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Inequalities For Gauss Hypergeometric Functions Via Discrete Chebychev Inequality*

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Abstract

Intensive studies aiming to extend the gamma and beta functions and to establish some properties for these extensions have been recently carried out. The fundamental purpose of this paper is to investigate some inequalities for a class of Gauss hypergeometric functions when investing the discrete Chebychev inequality.

1 Introduction

Recently, the gamma and beta functions attracted many researchers by virtue of their nice properties and interesting applications. Several generalizations and extensions for these special functions have been investigated by many researchers, see [3, 4, 11, 16, 17, 18] for instance and the related references cited therein. In this direction, the hypergeometric functions have been introduced in literature and intensive studies related to these new functions and their generalizations have been undertaken.

The Gauss hypergeometric function (GHF) was defined in [18, p.1]

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \ \ \Re e(a)>0, \ \Re e(b)>0, \ \Re e(c)>0, \tag{1} $$

provided that this series is convergent. Here, the notation $(\lambda)_n$, for $\Re e(\lambda)>0$, refers to the Pochhammer symbol defined by

$$ (\lambda)_n := \lambda(\lambda+1)...(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \text{ and } (\lambda)_0 = 1, $$

where $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$, $\Re e(x)>0$, is the standard gamma function. The function (GHF) satisfies the following hypergeometric differential equation [18, p.2]:

$$ z(1-z)\frac{d^2y}{d^2z} + \Big(c-(a+b+1)z\Big)\frac{dy}{dz} - aby = 0, \text{ with } y=y(z) ={}_2F_1(a,b;c;z). $$

The (GHF) can be written in terms of the classical beta function as follows [3]

$$ {}_2F_1(a,b;c;z) := \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n B(b+n,c-b) \frac{z^n}{n!}, \tag{2} $$

provided that $0<\Re e(b)<\Re e(c)$, where $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, $\Re e(x)>0, \Re e(y)>0$, is the standard beta function. The formula (2) was deduced from the following integral representation, [18, p.20]

$$ {}_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \ \ |z|\le 1. \tag{3} $$

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The confluent hypergeometric function (CHF), also called Kummer's function, was defined in [9] by the following formula

$$ {}_1F_1(b;c;z) := \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{z^n}{n!}, \quad \Re e(b) > 0,\ \Re e(c) > 0, \tag{4} $$

provided that this series is convergent. It can be written in terms of the classical beta function as follows [1, p.504]

$$ {}_1F_1(b;c;z) = \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} B(b+n,c-b) \frac{z^n}{n!}, \quad 0 < \Re e(b) < \Re e(c). \tag{5} $$

In integral form we have, [11]

$$ {}_1F_1(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1} e^{zu}\, du. \tag{6} $$

The definition of the confluent hypergeometric function given by (4) permits Özergin et al. to introduce in [11] the following generalized beta function

$$ B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt, \tag{7} $$

where $\Re e(\alpha) > 0, \Re e(\beta) > 0, \Re e(p) > 0$.

Based on (7), Özergin et al. defined in [11] the generalized Gauss hypergeometric function (GGHF) and the generalized confluent hypergeometric function (GCHF), respectively, as follows

$$ {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) := \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n\, B_p^{(\alpha,\beta)}(b+n,c-b) \frac{z^n}{n!}, \tag{8} $$

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} B_p^{(\alpha,\beta)}(b+n,c-b) \frac{z^n}{n!}. \tag{9} $$

It is clear that

$$ {}_2F_1^{(\alpha,\beta;0)}(a,b;c;z) = {}_2F_1(a,b;c;z) \text{ and } {}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z). \tag{10} $$

In [11], the authors gave integral representations of (GGHF) and (GCHF) as follows

$$ {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} (1-zt)^{-a}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt, \quad |z| \le 1, \tag{11} $$

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt. \tag{12} $$

The following remark will be of interest in the sequel.

