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Hypersurfaces Admitting A Given Curve As A Line Of Curvature And Vice Versa*

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Abstract

In this paper, we develop a method to obtain the lines of curvature on parametric hypersurfaces in Euclidean 4-space. We show that such lines are the solutions to systems of first order triplet non-linear differential equations. Even if these curves cannot be obtained analytically in general, it is also shown that it is possible to compute all curvatures of a Frenet line of curvature by using the extended Darboux frame along the curve. We obtain the Frenet vectors and extended Darboux vectors of the line of curvature without encountering any singular case. In addition, we construct a developable ruled hypersurface whose base curve is always a line of curvature. We provide an example to show the applicability of the given method.

1 Introduction

A curve on a surface is called a line of curvature if it is always tangent to a principal direction. Lines of curvature on surfaces have always been the focus of attention not only in differential geometry (e.g. [4, 16]) but also in geometric modeling (e.g. [12]). Differential geometrical properties of such curves on parametric surfaces and hypersurfaces can be found in [4, 13, 16] and [2], respectively. It is known that a line of curvature on a parametric surface in Euclidean 3-space $\mathbb{E}^3$ satisfies the following differential equation [4]

$$(\mathsf{LE}-\mathsf{NF})(u'^2+(\mathsf{ME}-\mathsf{NG})u'v'+(\mathsf{MF}-\mathsf{LG})(v')^2=0.$$

If the above differential equation can be solved explicitly, then the line of curvature on the given surface can be obtained. In this case, it is easy to compute its Frenet apparatus. If we have an approximate solution for the line of curvature, then we need new techniques to calculate its curvatures and Frenet vectors. In 2007, Che et al. studied lines of curvature and their differential geometric properties for implicit surfaces in $\mathbb{E}^3$ [5]. In 2014, Joo et al. presented algorithms for computing the differential geometric properties of lines of curvature of parametric surfaces in $\mathbb{E}^3$. They derived the unit tangent vector, curvature vector, binormal vector and torsion of such lines. They also derived algorithms for evaluating the higher-order derivatives of lines of curvature of parametric surfaces [10] (The previous studies including the applications of lines of curvature have been reviewed in [5] and [10]).

Lines of curvature have also been studied in $\mathbb{E}^4$. The differential equation of the lines of curvature for immersions of surfaces into $\mathbb{E}^4$ has been established by [9]. In [11], the authors establish the geometric structure of the lines of curvature of a hypersurface immersed in $\mathbb{E}^4$ in a neighborhood of the set of its principal curvature singularities, consisting of the points at which at least two principal curvatures are equal. The geometric structures of the lines of curvature and the partially umbilic singularities of the three-dimensional non-compact generic quadric hypersurfaces of $\mathbb{E}^4$ have been studied in [14].

In this paper, we present a method to compute the lines of curvature and their differential geometric properties on parametric hypersurfaces in $\mathbb{E}^4$. By using the extended Darboux frame along a curve lying on a hypersurface, we obtain the curvatures, Frenet vectors and extended Darboux vectors of the obtained lines

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of curvature. In addition, we construct a developable ruled hypersurface whose base curve is always a line of curvature.

This paper is organized as follows: In section 2, we introduce some necessary notations and give some definitions for curves lying on hypersurfaces. The extended Darboux frame is also given in section 2. In section 3, we give a method to compute the lines of curvature of parametric hypersurfaces in $\mathbb{E}^4$. Also, we provide a method for computing the curvatures, Frenet vectors and extended Darboux vectors of the line of curvature. In section 4, we construct a special developable ruled hypersurface whose base curve is always a line of curvature. An illustrative example is presented in section 5.

2 Preliminaries

2.1 Curves on Hypersurfaces in $\mathbb{E}^4$

Definition 1 *The ternary product of the vectors*

$$\mathbf{x} = \sum_{i=1}^{4} x_i \mathbf{e_i}, \quad \mathbf{y} = \sum_{i=1}^{4} y_i \mathbf{e_i}, \quad \textit{and} \quad \mathbf{z} = \sum_{i=1}^{4} z_i \mathbf{e_i}$$

is defined by [15]

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ denotes the standard basis of $\mathbb{R}^4$.

