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A General Reverse Of Young's Inequalities*

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Abstract

In this paper, we obtain a general reverse of Young's inequalities, and then some inequalities for operators and Hilbert-Schmidt norms will be presented.

1 Introduction and Preliminaries

Let $B(\mathcal{H})$ denote the $\mathbb{C}^*$-Algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, denoted by $A \geq 0$. Moreover, we denote $B(\mathcal{H})^+$ as the set of all positive operators, and $B(\mathcal{H})^{++}$ as the set of all invertible operators in $B(\mathcal{H})^+$. Let $\mathbf{M}_n$ be the algebra of all complex matrices of order $n \times n$ and $\mathbf{M}_n(\mathbb{C})^+$ be the class of strictly positive matrices in $\mathbf{M}_n(\mathbb{C})$. A matrix norm $|||.|||$ is unitarily invariant if $|||UAV||| = |||A|||$ for every $A \in \mathbf{M}_n$ and all unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$. For $A = (a_{i,j}) \in \mathbf{M}_n(\mathbb{C})$ the Hilbert-Schmidt norm is defined by $||A||_2 = \sqrt{\sum_{i,j=1}^n a_{i,j}^2}$. It is well known that the norm $||.||_2$ is unitarily invariant.

Assume that A, B are positive operators on a complex Hilbert space $\mathcal{H}$ and $\nu \in (0, 1)$. The weighted operator arithmetic mean for the pair (A, B) is defined by

$$A\nabla_\nu B := (1-\nu)A + \nu B.$$

In 1980, Kubo and Ando introduced the weighted operator geometric mean for the pair (A, B) with A positive and invertible and B positive by

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}.$$

These means can be rewritten by simplification as $A\nabla B$ and $A\sharp B$ for our convenience when $\nu = \frac{1}{2}$.

In the rest of this paper, if there is no special explanation, we default to $a, b > 0$ and $v \in [0, 1]$. Young's inequality is one of the most basic inequalities in Mathematics, which states

$$a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b,$$

the equality holds if and only if $a = b$. When $\nu \in [0, 1]$ we have the following fundamental operator means inequalities, or Young's inequality for operators

$$A\sharp_\nu B \leq A\nabla_\nu B.$$

Kittaneh and Manasrah [16] refined Young's inequality as follows

$$a^{1-\nu}b^\nu + r_0(\sqrt{a} - \sqrt{b})^2 \leq (1-\nu)a + \nu b, \tag{1}$$

where $r_0 = \min\{\nu, 1-\nu\}$. Earlier, the following squared version was shown in [10]

$$(a^{1-\nu}b^\nu)^2 + r_0^2(a-b)^2 \leq ((1-\nu)a + \nu b)^2, \tag{2}$$

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where $r_0 = \min\{\nu, 1-\nu\}$.

Reverses of (1) and (2) were shown in [17] as follows

$$((1-\nu)a + \nu b)^2 \le (a^{1-\nu}b^{\nu})^2 + R_0^2(a-b)^2, \tag{3}$$

and

$$(1-\nu)a + \nu b \le a^{1-\nu}b^{\nu} + R_0(\sqrt{a}-\sqrt{b})^2, \tag{4}$$

where $R_0 = \max\{\nu, 1-\nu\}$. It turns out that both (1) and (2) are special cases of the following inequality [1]

$$(a^{1-\nu}b^{\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \le ((1-\nu)a + \nu b)^m,$$

where $r_0 = \min\{\nu, 1-\nu\}$ for all positive integers m. The inequalities (1), (2), (3) and (4), then were shown to be special cases of the following more general result [2, 8].

Theorem 1 **([2, 8])** *Let $a, b > 0$ and $0 < \nu < 1$. Then for all positive integers m,*

$$r_0^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big) \le ((1-\nu)a+\nu b)^m - (a^{1-\nu}b^{\nu})^m \le R_0^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big), \tag{5}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $R_0 = \max\{\nu, 1-\nu\}$.

