \frac{2G_1'}{G_1 - 1}\right) \not\equiv 0,$$

then from (4)–(7) and Lemma 7, we see that

$$\begin{aligned}
(n+m-\sigma)T(r,f) \leq & T(r,F_1) - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) \\
\leq & N_2\left(r,\frac{1}{F_1}\right) + N_2\left(r,\frac{1}{G_1}\right) + 2\overline{N}\left(r,\frac{1}{F_1}\right) + \overline{N}\left(r,\frac{1}{G_1}\right) \\
& - N_2\left(r,\frac{1}{F_1}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g) \\
\leq & k\overline{N}(r,G) + N_{2+k}\left(r,\frac{1}{G}\right) + 2\left(k\overline{N}(r,F) + N_{1+k}\left(r,\frac{1}{F}\right)\right) \\
& + k\overline{N}(r,G) + N_{1+k}\left(r,\frac{1}{G}\right) + N_{2+k}\left(r,\frac{1}{F}\right) + S(r,f) + S(r,g) \\
\leq & \big[k(1+d) + (2+k+m+\sigma) + k(1+d) + (1+k+m+\sigma)\big]T(r,g) \\
& + \big[2k(1+d) + 2(1+k+m+\sigma) + (2+k+m+\sigma)\big]T(r,f) \\
& + S(r,f) + S(r,g) \\
= & (2\sigma + 2m + 2kd + 4k + 3)T(r,g) \\
& + (3\sigma + 3m + 2kd + 5k + 4)T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

Similary, we have

$$\begin{aligned}
(n+m-\sigma)T(r,g) \leq & (2\sigma + 2m + 2kd + 4k + 3)T(r,f) \\
& + (3\sigma + 3m + 2kd + 5k + 4)T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

The above two inequalities imply that

$$(n - 6\sigma - 4m - 4kd - 9k - 7)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which contradicts with $n \geq 6\sigma + 4m + 4kd + 9k + 8$. Hence $H \equiv 0$.

To complete the proof, we just need a similar discussion as done in the proof of Theorem 7. Hence we omit the details. ■

Proof of Theorem 10. Since $F^{(k)}$ and $G^{(k)}$ share $(1,\infty)$ and (∞,∞), there must be a nonzero constant C such that $\dfrac{F^{(k)}-1}{G^{(k)}-1} = C$ and hence $F^{(k)} = CG^{(k)} - C + 1$. Integrate both sides of this equation k times, then we see that

$$F(z) = CG(z) + \frac{1-C}{k!}z^k + p(z), \tag{15}$$

where $p(z)$ is a polynomial of degree $\deg(p(z)) \leq k-1$. Denote $q(z) := \dfrac{1-C}{k!}z^k + p(z)$, then $\overline{N}\left(r,\dfrac{1}{F-q}\right) =$

$\overline{N}\left(r,\dfrac{1}{G}\right)$. Now assert that $q(z)\equiv 0$. Otherwise, by the second fundamental theorem we get

$$\begin{aligned}
T(r,F) \leq& \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-q}\right)+S(r,F)\\
\leq& \overline{N}(r,f)+\overline{N}_O\left(r,\infty;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)+\overline{N}\left(r,\frac{1}{f}\right)\\
&+\overline{N}_O\left(r,0;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{P(g)}\right)\\
&+\sum_{j=1}^{d}\overline{N}\left(r,\frac{1}{g(q_jz+c_j)}\right)+S(r,f)+S(r,g)\\
\leq& 2T(r,f)+(1+m+d)T(r,g)+\overline{N}_O\left(r,\infty;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)\\
&+\overline{N}_O\left(r,0;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)+S(r,f)+S(r,g). \qquad (16)
\end{aligned}$$

