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The Natural Density Of Some Sets Of r-Free Numbers*

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Abstract

Let T be a possibly infinite set of prime numbers none of which divide an r-free number s. A simple formula is given for the natural density of the set of r-free numbers n divisible by s with $\gcd(s, n/s) = 1$ and not divisible by any of the primes in T.

1 Main Results

In 1885 Gegenbauer proved that the natural density of the set of square-free numbers, i.e., the proportion of natural numbers which are square-free, is $6/\pi^2$ [2, Theorem 333; reference on page 272]. By adapting an argument giving the natural density of the set of square-free numbers, G. J. O. Jameson in 2008 was able to prove that one third of the square-free numbers are even [5]. (Jameson gives a more elementary argument in [6].) This result has been recently generalized in several ways. Brown determined the proportion of square-free numbers which are divisible by every member of a given finite set of primes and at the same time not divisible by any of the primes in a given possibly infinite set of primes [1, Theorem 1], while Srisopha, Srichan and Mavecha determined the proportion of r-free numbers (i.e., numbers which are divisible by no r-th power greater than one) which are even [9, Theorem 1] and then went on in [10] to prove for r-free numbers a generalizaton of an important special case of Brown's work cited above (see Example 5 below). The following theorem generalizes all of these results. In the theorem ζ denotes the Riemann zeta function; it is known that the natural density of the set of r-free numbers is $1/\zeta(r)$ [4, Equation 14.24].

Theorem 1 *Let S and T be disjoint sets of prime numbers with S finite and let $r > 1$ be a positive integer. Then the natural density of the set of r-free numbers which are divisible by all the prime numbers in S and by none of those in T is*

$$\frac{1}{\zeta(r)} \prod_{p\in T} \frac{p^r - p^{r-1}}{p^r - 1} \prod_{p\in S} \frac{p^{r-1} - 1}{p^r - 1}.$$

We will remark in Section 4 that the first product in the theorem above is well-defined even if T is not finite.

Some comments about our notation. Throughout the paper the letter p will only be used to denote a prime number. The term *number* will always refer to a positive integer (but *positive real number* will have its usual meaning). Empty products, such as occur above if S or T is empty, are understood to equal 1. For any real number x and set B of numbers, we let $B\{x\}$ denote the set of elements of B less than or equal to x and let $B[x]$ denote the number of elements of $B\{x\}$. Recall that if $\lim_{x\to\infty} B[x]/x$ exists, then it is by definition the *natural density* of B [7, Definition 11.1]. In case it exists we will denote the natural density of B by B^*.

Examples 1 *1. Applying Theorem 1 with $r = 2$, $S = \{2\}$ and T empty, we see that the natural density of the set of even square-free numbers is $\frac{6}{\pi^2}\frac{2-1}{2^2-1}$. Thus one third of square-free numbers are even and two thirds are odd. (These are Jameson's results of course.)*

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2. Take $r = 2$ and let S and T be as in the theorem above. Then the natural density of the set of square-free numbers which are divisible by all the primes in S and by none of those in T is $\frac{6}{\pi^2}\prod_{p\in T}\frac{p}{1+p}\prod_{p\in S}\frac{1}{1+p}$. This fact is Theorem 1 of [1].

3. For all choices of r the natural density of the set O of odd r-free numbers is

$$\frac{1}{\zeta(r)}\frac{2^{r-1}}{2^r-1}$$

(we apply the above theorem with S empty and $T = \{2\}$). Thus the natural density of the set E of of even r-free numbers is

$$\frac{1}{\zeta(r)} - \frac{1}{\zeta(r)}\frac{2^{r-1}}{2^r-1}$$

so

$$\frac{O^*}{E^*} = \frac{2^{r-1}}{2^{r-1}-1} = \frac{2^r}{2^r-2}.$$

This fact is Theorem 1 of [9].