Definition 1 **([9])** *Let n be a positive integer. We consider the sequence $(r_n(\nu))$ of functions on $[0,1]$ defined by.*

$$r_0(\nu) = \min\{\nu, 1-\nu\} \quad \text{and} \quad r_n(\nu) = \min\{2r_{n-1}(\nu), 1-2r_{n-1}(\nu)\}.$$

Definition 2 **([9])** *Let $a, b > 0$, for $l, k \in \mathbb{N}$, we define the functions $f_{l,k}(a,b)$ by*

$$f_{l,k}(a,b) = \left(\sqrt{a^{1-\frac{k-1}{2^l}}b^{\frac{k-1}{2^l}}} - \sqrt{a^{1-\frac{k}{2^l}}b^{\frac{k}{2^l}}}\right)^2.$$

Choi in [9] proved the following multiple term refinement and the reverses of Young's inequality as follows

Theorem 2 *Let $a, b > 0$ and $0 \le \nu \le 1$. Then for all a positive integer N, we have*

$$(1-\nu)a + \nu b \quad \ge \quad a^{1-\nu}b^{\nu} + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} f_{l,k}(a,b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu) \tag{6}$$

and

$$(1-\nu)a + \nu b \quad \le \quad a^{1-\nu}b^{\nu} + (\sqrt{a}-\sqrt{b})^2 - \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} f_{l,k}(b,a)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu). \tag{7}$$

We refer the interested reader to [2, 11, 12, 13, 14, 15, 20] as a sample of recent progress in this direction.

In this paper, we obtain some general inequalities of (7) in Theorem 2, and then some general inequalities for operators and Hilbert-Schmidt norms are obtained using our new scalars results.

2 Main Results

2.1 Scalar Inequalities

In the section, we firstly show the scalars inequalities which are the base of this paper, before that, we list some lemmas that we will need in our analysis.

Lemma 1 ([9]) *Let $a, b > 0$, $0 \leq \nu \leq 1$, and N be a positive integer, define $R_N(a,b,\nu)$ by*

$$R_N(a,b,\nu) = (1-\nu)a + \nu b - \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} f_{l,k}(a,b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu).$$

Then we have

$$R_N(a,b,\nu) = \sum_{k=1}^{2^N}\Big((k-2^N\nu)a^{1-\frac{k-1}{2^N}}b^{\frac{k-1}{2^N}} + (2^N\nu - k+1)a^{1-\frac{k}{2^N}}b^{\frac{k}{2^N}}\Big)\chi_{(\frac{k-1}{2^N},\frac{k}{2^N})}(\nu).$$

Lemma 2 *Let $a, b > 0$ and N, m be two positive integers, and $0 \leq \nu \leq 1$. Then we have*

$$\begin{aligned}&(1-\nu^m)b^m + \nu^m a^m - \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l} f_{l,k}(b^m,a^m)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m)\\ =\;& \sum_{k=1}^{2^N}\Big((k-2^N\nu^m)a^{\frac{m(k-1)}{2^N}}b^{m(1-\frac{k-1}{2^N})} + (2^N\nu^m-k+1)a^{\frac{mk}{2^N}}b^{m(1-\frac{k}{2^N})}\Big)\chi_{(\frac{k-1}{2^N},\frac{k}{2^N})}(\nu^m).\end{aligned}$$

Proof. Taking a, b and ν by b^m, a^m and ν^m, respectively, in Lemma 1, one gets the desired result. ■

Lemma 3 *Let n be a positive integer. For $k = 1, 2, \ldots, n$, let $x_k > 0$ and $\nu_k \geq 0$ which satisfy $\sum_{k=1}^n \nu_k = 1$. Then*

$$\prod_{k=1}^n x_k^{\nu_k} \leq \sum_{k=1}^n \nu_k x_k. \tag{8}$$

Lemma 4 ([13]) *Let m be a positive integer and let ν be a positive number, such that $0 \leq \nu \leq 1$. Then*

$$\sum_{k=1}^m \binom{m}{k} k\nu^k(1-\nu)^{m-k} = m\nu, \tag{9}$$

and

$$\sum_{k=0}^{m-1}\binom{m}{k}(m-k)\nu^k(1-\nu)^{m-k} = m(1-\nu), \tag{10}$$

where $\binom{m}{k}$ is the binomial coefficient.

Now we are ready to state and prove our first main result about Young's inequalities.

Theorem 3 *Let $a, b > 0$ and $0 \leq \nu \leq 1$. Then for all positive integer N, we have*

$$\begin{aligned}((1-\nu)a+\nu b)^m \;\leq\;& (a^{1-\nu}b^\nu)^m + \Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big)\\ &- \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l} f_{l,k}(b^m,a^m)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m).\end{aligned} \tag{11}$$

Proof. Suppose that $0 \leq \nu \leq 1$. The inequality (11), is equivalent to

$$2^m(ab)^{\frac{m}{2}} \leq (a+b)^m - (\nu a + (1-\nu)b)^m + (a^{1-\nu}b^\nu)^m - \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l} f_{l,k}(b^m,a^m)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m).$$