Remark 1 *All the previous series as well as their related improper integrals are uniformly convergent. This implies that, whenever we have an infinite summation with one of these improper integrals then the order of the integral with the summation can be interchanged. We can also differentiate and take the limit under these infinite summations or improper integrals.*

The computation of confluent hypergeometric functions and Gauss hypergeometric functions is important in a wide variety of areas such as photon scattering from atoms [6], networks [19], Coulomb wave functions [5], binary stars [14], non-Newtonian fluids [20], and more. In [10, 13], the authors consider the general solution of the stationary state Schrödinger equation in terms of confluent hypergeometric functions and in

[8, 15], these class of functions are used in a discussion of the bound and continuum states of the hydrogen atom. In [2], Andrews provides an expanded survey on some significant applications of basic hypergeometric functions to theory of partitions, number theory, finite vector spaces, combinatorial identities.

But, in practice, computing these functions is not an easy task as revealed for example by authors in [12], where they developed some numerical methods for the computation of the confluent and Gauss hypergeometric functions. Motivated by the intention to get tools to overcome this difficulty, we aim in the present work to establish some approximations for the previous generalized hypergeometric functions. To achieve this fundamental target, we will mainly use the discrete Chebychev inequality that will be recalled in the following section.

2 Results

As mentioned earlier, we will investigate here some inequalities involving the functions (GCHF) and (GHF) by employing the discrete Chebychev inequality (DCI). This inequality is still attractive due to its numerous applications. It plays a fundamental role in the establishment of important results, as known for example in the field of probability. DCI reads as follows.

Theorem 1 *Let $(x_k), (y_k), (m_k), 1 \leq k \leq n$, be three families of real numbers such that $m_k \geq 0$ for any $k = 1, 2, ..., n$. Assume that (x_k) and (y_k) are monotonic in the same (resp. opposite) sense. Then we have*

$$\sum_{k=1}^{n} m_k \times \sum_{k=1}^{n} m_k \, x_k \, y_k \geq (\leq) \sum_{k=1}^{n} m_k \, x_k \times \sum_{k=1}^{n} m_k \, y_k. \tag{13}$$

A proof of this Theorem can be found in [7].

Now, we are in the position to state our first main result which concerns an inequality involving simultaneously the three generalized functions, namely $B_p^{(\alpha,\beta)}$, ${}_2F_1^{(\alpha,\beta;p)}$ and ${}_1F_1^{(\alpha,\beta;p)}$.

Theorem 2 *Let $a, b, c, \alpha, \beta, p > 0$ with $b < c$ and $0 \leq z < 1$. Then we have the following inequality*

$$(e^z - 1)\Big\{B(b, c-b)\, {}_2F_1^{(\alpha,\beta;p)}(a, b; c; z) - B_p^{(\alpha,\beta)}(b, c-b)\Big\} \leq \Big((1-z)^{-a} - 1\Big)\Big\{B(b, c-b)\, {}_1F_1^{(\alpha,\beta;p)}(b; c; z) - B_p^{(\alpha,\beta)}(b, c-b)\Big\}. \tag{14}$$

If moreover $a \geq 1$ then we have

$${}_2F_1^{(\alpha,\beta;p)}(a, b; c; z) \leq e^{-z}(1-z)^{-a}\, {}_1F_1^{(\alpha,\beta;p)}(b; c; z). \tag{15}$$

Proof. We consider the following sequences,

$$x_k = (a)_k, \; y_k = B_p^{(\alpha,\beta)}(b+k, c-b) \;\text{ and }\; m_k = \frac{z^k}{k!}, \;\; k \geq 0.$$