4. Again let r be arbitrary and suppose that b and m are relatively prime numbers. Let $T = \{p : p \equiv b \pmod{m}\}$ and let S be empty. Note that

$$\prod_{p\in T}\frac{p^r-p^{r-1}}{p^r-1} \le \prod_{p\in T}\frac{p}{1+p} = 0$$

[1, Section 5]. Therefore the natural density of the set of r-free numbers which are not divisible by any prime number congruent to b modulo m is zero. This fact generalizes Theorem 2 of [1].

5. Let P denote a finite set of prime numbers. We use some notation of [10]: let C_P denote the set of r-free numbers which are divisible by none of the primes of P and let C'_P denote its complement in the set of $r-$free numbers, i.e., the set of r-free numbers which are divisible by at least one of the primes in P. Then by the above theorem we have

$$\frac{{C'_P}^*}{C_P^*} = \frac{\frac{1}{\zeta(r)} - \frac{1}{\zeta(r)}\prod_{p\in P}\frac{p^r-p^{r-1}}{p^r-1}}{\frac{1}{\zeta(r)}\prod_{p\in P}\frac{p^r-p^{r-1}}{p^r-1}} = \prod_{p\in P}\frac{p^r-1}{p^r-p^{r-1}} - 1,$$

which is equivalent to the assertion of Theorem 1.1 of [10].

The above theorem will be proven as a corollary to a more subtle theorem which we will now state using the notion of a Steinitz number. A *Steinitz number* is a formal product of a possibly infinite number of possibly infinite prime powers; such a formal product is called *finite* if it is a product of a finite number of finite prime powers (such Steinitz numbers of course correspond precisely to positive integers). For a more formal definition of a Steinitz number (and for related definitions such as that of divisibility of Steinitz numbers) the reader might consult [8, p. 171].

Theorem 2 *Let r be a number greater than 1. Suppose that s and t are relatively prime r-free Steinitz numbers with s finite. Then the natural density of the set of r-free numbers n which are divisible by s and have $\gcd(s, n/s) = 1$ and which are also relatively prime to t is*

$$\frac{1}{s\zeta(r)}\prod_{p|st}\frac{p^r-p^{r-1}}{p^r-1}.$$

The condition $\gcd(s, n/s) = 1$ in the above theorem says roughly that the factorization of s into a product of prime powers is part of the factorization of n into a product of prime powers. Again, in Section 4 we will remark that the product above is well defined. The five examples above could have been easily derived directly from Theorem 2.

The "simple formula" referred to in the Abstract of this paper can be taken to be the formula of Theorem 2 where we let $t = \prod_{p \in T} p$.

We end this section by proving Theorem 1 while assuming Theorem 2. The proof of Theorem 2 itself will occupy the next three sections. In this argument and in the remainder of the paper we will let $A(s,t)$ denote the set of r-free numbers n relatively prime to t and divisible by s with $\gcd(s, n/s) = 1$ (for any Steinitz numbers s and t). Thus, for example, $A(1,1)$ is the set of all r-free numbers.

Proof. Write $S = \{p_1, \cdots, p_m\}$ and let t be the product of the primes in T. For any r-free number divisible by a prime p the highest power of p dividing it has the form p^i where $1 \leq i < r$. Thus the set of r-free numbers divisible by all the numbers in S and none of those in T is a disjoint union

$$\cup_{i_1=1}^{r-1} \cdots \cup_{i_m=1}^{r-1} A(p_1^{i_1} \cdots p_m^{i_m}, t)$$

so that by Theorem 2 its natural density is

$$\begin{aligned} & \frac{1}{\zeta(r)} \sum_{i_1=1}^{r-1} \cdots \sum_{i_m=1}^{r-1} \frac{1}{p_1^{i_1} \cdots p_m^{i_m}} \prod_{p \in S \cup T} \frac{p^r - p^{r-1}}{p^r - 1} \\ = \quad & \frac{1}{\zeta(r)} \prod_{p \in T} \frac{p^r - p^{r-1}}{p^r - 1} \prod_{p \in S} \left(\frac{p^r - p^{r-1}}{p^r - 1} \sum_{i=1}^{r-1} \frac{1}{p^i} \right) \\ = \quad & \frac{1}{\zeta(r)} \prod_{p \in T} \frac{p^r - p^{r-1}}{p^r - 1} \prod_{p \in S} \frac{p^{r-1} - 1}{p^r - 1}. \end{aligned}$$

■

2 A Basic Lemma

For the remainder of this paper r will denote a fixed number larger than 1 and s and t will denote relatively prime r-free Steinitz numbers with s finite.