By using Lemma 2, we have the following identities

$$
\begin{aligned}
&(a+b)^m-((1-\nu)a+\nu b)^m+(a^{1-\nu}b^\nu)^m-\sum_{l=0}^{N-1}r_l(\nu^m)\sum_{k=1}^{2^l}f_{l,k}(b^m,a^m)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m)\\
=\;&\sum_{k=0}^{m}\binom{m}{k}\Big(1-(1-\nu)^k\nu^{m-k}\Big)a^kb^{m-k}+(a^{1-\nu}b^\nu)^m\\
&-\sum_{l=0}^{N-1}r_l(\nu^m)\sum_{k=1}^{2^l}f_{l,k}(b^m,a^m)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m)\\
=\;&\sum_{k=0}^{m}\binom{m}{k}\Big(1-(1-\nu)^k\nu^{m-k}\Big)a^kb^{m-k}+(a^{1-\nu}b^\nu)^m-(1-\nu^m)b^m-\nu^ma^m\\
&+\sum_{k=1}^{2^N}\Big((k-2^N\nu^m)a^{\frac{m(k-1)}{2^N}}b^{m(1-\frac{k-1}{2^N})}+(2^N\nu^m-k+1)a^{m(\frac{k}{2^N})}b^{m(1-\frac{k}{2^N})}\Big)\chi_{(\frac{k-1}{2^N},\frac{k}{2^N})}(\nu^m)\\
=\;&\sum_{k=1}^{m-1}\binom{m}{k}\Big(1-(1-\nu)^k\nu^{m-k}\Big)a^kb^{m-k}+(a^{1-\nu}b^\nu)^m+\Big(1-\nu^m-(1-\nu)^m\Big)a^m\\
&+\sum_{k=1}^{2^N}\Big((k-2^N\nu^m)a^{\frac{m(k-1)}{2^N}}b^{m(1-\frac{k-1}{2^N})}+(2^N\nu^m-k+1)a^{\frac{mk}{2^N}}b^{m(1-\frac{k}{2^N})}\Big)\chi_{(\frac{k-1}{2^N},\frac{k}{2^N})}(\nu^m).
\end{aligned}
$$

Hence, it suffices to prove that, if $\nu^m\in(\frac{k-1}{2^N},\frac{k}{2^N})$, then

$$
\begin{aligned}
&\frac{1}{2^m}\Big[\sum_{k=1}^{m-1}\binom{m}{k}\Big(1-(1-\nu)^k\nu^{m-k}\Big)a^kb^{m-k}+(a^{1-\nu}b^\nu)^m+\Big(1-\nu^m-(1-\nu)^m\Big)a^m\\
&+(k-2^N\nu^m)a^{\frac{m(k-1)}{2^N}}b^{m(1-\frac{k-1}{2^N})}+(2^N\nu^m-k+1)a^{\frac{mk}{2^N}}b^{m(1-\frac{k}{2^N})}\Big]\ge(ab)^{\frac{m}{2}}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{1}{2^m}\Big[\sum_{k=1}^{m-1}\binom{m}{k}\Big(1-(1-\nu)^k\nu^{m-k}\Big)a^kb^{m-k}+(a^{1-\nu}b^\nu)^m+\Big(1-\nu^m-(1-\nu)^m\Big)a^m\\
&+(k-2^N\nu^m)a^{\frac{m(k-1)}{2^N}}b^{m(1-\frac{k-1}{2^N})}+(2^N\nu^m-k+1)a^{\frac{mk}{2^N}}b^{m(1-\frac{k}{2^N})}\Big]\\
=\;&\sum_{k=1}^{m+3}2^{-m}\nu_kx_k,
\end{aligned}
$$

where

$$
x_k=\begin{cases}
a^kb^{m-k} & 1\le k\le m-1,\\
a^m & k=m,\\
(a^{1-\nu}b^\nu)^m & k=m+1,\\
a^{\frac{m(k-1)}{2^N}}b^{m(1-\frac{k-1}{2^N})} & k=m+2,\\
a^{\frac{mk}{2^N}}b^{m(1-\frac{k}{2^N})} & k=m+3,
\end{cases}
$$

and

$$
\nu_k=\begin{cases}
\binom{m}{k}\Big(1-(1-\nu)^k\nu^{m-k}\Big) & 1\le k\le m-1,\\
1-(1-\nu)^m-\nu^m & k=m,\\
1 & k=m+1,\\
(k-2^N\nu^m) & k=m+2,\\
(2^N\nu^m-k+1) & k=m+3.
\end{cases}
$$

Therefore

1. $x_k > 0$ for all $k \in \{1, \ldots, m+3\}$,
2. $\nu_k \geq 0$ for all $k \in \{1, \ldots, m+3\}$, with $\sum_{k=1}^{m+3} 2^{-m}\nu_k = 1$.