By using the celebrated formula $\Gamma(x+1) = x\Gamma(x), \; x > 0$, it is not hard to see that

$$\frac{x_{k+1}}{x_k} := \frac{(a)_{k+1}}{(a)_k} = \frac{\Gamma(a+k+1)}{\Gamma(a+k)} = a + k > 1, \;\; \forall k \geq 1, \tag{16}$$

and so $(x_k)_{k\geq 1}$ is increasing. Otherwise, for $0 < t < 1$ fixed, the function $x \longmapsto t^{x-1}$ is decreasing on $(0, \infty)$ and thus the map $x \longmapsto B_p^{(\alpha,\beta)}(x, y)$, for fixed $y > 0$, is also decreasing. It follows that $(y_k)_{k\geq 0}$ is decreasing too. According to DCI (13), we have

$$\sum_{k=1}^{n} m_k \times \sum_{k=1}^{n} m_k \, x_k \, y_k \leq \sum_{k=1}^{n} m_k \, x_k \times \sum_{k=1}^{n} m_k \, y_k. \tag{17}$$

In another part, we have

$$\sum_{k=1}^{\infty} m_k\, x_k = \sum_{k=1}^{\infty} (a)_k\, \frac{z^k}{k!} = (1-z)^{-a} - 1,\ \sum_{k=1}^{\infty} m_k = \sum_{k=1}^{\infty} \frac{z^k}{k!} = e^z - 1, \tag{18}$$

and from (8) and (9) we get, respectively,

$$\sum_{k=1}^{\infty} m_k\, x_k\, y_k = \sum_{k=1}^{\infty} (a)_k B_p^{(\alpha,\beta)}(b+k, c-b)\, \frac{z^k}{k!} = B(b, c-b)\, {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) - B_p^{(\alpha,\beta)}(b, c-b), \tag{19}$$

$$\sum_{k=1}^{\infty} m_k\, y_k = \sum_{k=1}^{\infty} \frac{z^k}{k!}\, B_p^{(\alpha,\beta;p)}(b+k, c-b) = B(b, c-b)\ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) - B_p^{(\alpha,\beta)}(b, c-b). \tag{20}$$

Letting $n \longrightarrow \infty$ in (17) and substituting (18),(19) and (20) therein we obtain (14).

If moreover $a \geq 1$ then (x_k) is increasing from $k = 0$ and so (17) holds for $k = 0, 1, ..., n$. We then deduce (15) in a similar manner. ■

Taking $p = 0$ in Theorem 2, with the help of (10), we immediately obtain the following result.

Corollary 1 *Let $a, b, c > 0$ with $b < c$ and $0 \leq z < 1$. Then we have the following inequality*

$$(e^z - 1)\Big(\ {}_2F_1(a,b;c;z) - 1\Big) \leq \big((1-z)^{-a} - 1\big)\Big(\ {}_1F_1(b;c;z) - 1\Big). \tag{21}$$

If moreover $a \geq 1$ then we have

$${}_2F_1(a,b;c;z) \leq e^{-z}(1-z)^{-a}\ {}_1F_1(b;c;z). \tag{22}$$

The following remark may be of interest for the reader.

Remark 2 *(i) From (3) and (6), it is easy to see that the real-functions $z \longmapsto {}_1F_1(b;c;z)$ for $z \in \mathbb{R}$, and $z \longmapsto {}_2F_1(a,b;c;z)$ for $|z| < 1$, are positive increasing. This, with the fact that ${}_1F_1(b;c;0) = {}_2F_1(a,b;c;0) = 1$, implies that*

$${}_1F_1(b;c;z) \geq 1,\ {}_2F_1(a,b;c;z) \geq 1,\ \textit{for } z \geq 0.$$

It follows that (21) is equivalent to:

$$0 \leq \frac{{}_2F_1(a,b;c;z) - 1}{{}_1F_1(b;c;z) - 1} \leq \frac{(1-z)^{-a} - 1}{e^z - 1},\ \ 0 < z < 1.$$

(ii) By similar way as in (i), (14) is equivalent to

$$0 \leq \frac{B(b, c-b)\ {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) - B_p^{(\alpha,\beta)}(b, c-b)}{B(b, c-b)\ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) - B_p^{(\alpha,\beta)}(b, c-b)} \leq \frac{(1-z)^{-a} - 1}{e^z - 1},\ \ 0 < z < 1. \tag{23}$$