It is apparent that

$$nm(r,f)=m(r,f^n)\leq m(r,F)+m\left(r,\frac{1}{P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)}\right),$$

$$\begin{aligned}
nN(r,f)=&N(r,f^n)=N\left(r,\frac{F}{P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)}\right)\\
\leq& N(r,F)+N\left(r,\frac{1}{P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)}\right)\\
&-\overline{N}_O\left(r,\infty;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)-\overline{N}_O\left(r,0;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)\\
&+S(r,f).
\end{aligned}$$

Combine the above two inequalities, we obtain

$$\begin{aligned}
nT(r,f)\leq& T(r,F)+T\left(r,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)-\overline{N}_O\left(r,\infty;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)\\
&-\overline{N}_O\left(r,0;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)+S(r,f)\\
\leq& T(r,F)+(m+\sigma)T(r,f)-\overline{N}_O\left(r,\infty;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)\\
&-\overline{N}_O\left(r,0;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)+S(r,f)
\end{aligned}$$

for the reason that Lemmas 1 and 3 holds. Furthermore,

$$\begin{aligned}(n-m-\sigma)T(r,f) \leq & T(r,F)-\overline{N}_O\left(r,\infty;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)\\ & -\overline{N}_O\left(r,0;F,P(f)\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\right)+S(r,f).\end{aligned}$$

This together with inequality (16) implies

$$(n-m-\sigma-2)T(r,f)\leq(1+m+d)T(r,g)+S(r,f)+S(r,g).$$

Similarly,

$$(n-m-\sigma-2)T(r,g)\leq(1+m+d)T(r,f)+S(r,f)+S(r,g).$$

Hence

$$(n-\sigma-2m-d-3)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

which contradicts with $n\geq\sigma+2m+d+4$. As a result, $q(z)\equiv 0$, i.e., $p(z)=\dfrac{C-1}{k!}z^k$. Note that $\deg(p(z))\leq k-1$, we have $C=1$ and thus $p(z)\equiv 0$. From (15) we conclude that

$$f^n(z)P(f(z))\prod_{j=1}^{d}f^{v_j}(q_jz+c_j)\equiv g^n(z)P(g(z))\prod_{j=1}^{d}g^{v_j}(q_jz+c_j). \tag{17}$$

Set $t=\dfrac{f}{g}$. If t is a constant, then we have

$$\begin{aligned}g^n\prod_{j=1}^{d}g^{v_j}(q_jz+c_j)\Big[&a_mg^m(t^{\sigma+m+n}-1)+a_{m-1}g^{m-1}(t^{\sigma+m+n-1}-1)\\ &+\cdots+a_1g(t^{\sigma+n+1}-1)+a_0(t^{\sigma+n}-1)\Big]\equiv 0\end{aligned} \tag{18}$$

after substituting $f=tg$ into equation (17). Notice that g is transcendental, obviously $g^n\prod_{j=1}^{d}g^{v_j}(q_jz+c_j)\not\equiv 0$, and it follows from (18) that

$$\begin{aligned}&a_mg^m(t^{\sigma+m+n}-1)+a_{m-1}g^{m-1}(t^{\sigma+m+n-1}-1)\\ &+\cdots+a_1g(t^{\sigma+n+1}-1)+a_0(t^{\sigma+n}-1)\equiv 0.\end{aligned}$$

This equality implies $t^\tau=1$, where

$$\tau=GCD\{n+\sigma+m,n+\sigma+\eta_{m-1},\ldots,n+\sigma+\eta_1,n+\sigma+\eta_0\}$$

with $\eta_l=l$ when $a_l\not\equiv 0$ and $\eta_l=m$ when $a_l\equiv 0$ $(l=0,1,\ldots,m-1)$. As a result, $f=tg$ with such a constant t.

If t is not a constant, then $f,\ g$ satisfy the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2)=\omega_1^nP(\omega_1)\prod_{j=1}^{d}\omega_1^{v_j}(q_jz+c_j)-\omega_2^nP(\omega_2)\prod_{j=1}^{d}\omega_2^{v_j}(q_jz+c_j).$$

■

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