By the arithmetic-geometric mean inequality (8), we get

$$\sum_{k=1}^{m+3} 2^{-m}\nu_k x_k \geq \prod_{k=1}^{m+3} x_k^{2^{-m}\nu_k} = a^{\alpha(m)} b^{\beta(m)},$$

where

$$\begin{aligned}
\alpha(m) &= 2^{-m}\Big[\sum_{k=1}^{m-1}\binom{m}{k} k\Big(1-(1-\nu)^k\nu^{m-k}\Big) + m(1-\nu) \\
&\quad + m\Big(1-(1-\nu)^m-\nu^m\Big) \\
&\quad + (k-2^N\nu^m)\frac{m(k-1)}{2^N} \\
&\quad + (2^N\nu^m-k+1)\frac{mk}{2^N}\Big] \\
&= 2^{-m}\Big[m(2^{m-1}-1) - m(1-\nu) + m(1-\nu)^m + m(1-\nu) \\
&\quad + m\Big(1-(1-\nu)^m-\nu^m\Big) + m\nu^m\Big] \\
&= \frac{m}{2}, \quad \text{(by Lemma 4)}
\end{aligned}$$

and

$$\begin{aligned}
\beta(m) &= 2^{-m}\Big[\sum_{k=1}^{m-1}\binom{m}{k}(m-k)\Big(1-(1-\nu)^k\nu^{m-k}\Big) + m\nu \\
&\quad + (k-2^N\nu^m)m(1-\frac{k-1}{2^N}) \\
&\quad + (2^N\nu^m-k+1)m(1-\frac{k}{2^N})\Big] \\
&= 2^{-m}\Big[m(2^{m-1}-1) - m\nu + m\nu^m + m\nu \\
&\quad + m(1-\nu^m)\Big] \\
&= \frac{m}{2}, \quad \text{(by Lemma 4).}
\end{aligned}$$

This completes the proof. ■

Remark 1 *By taking $m = 1$ in Theorem 3, then we recapture inequality (7) in Theorem 2.*

2.2 Operator Inequalities

Based on the scalar inequalities mentioned above and the monotonic property of operator functions, we obtain the operators versions of these inequalities.

Lemma 5 ([21, p. 3]) *Let $T \in B(\mathcal{H})$ be a self-adjoint operator. If f and g are both continuous real valued functions with $f(t) \geq g(t)$ for $t \in Sp(T)$ (where the sign $Sp(T)$ denotes the spectrum of T), then $f(T) \geq g(T)$.*

An analogue of Theorem 3 for operators is given by the following theorem.

Theorem 4 *Let $A, B \in B(\mathcal{H})^{++}$, $0 \le \nu \le 1$ and let m be a positive integer. Then*

$$\begin{aligned} A\sharp_m(A\nabla_\nu B) &\le A\sharp_{m\nu}B + \Big(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\Big) \\ &\quad - \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l}\Big(A\sharp_{m\frac{(2^l-k+1)}{2^l}}B + A\sharp_{m\frac{(2^l-k)}{2^l}}B - 2A\sharp_{m\frac{(2^l-2k+1)}{2^{l+1}}}B\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m). \end{aligned}$$

Proof. Let $a = 1$ in Theorem 3. Then

$$\begin{aligned} ((1-\nu)+\nu b)^m &\le b^{m\nu} + ((b+1)^m - 2^m b^{\frac{m}{2}}) \\ &\quad - \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l}\Big(b^{m\frac{(2^l-k+1)}{2^l}} + b^{m\frac{(2^{l+1}-k)}{2^l}} - 2b^{m\frac{(2^{l+1}-2k+1)}{2^{l+1}}}\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m). \quad (12) \end{aligned}$$

Since the operator $C = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, Lemma 5 and inequality (12) imply that

$$\begin{aligned} ((1-\nu)I+\nu C)^m &\le C^{m\nu} + ((C+I)^m - 2^m C^{\frac{m}{2}}) \\ &\quad - \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l}\Big(C^{m\frac{(2^l-k+1)}{2^l}} + C^{m\frac{(2^l-k)}{2^l}} - 2C^{m\frac{(2^{l+1}-2k+1)}{2^{l+1}}}\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m). \quad (13) \end{aligned}$$