Let $M \subset \mathbb{E}^4$ be a regular hypersurface given by its parametric equation $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$ and $\alpha : I \subset \mathbb{R} \to M$ be a unit speed curve. If $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ denotes the moving Frenet frame along α, then the Frenet formulas are given by [1]

$$\mathbf{t}' = k_1 \mathbf{n}, \qquad \mathbf{n}' = -k_1 \mathbf{t} + k_2 \mathbf{b}_1, \qquad \mathbf{b}_1' = -k_2 \mathbf{n} + k_3 \mathbf{b}_2, \qquad \mathbf{b}_2' = -k_3 \mathbf{b}_1,$$

where $\mathbf{t}, \mathbf{n}, \mathbf{b}_1$, and $\mathbf{b}_2$ denote the tangent, the principal normal, the first binormal, and the second binormal vector fields, respectively, and $k_i (i = 1, 2, 3)$ denotes the ith curvature function of the curve α. The Frenet vectors and curvatures of the curve are given by [1]

$$\mathbf{t} = \alpha', \quad \mathbf{n} = \frac{\alpha''}{\|\alpha''\|}, \quad \mathbf{b}_2 = \frac{\alpha' \otimes \alpha'' \otimes \alpha'''}{\|\alpha' \otimes \alpha'' \otimes \alpha'''\|}, \quad \mathbf{b}_1 = \mathbf{b}_2 \otimes \mathbf{t} \otimes \mathbf{n}, \tag{1}$$

$$k_1 = \|\alpha''\|, \quad k_2 = \frac{\langle \mathbf{b}_1, \alpha''' \rangle}{k_1}, \quad k_3 = \frac{\langle \mathbf{b}_2, \alpha^{(4)} \rangle}{k_1 k_2}.$$

The derivatives of the curve α are obtained as

$$\alpha' = \mathbf{t}, \quad \alpha'' = \mathbf{t}' = k_1 \mathbf{n}, \quad \alpha''' = -k_1^2 \mathbf{t} + k_1' \mathbf{n} + k_1 k_2 \mathbf{b}_1,$$

$$\alpha^{(4)} = -3k_1 k_1' \mathbf{t} + (-k_1^3 + k_1'' - k_1 k_2^2)\mathbf{n} + (2k_1' k_2 + k_1 k_2')\mathbf{b}_1 + k_1 k_2 k_3 \mathbf{b}_2.$$

In addition, since α lies on M, we can write $\alpha(s) = \mathbf{R}(u_1(s), u_2(s), u_3(s))$. Thus, we have $\alpha'(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i'$, where $\mathbf{R}_i = \frac{\partial \mathbf{R}}{\partial u_i}$, $i = 1, 2, 3$, and

$$\alpha''(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i'' + \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i' u_j', \tag{2}$$

$$\alpha'''(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i''' + 3 \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i'' u_j' + \sum_{i,j,k=1}^{3} \mathbf{R}_{ijk} u_i' u_j' u_k', \tag{3}$$

$$\alpha^{(4)}(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i^{(4)} + 4 \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i''' u_j' + 3 \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i'' u_j'' + 6 \sum_{i,j,k=1}^{3} \mathbf{R}_{ijk} u_i'' u_j' u_k' + \sum_{i,j,k,\ell=1}^{3} \mathbf{R}_{ijk\ell} u_i' u_j' u_k' u_\ell'. \tag{4}$$

Definition 2 *A unit speed curve $\beta : I \to \mathbb{E}^n$ of class C^n is called a* Frenet curve *if the vectors $\beta'(s), \beta''(s), ...,$ $\beta^{(n-1)}(s)$ are linearly independent at each point along the curve.*

2.2 Extended Darboux Frame in $\mathbb{E}^4$

Let $M \subset \mathbb{E}^4$ be a regular hypersurface and $\alpha : I \subset \mathbb{R} \to M$ be a unit speed Frenet curve. Let $\mathbf{T}$ denote the unit tangent vector field along α, and $\mathbf{N}$ denote the unit normal vector field of M restricted to the curve α. Then the extended Darboux frame of first kind along α is given by $\{\mathbf{T}, \mathbf{E}, \mathbf{D}, \mathbf{N}\}$, where

$$\mathbf{E} = \frac{\mathbf{T}' - \langle \mathbf{T}', \mathbf{N} \rangle \mathbf{N}}{\| \mathbf{T}' - \langle \mathbf{T}', \mathbf{N} \rangle \mathbf{N} \|}, \quad \mathbf{D} = \mathbf{N} \otimes \mathbf{T} \otimes \mathbf{E}.$$

This frame satisfies the following system of differential equations [8]