(iii) By normalizing (7) and setting

$$B^{(\alpha,\beta;p)}(x,y) := \frac{1}{B(x,y)} \int_0^1 t^{x-1}(1-t)^{y-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt, \tag{24}$$

then (23) is equivalent to

$$0 \leq \frac{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) - B^{(\alpha,\beta;p)}(b, c-b)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z) - B^{(\alpha,\beta;p)}(b, c-b)} \leq \frac{(1-z)^{-a} - 1}{e^z - 1},\ \ 0 < z < 1. \tag{25}$$

The following result gives a refinement of Theorem 2. For the sake of simplicity, we use the notation (24) and we set

$$\Lambda^{(\alpha,\beta;p)}(a,b;c;z) = \frac{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) - B^{(\alpha,\beta;p)}(b,c-b)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z) - B^{(\alpha,\beta;p)}(b,c-b)}, \quad 0 < z \leq 1. \tag{26}$$

We have the following result.

Theorem 3 *Let $a,b,c,\alpha,\beta,p > 0$ with $b < c$. Then we have*

$$\max_{0<z\leq 1} \Lambda^{(\alpha,\beta;p)}(a,b;c;z) = \Lambda^{(\alpha,\beta;p)}(a,b;c;1). \tag{27}$$

If moreover $a \geq 1$ then we have

$$\max_{0<z\leq 1} \frac{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;z)} = \frac{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;1)}{{}_1F_1^{(\alpha,\beta;p)}(b;c;1)}. \tag{28}$$

Proof. We consider the following sequences

$$x_k = (a)_k, \; y_k = z^k, \; m_k = \frac{B_p^{(\alpha,\beta)}(b+k,c-b)}{k!}, \quad k \geq 0.$$

As previous, $(x_k)_{k\geq 1}$ is increasing for any $a > 0$ and (y_k) is decreasing. Here we have

$$\sum_{k=1}^{\infty} m_k x_k = B(b,c-b)\; {}_2F_1^{(\alpha,\beta;p)}(a,b;c;1) - B_p^{(\alpha,\beta)}(b,c-b), \tag{29}$$

$$\sum_{k=1}^{\infty} m_k = B(b,c-b)\; {}_1F_1^{(\alpha,\beta;p)}(b;c;1) - B_p^{(\alpha,\beta)}(b,c-b). \tag{30}$$

Applying DCI (13) and taking into account (19), (20), (29) and (30), with the help of (26), we get

$$\begin{aligned} &\left\{{}_1F_1^{(\alpha,\beta;p)}(b;c;1) - B^{(\alpha,\beta;p)}(b,c-b)\right\}\left\{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) - B^{(\alpha,\beta;p)}(b,c-b)\right\} \\ &\qquad \leq \left\{{}_2F_1^{(\alpha,\beta;p)}(a,b;c;1) - B^{(\alpha,\beta;p)}(b,c-b)\right\}\left\{{}_1F_1^{(\alpha,\beta;p)}(b;c;z) - B^{(\alpha,\beta;p)}(b,c-b)\right\}. \end{aligned}$$

Whence (27). If moreover $a \geq 1$ then we proceed as in the proof of the preceding theorem and we get (28). ■

Remark 3 *In the above, we claimed that Theorem 3 refines Theorem 2. In fact, this follows from (27) when combined with (25) and (26).*

We have the following result as well.

Theorem 4 *Let $a,b,c,\alpha,\beta,p > 0$ with $c > b+1$ and $0 \leq z \leq 1$. If $a \geq 1$ then the following inequality holds*

$$B_p^{(\alpha,\beta)}(b,c-b-1)\; {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) \leq \; {}_2F_1^{(\alpha,\beta;p)}(a,b;c;1) \times \; {}_2F_1^{(\alpha,\beta;p)}(1,b;c;z). \tag{31}$$

If $a \leq 1$ then (31) is reversed.