Multiplying inequality (13) by $A^{\frac{1}{2}}$ on the left and right sides, we get

$$\begin{aligned} A\sharp_m(A\nabla_\nu B) &\le A\sharp_{m\nu}B + \Big(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\Big) \\ &\quad - \sum_{l=0}^{N-1} r_l(\nu^m)\sum_{k=1}^{2^l}\Big(A\sharp_{m\frac{(2^l-k+1)}{2^l}}B + A\sharp_{m\frac{(2^l-k)}{2^l}}B - 2A\sharp_{m\frac{(2^l-2k+1)}{2^{l+1}}}B\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^m). \end{aligned}$$

The proof is then completed. ■

2.3 Refinement of Young's Inequality for Hilbert-Schmidt Norms

In this subsection, we are concerned by establishing a new refinement of Young's inequality for Hilbert-Schmidt norms. Precisely, we show the following result.

Theorem 5 *Let $0 \le \nu \le 1$ and let $A, B \in \mathbf{M}_n(\mathbb{C})^+$ and $X \in \mathbf{M}_n(\mathbb{C})$. Then for all positive integer N, we have*

$$\begin{aligned} ||(1-\nu)AX + \nu XB||_2^2 &\le ||A^{1-\nu}XB^\nu||_2^2 + ||AX - XB||_2^2 \\ &\quad + \sum_{l=0}^{N-1} r_l(\nu^2)\sum_{k=1}^{2^l}||B^{1-\frac{(k-1)}{2^l}}XA^{\frac{(k-1)}{2^l}} - B^{1-\frac{k}{2^l}}XA^{\frac{k}{2^l}}||_2^2\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^2). \end{aligned}$$

Proof. Since A and B are positive matrices, then by the spectral decomposition theorem, there exist unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$ satisfying $A = UD_1U^*$, $B = VD_2V^*$, where $D_1 = diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $D_2 = diag(\beta_1, \beta_2, \ldots, \beta_n)$, $(\alpha_i \ge 0, \beta_i \ge 0,\ i = 1, 2, \ldots, n)$. Suppose that $Y = U^*XV = [y_{i,j}]$, we have

$$(1-\nu)AX + \nu XB = U((1-\nu)D_1Y + \nu YD_2)V^* = U((((1-\nu)\alpha_i + \nu\beta_j))y_{i,j})V^*,$$

$$A^{1-\nu}XB^\nu = U(\alpha_i^{1-\nu}\beta_j^\nu y_{i,j})V^*,$$
$$AX - XB = U[(\alpha_i - \beta_j)y_{i,j}]V^*,$$

and

$$B^{1-\frac{k-1}{2^l}}XA^{\frac{k-1}{2^l}} - B^{1-\frac{k}{2^l}}XA^{\frac{k}{2^l}} = U\Big(\Big(\beta_i^{\frac{k-1}{2^l}}\alpha_i^{1-\frac{k-1}{2^l}} - \beta_i^{\frac{k}{2^l}}\alpha_i^{1-\frac{k}{2^l}}\Big)y_{i,j}\Big)V^*.$$

Now by the Theorem 3 for $m = 2$ and the unitarily invariant of the Hilbert-Schmidt norm, we have

$$\begin{aligned}
||(1-\nu)AX + \nu XB||_2^2 &= \sum_{i,j=1}^{n}((1-\nu)\alpha_i + \nu\beta_j))^2|y_{i,j}|^2 \\
&\leq \sum_{i,j=1}^{n}(\alpha_i^{1-\nu}\beta_j^{\nu})^2|y_{i,j}|^2 \\
&\quad + \sum_{i,j=1}^{n}(\alpha_i - \beta_j)^2|y_{i,j}|^2 + \sum_{l=0}^{N-1} r_l(\nu^2)\sum_{k=1}^{2^l}\sum_{i,j=1}^{n} f_{l,k}(\beta_j^2, \alpha_i^2)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^2)|y_{i,j}|^2 \\
&= ||A^{1-\nu}XB^{\nu}||_2^2 + ||AX - XB||_2^2 \\
&\quad + \sum_{l=0}^{N-1} r_l(\nu^2)\sum_{k=1}^{2^l}||B^{1-\frac{(k-1)}{2^l}}XA^{\frac{(k-1)}{2^l}} - B^{1-\frac{k}{2^l}}XA^{\frac{k}{2^l}}||_2^2\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\nu^2).
\end{aligned}$$

This completes the proof of our result. ■

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