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{E}' \\ \mathbf{D}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & k_g^1 & 0 & k_n \\ -k_g^1 & 0 & k_g^2 & \tau_g^1 \\ 0 & -k_g^2 & 0 & \tau_g^2 \\ -k_n & -\tau_g^1 & -\tau_g^2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{E} \\ \mathbf{D} \\ \mathbf{N} \end{pmatrix}, \tag{5}$$

where k_g^i and τ_g^i denote the geodesic curvature and geodesic torsion of order i, respectively, and k_n denotes the normal curvature of the hypersurface in the direction of the tangent vector $\mathbf{T}$. Then we have [8]

$$k_g^1 = \langle \mathbf{T}', \mathbf{E} \rangle, \quad k_g^2 = \langle \mathbf{E}', \mathbf{D} \rangle, \quad \tau_g^1 = \langle \mathbf{E}', \mathbf{N} \rangle, \quad \tau_g^2 = \langle \mathbf{D}', \mathbf{N} \rangle, \quad k_n = \langle \mathbf{T}', \mathbf{N} \rangle.$$

2.3 Ruled Hypersurface in $\mathbb{E}^4$

A ruled hypersurface in $\mathbb{E}^4$ can be parametrized by the map

$$\psi : I \times \mathbb{R}^2 \to \mathbb{E}^4, \ \psi(s, u, v) = \beta(s) + u\mathbf{e}_1(s) + v\mathbf{e}_2(s),$$

where $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ denotes the base curve with unit tangent vector $\mathbf{e}_0$, and $\{\mathbf{e}_1(s), \mathbf{e}_2(s)\}$ denotes an orthonormal basis of generating plane along β. Let

$$\text{rank}[\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1', \mathbf{e}_2'] = 4 - k. \tag{6}$$

If $k = 0$ (resp. $k = 1$) in (6), then the ruled hypersurface is called non-developable (resp. developable) [3].

3 Lines of Curvature of Parametric Hypersurfaces in $\mathbb{E}^4$

In this section, we first show how we can obtain the lines of curvature of a parametric hypersurface in $\mathbb{E}^4$. Then, we show that, even if such curves cannot be obtained analytically in general, it is still possible to compute their curvatures and Frenet vectors.

Let $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$ denote a regular hypersurface M defined on a domain B. Then, the unit normal vector field of M is given by $\mathbf{N} = \frac{\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3}{\| \mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3 \|}$. Since the normal curvature of M at a point is given by $k_n = \frac{\mathrm{II}}{\mathrm{I}}$, we have

$$k_n(\lambda, \mu) = \frac{h_{11} + 2h_{12}\lambda + 2h_{13}\mu + 2h_{23}\lambda\mu + h_{22}\lambda^2 + h_{33}\mu^2}{g_{11} + 2g_{12}\lambda + 2g_{13}\mu + 2g_{23}\lambda\mu + g_{22}\lambda^2 + g_{33}\mu^2}, \tag{7}$$

where I and II denote the first and second fundamental forms of M, respectively, $(\lambda,\mu)=\left(\frac{du_2}{du_1},\frac{du_3}{du_1}\right)$ denotes the tangent direction for $du_1\neq 0$, and g_{ij}, h_{ij} denote the coefficients of the first and second fundamental forms, respectively. It is well-known that the extremal values of normal curvature are principal curvatures [2]. If we take the partial derivatives of k_n with respect to λ and μ, respectively, we have

$$\frac{\partial k_n}{\partial\lambda}=\frac{\Big(2h_{12}+2h_{23}\mu+2h_{22}\lambda\Big)\mathrm{I}-\mathrm{II}\Big(2g_{12}+2g_{23}\mu+2g_{22}\lambda\Big)}{\Big(g_{11}+2g_{12}\lambda+2g_{13}\mu+2g_{23}\lambda\mu+g_{22}\lambda^2+g_{33}\mu^2\Big)^2},$$

$$\frac{\partial k_n}{\partial\mu}=\frac{\Big(2h_{13}+2h_{23}\lambda+2h_{33}\mu\Big)\mathrm{I}-\mathrm{II}\Big(2g_{13}+2g_{23}\lambda+2g_{33}\mu\Big)}{\Big(g_{11}+2g_{12}\lambda+2g_{13}\mu+2g_{23}\lambda\mu+g_{22}\lambda^2+g_{33}\mu^2\Big)^2}.$$