The next lemma shows how the calculation of the natural density of the sets $A(s,t)$ reduces to the calculation of the natural density of sets of the form $A(1,t)$ and, when t is finite, to the calculation of the natural density of sets of the form $A(s,1)$. The lemma generalizes [1, Lemma 1].

Lemma 1 *Suppose that a is an r-free number relatively prime to st. Then for any positive real number x we have $A(s,at)[x] = A(as,t)[ax]$. Moreover $A(s,at)$ has a natural density if and only if $A(as,t)$ has a natural density, in which case we have $A(s,at)^* = aA(as,t)^*$.*

This lemma is trivial in the case that $x \leq 1$ where the sets $A(s,at)\{x\}$ and $A(as,t)\{ax\}$ are empty unless $x = s = 1$ in which case they equal $\{1\}$ and $\{a\}$ respectively. We have included these trivial cases in the statement of the lemma to facilitate its application in the proof of Lemma 2.

Proof. We first claim that for any positive real number x multiplication by a gives a bijection

$$A(s,at)\{x\} \longrightarrow A(as,t)\{ax\}.$$

First suppose that $c \in A(s,at)\{x\}$. Then $1 \leq c \leq x$ and $ac \leq ax$. Also, $\gcd(c,t) = \gcd(a,t) = 1$, so $\gcd(ac,t) = 1$. Next, $s|c$ and

$$\gcd(s, c/s) = \gcd(a, c/s) = 1,$$

so $as|ac$ and $\gcd(as, ac/as) = 1$. Hence ac is in $A(as,t)\{ax\}$, so multiplication by a is a map as indicated and it is clearly injective. Now suppose that $c \in A(as,t)\{ax\}$. Then $a|c$; to prove our claim it remains to show that $a/c \in A(s,at)\{x\}$. Since $as|c$ and $\gcd(c/(as), as) = 1$, then $s|(c/a)$ and $\gcd(s, (c/a)/s) = 1$. Finally,

$$\gcd(a, c/(as)) = \gcd(a, s) = \gcd(c/a, t) = 1,$$

so $\gcd(at, c/a) = 1$. Hence our map is indeed a bijection, which proves the first assertion of our lemma.

By the above paragraph, for any positive real number x, we have

$$\frac{A(s,at)[x]}{x} = a\frac{A(as,t)[ax]}{ax}.$$

The lemma follows by taking limits as x (and hence ax) goes to infinity. ■

The reader may have noted that $A(s,t) = A(s, \prod_{p|t} p)$; we will often use this fact below. Allowing t to be a Steinitz number which is not necessarily square-free simplifies the statement, the proof, and the applications of Lemma 1.

3 Proof of Theorem 2 When t is Finite

Lemma 2 *Let p be a prime number not dividing s. If the set $A(s,1)$ has a natural density, then so does $A(s,p)$ and*

$$A(s,p)^* = \frac{p^r - p^{r-1}}{p^r - 1} A(s,1)^*.$$

Proof. Pick a positive real number ϵ and a number k with $1/p^{rk} < \epsilon/6$. It will be convenient to write $D = A(s,1)^*$ and $\mathcal{A} = A(s,p)$. Recall from Lemma 1 that for any number $i < r$ and any positive real number x we have

$$A(p^i s, 1)[x] = A(s,p)[x/p^i] = \mathcal{A}[x/p^i].$$

Note that $A(s,1)$ can be written as a disjoint union

$$A(s,1) = A(s,p) \cup A(ps,1) \cup A(p^2 s,1) \cup \cdots \cup A(p^{r-1}s,1)$$

so that for any positive real number x,

$$A(s,1)[x] = \mathcal{A}[x] + \mathcal{A}[x/p] + \cdots + \mathcal{A}[x/p^{r-1}].$$