Proof. Consider the following sequences

$$x_k = \frac{(a)_k}{k!}, \; y_k = z^k, \; m_k = B_p^{(\alpha,\beta)}(b+k,c-b); \quad k \geq 0.$$

As for (16), we have

$$\frac{x_{k+1}}{x_k} = \frac{a+k}{k+1}, \quad k \geq 0.$$

So, (x_k) is increasing if $a \geq 1$ and decreasing if $a \leq 1$. Clearly, (y_k) is decreasing. By DCI (13) we have

$$\sum_{k=0}^{n} m_k \times \sum_{k=0}^{n} m_k\, x_k\, y_k \leq \sum_{k=0}^{n} m_k\, x_k \times \sum_{k=0}^{n} m_k\, y_k, \tag{32}$$

provided that $a \geq 1$, with reversed inequality for $a \leq 1$. Utilizing (7), with Remark 1, we can write

$$\begin{aligned} \sum_{k=0}^{\infty} m_k := \sum_{k=0}^{\infty} B_p^{(\alpha,\beta)}(b+k, c-b) &= \sum_{k=0}^{\infty} \int_0^1 t^{b+k-1}(1-t)^{c-b-1}\ {}_1F_1\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)\, dt \\ &= \int_0^1 \Big(\sum_{k=0}^{\infty} t^k\Big) t^{b-1}(1-t)^{c-b-1}\ {}_1F_1\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)\, dt \\ &= \int_0^1 t^{b-1}(1-t)^{c-b-2}\ {}_1F_1\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)\, dt, \end{aligned} \tag{33}$$

which, with (7) again, yields

$$\sum_{k=0}^{\infty} m_k = B_p^{(\alpha,\beta)}(b, c-b-1). \tag{34}$$

In another part, by (8) we have

$$\sum_{k=0}^{\infty} m_k x_k = \sum_{k=0}^{\infty} (a)_k B_p^{(\alpha,\beta;p)}(c+k, c-b)\frac{1}{k!} = B(b, c-b)\ {}_2F_1^{(\alpha,\beta;p)}(a,b;c;1) \tag{35}$$

and, with the fact that $(1)_k = k!$,

$$\sum_{k=0}^{\infty} m_k y_k = \sum_{k=0}^{\infty} B_p^{(\alpha,\beta)}(b+k, c-b) z^k = \ {}_2F_1(1,b;c;z). \tag{36}$$

Finally, as for (11) we get

$$\sum_{k=0}^{\infty} m_k x_k y_k = \ {}_2F_1(a,b;c;z). \tag{37}$$

After letting $n \to \infty$ in (32) and then substituting (34), (35), (36) and (37) therein, we get (31) (resp. its reverse) when $a \geq 1$ (resp. $a \leq 1$). ■

The following result may be stated as well.

Theorem 5 *Let $a, b, c, \alpha, \beta, p, z > 0$ with $c > b$ and $0 \leq z < 1$. If $a \geq 1$ then the following inequality holds*

$$ {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z) \leq (1-z)\ {}_2F_1^{(\alpha,\beta;p)}(a,1;1;z) \times \ {}_2F_1^{(\alpha,\beta;p)}(1,b;c;z). \tag{38}$$

If $a \leq 1$ then (38) is reversed.

Proof. Let us consider the following

$$x_k = \frac{(a)_k}{k!},\ y_k = B_p^{(\alpha,\beta)}(b+k, c-b),\ m_k = z^k, \quad k \geq 0.$$

As in the proof of Theorem 4, (x_k) is increasing for $a \geq 1$ and decreasing when $a \leq 1$. As in the proof of Theorem 2, (y_k) is decreasing. Furthermore, using (1), we have

$$\sum_{k=0}^{\infty} m_k x_k = \ {}_2F_1(a,1;1;z) \ \text{ and, } \sum_{k=0}^{\infty} m_k = \frac{1}{1-z}.$$

In a similar manner as for the previous results, we can then conclude by utilizing (36) and (37). ■

Remark 4 *The right side in inequalities (15) and (22) is easier to compute than the left one. Thus, these inequalities can be used in providing other upper bounds in the inequalities (31) and (38).*

Now we will give another inequality, of multiplicative type, regrouping as well the three generalized functions $B_p^{(\alpha,\beta)}$, ${}_2F_1^{(\alpha,\beta;p)}$ and ${}_1F_1^{(\alpha,\beta;p)}$. It reads as follows.