Then, we obtain

$$k_n(\lambda,\mu)=\frac{\mathrm{II}}{\mathrm{I}}=\frac{h_{12}+h_{22}\lambda+h_{23}\mu}{g_{12}+g_{22}\lambda+g_{23}\mu}=\frac{h_{13}+h_{23}\lambda+h_{33}\mu}{g_{13}+g_{23}\lambda+g_{33}\mu}=\frac{h_{11}+h_{12}\lambda+h_{13}\mu}{g_{11}+g_{12}\lambda+g_{13}\mu}.$$

Thus, the principal curvatures satisfy the following homogeneous system [2]

$$\begin{cases}(h_{11}-k_ng_{11})du_1+(h_{12}-k_ng_{12})du_2+(h_{13}-k_ng_{13})du_3=0,\\(h_{12}-k_ng_{12})du_1+(h_{22}-k_ng_{22})du_2+(h_{23}-k_ng_{23})du_3=0,\\(h_{13}-k_ng_{13})du_1+(h_{23}-k_ng_{23})du_2+(h_{33}-k_ng_{33})du_3=0.\end{cases}\tag{8}$$

Let us denote the coefficient matrix of the above system by $\mathcal{A}$, i.e.

$$\mathcal{A}=\begin{pmatrix}h_{11}-k_ng_{11}&h_{12}-k_ng_{12}&h_{13}-k_ng_{13}\\h_{12}-k_ng_{12}&h_{22}-k_ng_{22}&h_{23}-k_ng_{23}\\h_{13}-k_ng_{13}&h_{23}-k_ng_{23}&h_{33}-k_ng_{33}\end{pmatrix}.$$

In case $rank(\mathcal{A})=0$, since all directions satisfy (8), the point is an umbilical point. Then, because of the properties of umbilical points, we obtain

$$k_n=\frac{h_{11}}{g_{11}}=\frac{h_{12}}{g_{12}}=\frac{h_{13}}{g_{13}}=\frac{h_{23}}{g_{23}}=\frac{h_{22}}{g_{22}}=\frac{h_{33}}{g_{33}}.$$

In case $rank(\mathcal{A})=3$, the system has the trivial solution only. For the principal directions, we need the nontrivial solutions of this system. This system has a nontrivial solution if and only if $\det\mathcal{A}=0$, i.e.

$$\begin{aligned}\det\mathcal{A}\;=\;&\Big(g_{12}^2g_{33}+g_{13}^2g_{22}+g_{23}^2g_{11}-2g_{12}g_{23}g_{13}-g_{11}g_{22}g_{33}\Big)k_n^3\\&+\Big(2h_{12}g_{13}g_{23}-2h_{12}g_{12}g_{33}+2h_{13}g_{12}g_{23}+2h_{23}g_{12}g_{13}+h_{22}g_{11}g_{33}+h_{33}g_{11}g_{22}\\&+h_{11}g_{22}g_{33}-2h_{13}g_{22}g_{13}-2h_{23}g_{11}g_{23}-h_{33}g_{12}^2-h_{22}g_{13}^2-h_{11}g_{23}^2\Big)k_n^2\\&+\Big(2h_{12}h_{33}g_{12}-2h_{12}h_{23}g_{13}-2h_{13}h_{23}g_{12}+2h_{11}h_{23}g_{23}+2h_{13}h_{22}g_{13}\\&-h_{11}h_{22}g_{33}-h_{22}h_{33}g_{11}-h_{11}h_{33}g_{22}-2h_{12}h_{13}g_{23}+h_{23}^2g_{11}+h_{13}^2g_{22}+h_{12}^2g_{33}\Big)k_n\\&+2h_{12}h_{13}h_{23}+h_{11}h_{22}h_{33}-h_{12}^2h_{33}-h_{13}^2h_{22}-h_{23}^2h_{11}=0.\end{aligned}$$

If we denote

$$\mathcal{K}_1=-\frac{2h_{12}h_{13}h_{23}+h_{11}h_{22}h_{33}-h_{12}^2h_{33}-h_{13}^2h_{22}-h_{23}^2h_{11}}{g_{12}^2g_{33}+g_{13}^2g_{22}+g_{23}^2g_{11}-2g_{12}g_{23}g_{13}-g_{11}g_{22}g_{33}},$$