But then we also have

$$A(s,1)[x/p] = \mathcal{A}[x/p] + \mathcal{A}[x/p^2] + \cdots + \mathcal{A}[x/p^r]$$

and hence

$$A(s,1)[x] - A(s,1)[x/p] = \mathcal{A}[x] - \mathcal{A}[x/p^r]$$

so

$$\frac{A(s,1)[x]}{x} - \frac{A(s,1)[x/p]}{x/p}\frac{1}{p} = \frac{\mathcal{A}[x]}{x} - \frac{\mathcal{A}[x/p^r]}{x}.$$

Therefore, by our hypothesis that $A(s,1)^* = D$, we see that $D - \frac{1}{p}D$ is the limit as x goes to ∞ of $\frac{\mathcal{A}[x]}{x} - \frac{\mathcal{A}[x/p^r]}{x}$, and hence that there exists a number $M > 0$ such that if $x > M$ then

$$\left| D(1 - \frac{1}{p}) - \frac{\mathcal{A}[x]}{x} + \frac{\mathcal{A}[x/p^r]}{x} \right| < \frac{\epsilon}{6}.$$

If $x > p^{rk}M$, then $x/p^{ri} > M$ whenever $0 \le i \le k$ and so

$$\left| D(1 - \frac{1}{p})\frac{x}{p^{ri}} - \mathcal{A}[x/p^{ri}] + \mathcal{A}[x/p^{r(i+1)}] \right| < \frac{\epsilon}{6}\frac{x}{p^{ri}}. \tag{1}$$

Using the triangle inequality to combine all the inequalities in (1) for $0 \le i < k$ and dividing through by x we obtain the inequality

$$\left| D(1 - \frac{1}{p})\sum_{i=0}^{k-1}\frac{1}{p^{ri}} - \frac{\mathcal{A}[x]}{x} + \frac{\mathcal{A}[x/p^{rk}]}{x} \right| < \frac{\epsilon}{6}\sum_{i=0}^{k-1}\frac{1}{p^{ri}} < \frac{\epsilon/6}{1 - (1/p^r)} < \frac{\epsilon}{3}$$

(the last inequality above follows from the fact that $p^r \geq 2$). Recalling the choice of k at the beginning of this proof we deduce that

$$\left| -\frac{\mathcal{A}[x/p^{rk}]}{x} \right| \leq \frac{x}{p^{rk}} \frac{1}{x} = \frac{1}{p^{rk}} < \frac{\epsilon}{6}$$

and that

$$\left| D(1-\frac{1}{p})\frac{p^r}{p^r-1} - D(1-\frac{1}{p})\sum_{i=0}^{k-1}\frac{1}{p^{ri}} \right| = D(1-\frac{1}{p})\frac{p^r}{p^r-1}\frac{1}{p^{rk}} < D\frac{p^r-p^{r-1}}{p^r-1}\frac{\epsilon}{6} < \frac{\epsilon}{6}.$$

Again combining inequalities with the triangle inequality we have

$$\left| D(1-\frac{1}{p})\frac{p^r}{p^r-1} - \frac{\mathcal{A}[x]}{x} \right| < \epsilon.$$

Hence

$$A(s,p)^* = \mathcal{A}^* = D(1-\frac{1}{p})\frac{p^r}{p^r-1} = \frac{p^r-p^{r-1}}{p^r-1}A(s,1)^*.$$

■

We now turn directly to the proof of Theorem 2 in the case that t is finite.