Theorem 6 *Let $b, c, \alpha, \beta, p > 0$ with $b+1 < c$ and $0 \le z \le 1$. Then we have the following inequality*

$$B_p^{(\alpha,\beta)}(b, c-b-1)\ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) \ge B(b, c-b)\ {}_1F_1^{(\alpha,\beta;p)}(b;c;1) \times\ {}_2F_1^{(\alpha,\beta;p)}(1,b;c;z). \tag{39}$$

In particular, we have

$${}_1F_1(b;c;z) \ge \frac{c}{c-b-1}\ {}_1F_1(b;c;1) \times\ {}_2F_1(1,b;c;z). \tag{40}$$

Proof. Here we consider the following sequences

$$x_k = z^k,\ y_k = \frac{1}{k!},\ \ m_k = B_p^{(\alpha,\beta)}(b+k, c-b);\ \ k \ge 0.$$

It is obvious that (x_k) and (y_k) are both decreasing. Using DCI (13) we have

$$\sum_{k=0}^{n} m_k \times \sum_{k=0}^{n} m_k\, x_k\, y_k \ge \sum_{k=0}^{n} m_k\, x_k \times \sum_{k=0}^{n} m_k\, y_k. \tag{41}$$

By (9), we obtain

$$\sum_{k=0}^{\infty} m_k\, y_k := \sum_{k=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+k, c-b)}{k!} = B(b, c-b)\ {}_1F_1(b;c;1). \tag{42}$$

Letting $n \to \infty$ in (41) and using (34), (36), (37) and (42) we get (39).

Taking $p = 0$ in (39) we immediately obtain (40), so completing the proof. ■

Another main result is recited in the following.

Theorem 7 *Let $b, c, \alpha, \beta, p > 0$ with $b < c$. Let z, z_0 be such that $0 \le z < z_0$. Then we have*

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) \ge e^{z-z_0}\ {}_1F_1^{(\alpha,\beta;p)}(b;c;z_0). \tag{43}$$

That is, the function $z \longmapsto e^{-z}\ {}_1F_1^{(\alpha,\beta;p)}(b;c;z)$ is decreasing on $(0,\infty)$. In particular, we have

$${}_1F_1(b;c;z) \ge e^{z-z_0}\ {}_1F_1(b;c;z_0),\ \ 0 \le z < z_0. \tag{44}$$

Proof. Let us consider the following sequences,

$$x_k = B_p^{(\alpha,\beta)}(b+k, c-b),\ y_k = \left(\frac{z}{z_0}\right)^k \text{ and } m_k = \frac{z_0^k}{k!},\ \text{ for } k \ge 0.$$

Since $0 < z < z_0$ then $(y_k)_{k\ge0}$ is decreasing and as in the proof of the previous theorem $(x_k)_{k\ge0}$ is decreasing as well. Using DCI (13) we obtain

$$\sum_{k=0}^{n} m_k \times \sum_{k=0}^{n} m_k\, x_k\, y_k \ge \sum_{k=0}^{n} m_k\, x_k \times \sum_{k=0}^{n} m_k\, y_k. \tag{45}$$

Otherwise, taking into account the formula (9) we have

$$\sum_{k=0}^{\infty} m_k = e^{z_0},\ \sum_{k=0}^{\infty} m_k\, x_k\, y_k = B(b, c-b){}_1F_1^{(\alpha,\beta;p)}(b;c;z),$$

$$\sum_{k=0}^{\infty} m_k\, x_k = B(b, c-b){}_1F_1^{(\alpha,\beta;p)}(b;c;z_0) \text{ and } \sum_{k=0}^{\infty} m_k\, y_k = e^{z}.$$