$$\mathcal{K}_2 = -\frac{2h_{12}g_{13}g_{23} - 2h_{12}g_{12}g_{33} + 2h_{13}g_{12}g_{23} + 2h_{23}g_{12}g_{13} + h_{22}g_{11}g_{33} + h_{33}g_{11}g_{22}}{3\left(g_{12}^2 g_{33} + g_{13}^2 g_{22} + g_{23}^2 g_{11} - 2g_{12}g_{23}g_{13} - g_{11}g_{22}g_{33}\right)}$$
$$- \frac{h_{11}g_{22}g_{33} - 2h_{13}g_{22}g_{13} - 2h_{23}g_{11}g_{23} - h_{33}g_{12}^2 - h_{22}g_{13}^2 - h_{11}g_{23}^2}{3\left(g_{12}^2 g_{33} + g_{13}^2 g_{22} + g_{23}^2 g_{11} - 2g_{12}g_{23}g_{13} - g_{11}g_{22}g_{33}\right)},$$

$$\mathcal{K}_3 = \frac{2h_{12}h_{33}g_{12} - 2h_{12}h_{23}g_{13} - 2h_{13}h_{23}g_{12} + 2h_{11}h_{23}g_{23} + 2h_{13}h_{22}g_{13} - h_{11}h_{22}g_{33}}{g_{12}^2 g_{33} + g_{13}^2 g_{22} + g_{23}^2 g_{11} - 2g_{12}g_{23}g_{13} - g_{11}g_{22}g_{33}}$$
$$+ \frac{-h_{22}h_{33}g_{11} - h_{11}h_{33}g_{22} - 2h_{12}h_{13}g_{23} + h_{23}^2 g_{11} + h_{13}^2 g_{22} + h_{12}^2 g_{33}}{g_{12}^2 g_{33} + g_{13}^2 g_{22} + g_{23}^2 g_{11} - 2g_{12}g_{23}g_{13} - g_{11}g_{22}g_{33}},$$

the last equation can be written as

$$k_n^3 - 3\mathcal{K}_2 k_n^2 + \mathcal{K}_3 k_n - \mathcal{K}_1 = 0, \tag{9}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ correspond to the Gauss and mean curvatures, respectively. Note that (9) is a third order equation with respect to k_n. If we use the Cardano's method [7] for cubic equations, we can express the principal curvatures in terms of $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$.

3.1 Computation of Line of Curvature

Let us now give a method for obtaining the unit speed line of curvature of a parametric hypersurface. Let us assume that $rank(\mathcal{A}) = 2$. In this case, we have at least one 2×2 submatrix of $\mathcal{A}$ which has nonzero determinant. Suppose that

$$a_1 = (h_{12} - k_n g_{12})(h_{23} - k_n g_{23}) - (h_{22} - k_n g_{22})(h_{13} - k_n g_{13}) \neq 0.$$

Note that a_1 corresponds to the determinant for coefficients of du_2 and du_3 in the first and second equations of (8). Let

$$a_2 = (h_{12} - k_n g_{12})(h_{13} - k_n g_{13}) - (h_{11} - k_n g_{11})(h_{23} - k_n g_{23}),$$
$$a_3 = (h_{11} - k_n g_{11})(h_{22} - k_n g_{22}) - (h_{12} - k_n g_{12})^2.$$

Then, if we choose

$$u_1' = \eta a_1, \quad u_2' = \eta a_2, \quad u_3' = \eta a_3, \tag{10}$$

(where η is a nonzero factor) it is easy to see that (10) satisfies (8). Note that a_2 and a_3 also correspond to the determinants of some submatrices obtained from the matrix $\mathcal{A}$. Since the line of curvature is unit speed, its first fundamental form is given by

$$\sum_{i,j=1}^{3} g_{ij} u_i' u_j' = 1. \tag{11}$$

Hence, substituting (10) into (11) determines η as

$$\eta = \mp \frac{1}{\sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + 2g_{13}a_1a_3 + 2g_{23}a_2a_3 + g_{22}a_2^2 + g_{33}a_3^2}}. \tag{12}$$

If we substitute (12) into (10), we obtain

$$\begin{cases} u_1' = \mp \frac{a_1}{\sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + 2g_{13}a_1a_3 + 2g_{23}a_2a_3 + g_{22}a_2^2 + g_{33}a_3^2}}, \\ u_2' = \mp \frac{a_2}{\sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + 2g_{13}a_1a_3 + 2g_{23}a_2a_3 + g_{22}a_2^2 + g_{33}a_3^2}}, \\ u_3' = \mp \frac{a_3}{\sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + 2g_{13}a_1a_3 + 2g_{23}a_2a_3 + g_{22}a_2^2 + g_{33}a_3^2}}. \end{cases} \tag{13}$$

Thus, the line of curvature $\alpha(s)$ can be obtained as a solution to the system (13) together with the initial values $u_1(0) = u_0, u_2(0) = v_0, u_3(0) = w_0$.