Let $u = p_1 p_2 \cdots p_m$ be the product of the distinct primes dividing st. Then by Lemma 1,

$$A(s,t)^* = \frac{1}{s}A(1,st)^* = \frac{1}{s}A(1,u)^* = \frac{u}{s}A(u,1)^* = \frac{u}{s}\frac{1}{p_1}A(\frac{u}{p_1},p_1)^*$$

which by Lemma 2 equals

$$\frac{u}{s}\frac{1}{p_1}\frac{p_1^r-p_1^{r-1}}{p_1^r-1}A(\frac{u}{p_1},1)^*.$$

Proceeding by induction on m we have

$$A(s,t)^* = \frac{u}{s}A(\frac{u}{p_1\cdots p_m},1)^* \prod_{i\leq m}\frac{1}{p_i}\prod_{i\leq m}\frac{p_i^r-p_i^{r-1}}{p_i^r-1} = \frac{1}{s\zeta(r)}\prod_{p|st}\frac{p^r-p^{r-1}}{p^r-1}.$$

4 Proof of Theorem 2 When t is not Finite

We suppose in this section that t is a Steinitz number which is not finite, say with prime divisors $p_1, p_2, \cdots$. (We may suppose of course without loss of generality that t has an infinite number of prime divisors.) Let us set $\alpha = \prod_{i=1}^{\infty}\frac{p_i^r-p_i^{r-1}}{p_i^r-1}$. Since each factor $\frac{p_i^r-p_i^{r-1}}{p_i^r-1}$ is positive and less than one, the partial products of the infinite product α form a decreasing sequence bounded below by 0, and hence the infinite product converges, and indeed its limit is independent of the order of the factors. (The independence can be shown in detail by the argument at the beginning of [1, Section 4].) Similar remarks are valid for the product in Theorem 1.

We first consider the case that $\alpha = 0$. Let $\epsilon > 0$. Then by hypothesis there exists a number M such that if $k > M$ then

$$\frac{1}{\zeta(r)}\prod_{i=1}^{k}\frac{p^r-p^{r-1}}{p^r-1} < \epsilon/2.$$

Fix a number $k > M$. Then by the case of the theorem proved in Section 3 there exists a number M' such that if $x > M'$ then

$$\frac{A(1,p_1\cdots p_k)[x]}{x} \leq A(1,p_1\cdots p_k)^* + \epsilon/2$$

and therefore

$$\begin{aligned}
\frac{A(1,p_1p_2\cdots)[x]}{x} &\le \frac{A(1,p_1\cdots p_k)[x]}{x} \\
&\le A(1,p_1p_2\cdots p_k)^* + \epsilon/2 = \frac{1}{\zeta(r)}\prod_{i=1}^{k}\frac{p^r-p^{r-1}}{p^r-1} + \epsilon/2 < \epsilon/2+\epsilon/2 = \epsilon.
\end{aligned}$$

Thus $A(1,p_1p_2\cdots)^*$ exists and equals 0.

Next let us suppose that $\alpha > 0$. We begin by proving the theorem in the case that $s = 1$. It will be convenient to set

$$\psi(j) = \frac{p_j^r - p_j^{r-1}}{p_j^r - 1}$$

for all j, so that α is the product of the $\psi(j)$.

Note that for any prime p we have $(p^r-1)/(p^r-p^{r-1}) \ge (p+1)/p$. Then

$$\infty > -2\log(\alpha) = 2\sum_{k=1}^{\infty} -\log(\psi(k)) \ge 2\sum_{k=1}^{\infty}\log(1+\frac{1}{p_k}) \ge \sum_{k=1}^{\infty}\frac{1}{p_k}.$$

For any $i < r$ we have

$$\sum_{k=1}^{\infty}\frac{A(p_k^i, p_1p_2\cdots p_{k-1})[n]}{n} \le \sum_{k=1}^{\infty}\frac{n/p_k^i}{n} = \sum_{k=1}^{\infty}\frac{1}{p_k^i},$$

which is finite for all $i < r$ (using the previous paragraph when $i = 1$). This fact will allow us to apply Tannery's Theorem [3, 11] in the next paragraph.