Substituting these, after letting $n \longrightarrow \infty$ in (45), we get (43). Taking $p = 0$ in (43), with the help of (10), we get (44), so completing the proof. ■

Corollary 2 *Let $b, c, \alpha, \beta, p, z > 0$ with $b < c$. Then we have*

$$
{}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z) \le \frac{c}{b}\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z) \tag{46}
$$

Proof. Following Theorem 7, $z \longmapsto e^{-z}\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z)$ is decreasing on $(0,\infty)$ and so we have

$$
\forall z > 0 \qquad \frac{d}{dz}\Big(e^{-z}\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z)\Big) \le 0,
$$

or, equivalently,

$$
\forall z > 0 \qquad \frac{d}{dz}\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z)\Big) \le\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z). \tag{47}
$$

Using (9) we have

$$
\frac{d}{dz}\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \sum_{n=1}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^{n-1}}{(n-1)!}.
$$

Now, using the celebrated formulas $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \Gamma(x+1) = \Gamma(x),\ x, y > 0$, we get

$$
\begin{aligned}
\frac{d}{dz}\, {}_1F_1^{(\alpha,\beta;p)}(b;c;z) &= \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+1+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \\
&= \frac{B(b+1,c-b)}{B(b,c-b)} \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+1+n,c-b)}{B(b+1,c-b)} \frac{z^n}{n!} = \frac{b}{c}\, {}_1F_1^{(\alpha,\beta;p)}(b+1;c+1;z).
\end{aligned}
$$

Substituting this in (47) we get (46). ■

We end this section by stating the following result.

Theorem 8 *Let $a, b, c, \alpha, \beta, p > 0$ and $0 \le z_1 < 1,\ 0 \le z_2 < 1$. Then we have the following inequality*

$$
\Big({}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_1z_2)\Big)^2 \ge (1-z_1)^a(1-z_2)^a\, {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_1) \times\, {}_2F_1^{(\alpha,\beta;p)}(a,b;c;z_2) \tag{48}
$$

Proof. Let us consider the following sequences

$$
x_k = B_p^{(\alpha,\beta)}(b+k,c-b),\ y_k = z_1^k,\ m_k = \frac{(a)_k}{k!} z_2^k, \quad k \ge 0.
$$

The sequences (x_k) and (y_k) are both decreasing. Otherwise, by similar way as previous we have

$$
\sum_{k=0}^{\infty} m_k = (1-z_2)^{-a},\ \sum_{k=0}^{\infty} m_k x_k y_k = B(b,c-b)\, {}_2F_1(a,b;c;z_1z_2),
$$
$$
\sum_{k=0}^{\infty} m_k x_k = B(b,c-b)\, {}_2F_1(a,b;c;z_2),\ \sum_{k=0}^{\infty} m_k y_k =\, {}_2F_1(a,1;1;z_1z_2).
$$

By DCI (13) we then obtain

$$
{}_2F_1(a,b;c;z_1z_2) \ge (1-z_2)^a\, {}_2F_1(a,b;c;z_2)\, {}_2F_1(a,1;1;z_1z_2). \tag{49}
$$

Interchanging the role of z_1 and z_2 in (49) we obtain

$$
{}_2F_1(a,b;c;z_1z_2) \ge (1-z_1)^a\, {}_2F_1(a,b;c;z_1)\, {}_2F_1(a,1;1;z_1z_2). \tag{50}
$$

Multiplying (49) and (50) side by side we get (48), since ${}_2F_1(a,1;1;z_1z_2) \ge 1$ for $z_1, z_2 \ge 0$. ■

3 Concluding Remarks

In this paper, we have investigated some approximations for the generalized Gauss hypergeometric and the generalized confluent hypergeometric functions. Investing the discrete Chebychev inequality, we established several analytic inequalities involving these functions. In particular, some approximations have been expressed in terms of the classical and generalized beta function and the confluent hypergeometric function. In perspective, we plan to study in a future research the investment of the inequalities highlighted in this paper in the study of the behavior of these classes of hypergeometric functions.

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