3.2 Curvatures of Line of Curvature

Now, we want to find all curvatures of the line of curvature $\alpha(s)$ obtained by the above method. For this purpose, we need to compute the higher order derivatives of $\alpha(s)$.

3.2.1 First Curvature (k_1)

If we use (2) and (5), we may write

$$\alpha''(s) = k_g^1 \mathbf{E} + k_n \mathbf{N} = \mathbf{R}_1 u_1'' + \mathbf{R}_2 u_2'' + \mathbf{R}_3 u_3'' + \Omega_1, \tag{14}$$

where $\Omega_1 = \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i' u_j'$. Since u_1', u_2' and u_3' are known from (13), Ω_1 is known. Taking inner product of (14) with $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$, respectively, we get the equations

$$k_g^1 \langle \mathbf{E}, \mathbf{R}_1 \rangle = g_{11} u_1'' + g_{12} u_2'' + g_{13} u_3'' + \langle \Omega_1, \mathbf{R}_1 \rangle, \tag{15}$$

$$k_g^1 \langle \mathbf{E}, \mathbf{R}_2 \rangle = g_{12} u_1'' + g_{22} u_2'' + g_{23} u_3'' + \langle \Omega_1, \mathbf{R}_2 \rangle, \tag{16}$$

$$k_g^1 \langle \mathbf{E}, \mathbf{R}_3 \rangle = g_{13} u_1'' + g_{23} u_2'' + g_{33} u_3'' + \langle \Omega_1, \mathbf{R}_3 \rangle. \tag{17}$$

Moreover, since $\alpha' = \sum_{i=1}^{3} \mathbf{R}_i u_i'$ and $\langle \mathbf{E}, \alpha' \rangle = 0$, for $u_1' \neq 0$ we have

$$\langle \mathbf{E}, \mathbf{R}_1 \rangle = -\frac{u_2' \langle \mathbf{E}, \mathbf{R}_2 \rangle}{u_1'} - \frac{u_3' \langle \mathbf{E}, \mathbf{R}_3 \rangle}{u_1'}. \tag{18}$$

If we substitute (18) into (15), using (16) and (17) we get

$$\left(\sum_{i=1}^{3} u_i' g_{1i} \right) u_1'' + \left(\sum_{i=1}^{3} u_i' g_{2i} \right) u_2'' + \left(\sum_{i=1}^{3} u_i' g_{3i} \right) u_3'' = -\Big\langle \Omega_1, \alpha' \Big\rangle. \tag{19}$$

Differentiating the first and second equations of (8), we obtain

$$(h_{11} - k_n g_{11}) u_1'' + (h_{12} - k_n g_{12}) u_2'' + (h_{13} - k_n g_{13}) u_3'' = \rho_1, \tag{20}$$

$$(h_{12} - k_n g_{12}) u_1'' + (h_{22} - k_n g_{22}) u_2'' + (h_{23} - k_n g_{23}) u_3'' = \rho_2, \tag{21}$$

where

$$\rho_1 = -\Big(h_{11}' - k_n' g_{11} - k_n g_{11}' \Big) u_1' - \Big(h_{12}' - k_n' g_{12} - k_n g_{12}' \Big) u_2' - \Big(h_{13}' - k_n' g_{13} - k_n g_{13}' \Big) u_3',$$
$$\rho_2 = -\Big(h_{12}' - k_n' g_{12} - k_n g_{12}' \Big) u_1' - \Big(h_{22}' - k_n' g_{22} - k_n g_{22}' \Big) u_2' - \Big(h_{23}' - k_n' g_{23} - k_n g_{23}' \Big) u_3'.$$

If we consider (16), (17) and (19)–(21), we obtain the following nonhomogeneous system of linear equations

$$\begin{pmatrix} \sum_{i=1}^{3} u_i' g_{1i} & \sum_{i=1}^{3} u_i' g_{2i} & \sum_{i=1}^{3} u_i' g_{3i} & 0 & 0 \\ g_{12} & g_{22} & g_{23} & -1 & 0 \\ g_{13} & g_{23} & g_{33} & 0 & -1 \\ h_{11} - k_n g_{11} & h_{12} - k_n g_{12} & h_{13} - k_n g_{13} & 0 & 0 \\ h_{12} - k_n g_{12} & h_{22} - k_n g_{22} & h_{23} - k_n g_{23} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1'' \\ u_2'' \\ u_3'' \\ k_g^1 \langle \mathbf{E}, \mathbf{R}_2 \rangle \\ k_g^1 \langle \mathbf{E}, \mathbf{R}_3 \rangle \end{pmatrix} = \begin{pmatrix} -\langle \Omega_1, \alpha' \rangle \\ -\langle \Omega_1, \mathbf{R}_2 \rangle \\ -\langle \Omega_1, \mathbf{R}_3 \rangle \\ \rho_1 \\ \rho_2 \end{pmatrix}. \tag{22}$$