If β is an r-free number not in $A(1,p_1p_2\cdots)$, then there exists a least number k with p_k dividing β. It follows that we have a disjoint union

$$A(1,1)\setminus A(1,p_1p_2\cdots) = \cup_{i=1}^{r-1}\cup_{k=1}^{\infty} A(p_k^i, p_1p_2\cdots p_{k-1}).$$

Therefore

$$(A(1,1)\setminus A(1,p_1p_2\cdots))^* = \lim_{n\to\infty}\sum_{k=1}^{\infty}\sum_{i=1}^{r-1}\frac{A(p_k^i,p_1\cdots p_{k-1})[n]}{n}$$

which by Tannery's theorem and the case of our theorem proven in the last section is equal to

$$\sum_{k=1}^{\infty}\sum_{i=1}^{r-1}\lim_{n\to\infty}\frac{A(p_k^i,p_1\cdots p_{k-1})[n]}{n} = \sum_{k=1}^{\infty}\sum_{i=1}^{r-1}A(p_k^i,p_1\cdots p_{k-1})^* = \sum_{k=1}^{\infty}\sum_{i=1}^{r-1}\frac{1}{\zeta(r)p_k^i}\prod_{j=1}^{k}\psi(j).$$

Observing that for each $k \ge 1$ we have

$$\psi(k)\sum_{i=1}^{r-1}\frac{1}{p_k^i} = 1-\psi(k),$$

we deduce that

$$\begin{aligned}(A(1,1)\setminus A(1,p_1p_2\cdots))^* &= \frac{1}{\zeta(r)}\left(1-\psi(1)+\sum_{k=2}^{\infty}(1-\psi(k))\prod_{i=1}^{k-1}\psi(i)\right)\\ &= \frac{1}{\zeta(r)}\left(1-\psi(1)+\sum_{k=2}^{\infty}(\prod_{i=1}^{k-1}\psi(i)-\prod_{i=1}^{k}\psi(i))\right)\\ &= \frac{1}{\zeta(r)}\left(1-\psi(1)+\lim_{L\to\infty}(\prod_{j=1}^{1}\psi(j)-\prod_{j=1}^{L}\psi(j))\right)\\ &= \frac{1}{\zeta(r)}(1-\prod_{j=1}^{\infty}\psi(j)),\end{aligned}$$

so that we have

$$A(1,p_1p_2\cdots)^* = \frac{1}{\zeta(r)} - \frac{1}{\zeta(r)}(1-\prod_{j=1}^{\infty}\psi(j)) = \frac{1}{\zeta(r)}\prod_{j=1}^{\infty}\psi(j).$$

This proves the theorem in the case that $s=1$. Applying Lemma 1 and the proof of the theorem in the case that $s=1$, we see that in general we have

$$A(s,t)^* = \frac{1}{s}A(1,st)^* = \frac{1}{s}A(1,\prod_{p|st}p)^* = \frac{1}{s\zeta(r)}\prod_{p|st}\frac{p^r-p^{r-1}}{p^r-1},$$

which completes the proof of Theorem 2.

References

[1] R. Brown, The natural density of some sets of square-free numbers, Integers, 21(2021), 1–9.

[2] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers (5th ed.), Oxford Univ. Press, Oxford, 1979.

[3] J. Hofbauer, A simple proof of $1+1/22+1/32+\cdots=\pi^2/6$ and related identities, American Math. Monthly, 109(2002), 196–200.

[4] A. Ivić, The Riemann zeta-Function, the Theory of the Riemann zeta-Function with Applications, Wiley, New York, 1985.

[5] G. J. O. Jameson, Even and odd square-free numbers, Math. Gaz., 94(2010), 123–127.

[6] G. J. O. Jameson, Revisiting even and odd square-free numbers, Math. Gaz., 105(2021), 299–300.

[7] I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers (4th ed.), Wiley, New York, 1980.

[8] S. Roman, Field Theory, Springer-Verlag, Berlin, 1995.

[9] S. Srisopha, T. Srichan and S. Mavecha, Odd/even r-free numbers, Applied Math. E-Notes, 22(2022), 656–659.

[10] S. Srisopha, T. Srichan and S. Mavecha, Note on the natural density of r-free numbers, Notes on Number Theory and Discrete Math., 28(2022), 435–440.

[11] J. Tannery, Introduction à la Théorie des Fonctions d'une Variable, 2 ed., Tome 1, Libraire Scientifique A, Hermann, Paris, 1904.