Since the determinant of the coefficient matrix of the above system is nonzero, i.e.

$$\sqrt{g_{11} a_1^2 + 2 g_{12} a_1 a_2 + 2 g_{13} a_1 a_3 + 2 g_{23} a_2 a_3 + g_{22} a_2^2 + g_{33} a_3^2} \neq 0,$$

the unknowns u_1'', u_2'', u_3'', $k_g^1\langle \mathbf{E}, \mathbf{R}_2\rangle$, and $k_g^1\langle \mathbf{E}, \mathbf{R}_3\rangle$ can be computed from (22). These solutions enable us to compute the curvature vector α'', the vectors $\mathbf{E} = \frac{\mathbf{T}' - \langle \mathbf{T}', \mathbf{N}\rangle \mathbf{N}}{\|\mathbf{T}' - \langle \mathbf{T}', \mathbf{N}\rangle \mathbf{N}\|}$ and $\mathbf{D} = \mathbf{N} \otimes \mathbf{T} \otimes \mathbf{E}$, and k_g^1. Hence, the first curvature of the line of curvature is obtained by

$$k_1 = \sqrt{(k_n)^2 + (k_g^1)^2}.$$

3.2.2 Second Curvature (k_2)

To obtain the second curvature, we need to determine the third order derivative of α.

Since α is a line of curvature, we have $\tau_g^1 = \tau_g^2 = 0$. If we take the derivative of $\alpha'' = k_g^1\mathbf{E} + k_n\mathbf{N}$ with respect to arc-length, we get

$$\alpha''' = (k_g^1)'\mathbf{E} - k_1^2\mathbf{T} + k_g^1 k_g^2\mathbf{D} + k_n'\mathbf{N}.$$

Thus, if we use (3) and (5), we have

$$(k_g^1)'\mathbf{E} - k_1^2\mathbf{T} + k_g^1 k_g^2\mathbf{D} + k_n'\mathbf{N} = \mathbf{R}_1 u_1''' + \mathbf{R}_2 u_2''' + \mathbf{R}_3 u_3''' + \Omega_2, \tag{23}$$

where

$$\Omega_2 = 3\sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i' u_j'' + \sum_{i,j,k=1}^{3} \mathbf{R}_{ijk} u_i' u_j' u_k'.$$

If we take the inner product of both sides of (23) with $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, respectively, we obtain the linear equations

$$g_{11}u_1''' + g_{12}u_2''' + g_{13}u_3''' - (k_g^1)'\langle \mathbf{E}, \mathbf{R}_1\rangle - k_g^1 k_g^2\langle \mathbf{D}, \mathbf{R}_1\rangle = -k_1^2\langle \mathbf{T}, \mathbf{R}_1\rangle - \langle \Omega_2, \mathbf{R}_1\rangle, \tag{24}$$

$$g_{12}u_1''' + g_{22}u_2''' + g_{23}u_3''' - (k_g^1)'\langle \mathbf{E}, \mathbf{R}_2\rangle - k_g^1 k_g^2\langle \mathbf{D}, \mathbf{R}_2\rangle = -k_1^2\langle \mathbf{T}, \mathbf{R}_2\rangle - \langle \Omega_2, \mathbf{R}_2\rangle,$$

$$g_{13}u_1''' + g_{23}u_2''' + g_{33}u_3''' - (k_g^1)'\langle \mathbf{E}, \mathbf{R}_3\rangle - k_g^1 k_g^2\langle \mathbf{D}, \mathbf{R}_3\rangle = -k_1^2\langle \mathbf{T}, \mathbf{R}_3\rangle - \langle \Omega_2, \mathbf{R}_3\rangle,$$

with the unknowns u_1''', u_2''', u_3''', $(k_g^1)'$ and k_g^2. So, we have to find two more equations to obtain these unknowns. Differentiating (20) and (21), we get

$$(h_{11} - k_n g_{11})u_1''' + (h_{12} - k_n g_{12})u_2''' + (h_{13} - k_n g_{13})u_3''' = \rho_3, \tag{25}$$

$$(h_{12} - k_n g_{12})u_1''' + (h_{22} - k_n g_{22})u_2''' + (h_{23} - k_n g_{23})u_3''' = \rho_4, \tag{26}$$

where

$$\begin{aligned}
\rho_3 \quad = \quad & -\Big(h_{11}'' - k_n'' g_{11} - 2k_n' g_{11}' - k_n g_{11}''\Big)u_1' - 2\Big(h_{11}' - k_n' g_{11} - k_n g_{11}'\Big)u_1'' \\
& -\Big(h_{12}'' - k_n'' g_{12} - 2k_n' g_{12}' - k_n g_{12}''\Big)u_2' - 2\Big(h_{12}' - k_n' g_{12} - k_n g_{12}'\Big)u_2'' \\
& -\Big(h_{13}'' - k_n'' g_{13} - 2k_n' g_{13}' - k_n g_{13}''\Big)u_3' - 2\Big(h_{13}' - k_n' g_{13} - k_n g_{13}'\Big)u_3'',
\end{aligned}$$

and

$$\begin{aligned}
\rho_4 \quad = \quad & -\Big(h_{12}'' - k_n'' g_{12} - 2k_n' g_{12}' - k_n g_{12}''\Big)u_1' - 2\Big(h_{12}' - k_n' g_{12} - k_n g_{12}'\Big)u_1'' \\
& -\Big(h_{22}'' - k_n'' g_{22} - 2k_n' g_{22}' - k_n g_{22}''\Big)u_2' - 2\Big(h_{22}' - k_n' g_{22} - k_n g_{22}'\Big)u_2'' \\
& -\Big(h_{23}'' - k_n'' g_{23} - 2k_n' g_{23}' - k_n g_{23}''\Big)u_3' - 2\Big(h_{23}' - k_n' g_{23} - k_n g_{23}'\Big)u_3''.
\end{aligned}$$

Then, the equations (24) through (26) constitute a linear equation system $\mathcal{Q}X = \mathcal{S}$, where

$$\mathcal{Q} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & -\langle \mathbf{E}, \mathbf{R}_1\rangle & -k_g^1\langle \mathbf{D}, \mathbf{R}_1\rangle \\ g_{12} & g_{22} & g_{23} & -\langle \mathbf{E}, \mathbf{R}_2\rangle & -k_g^1\langle \mathbf{D}, \mathbf{R}_2\rangle \\ g_{13} & g_{23} & g_{33} & -\langle \mathbf{E}, \mathbf{R}_3\rangle & -k_g^1\langle \mathbf{D}, \mathbf{R}_3\rangle \\ h_{11}-k_n g_{11} & h_{12}-k_n g_{12} & h_{13}-k_n g_{13} & 0 & 0 \\ h_{12}-k_n g_{12} & h_{22}-k_n g_{22} & h_{23}-k_n g_{23} & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} u_1''' \\ u_2''' \\ u_3''' \\ (k_g^1)' \\ k_g^2 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} -k_1^2\langle \mathbf{T}, \mathbf{R}_1\rangle - \langle \Omega_2, \mathbf{R}_1\rangle \\ -k_1^2\langle \mathbf{T}, \mathbf{R}_2\rangle - \langle \Omega_2, \mathbf{R}_2\rangle \\ -k_1^2\langle \mathbf{T}, \mathbf{R}_3\rangle - \langle \Omega_2, \mathbf{R}_3\rangle \\ \rho_3 \\ \rho_4 \end{pmatrix}.$$

We should here note that $k_g^1 \neq 0$. Otherwise, by (5), the equality $\mathbf{T}' = k_g^1\mathbf{E} + k_n\mathbf{N}$ becomes $\mathbf{T}' = k_n\mathbf{N}$ which implies that α is a geodesic curve. However, a Frenet curve which is a line of curvature cannot be a geodesic curve. Thus, since the determinant of the coefficient matrix $\mathcal{Q}$ is nonzero, i.e.

$$\det \mathcal{Q} = k_g^1\|\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3\|\sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + 2g_{13}a_1a_3 + 2g_{23}a_2a_3 + g_{22}a_2^2 + g_{33}a_3^2} \neq 0,$$

the unknowns u_1''', u_2''', u_3''', $(k_g^1)'$ and k_g^2 can be obtained. These solutions enable us to compute α'''. If we use (1), we can compute the Frenet vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of α. Therefore, the second curvature can be obtained by $k_2 = \frac{\langle \mathbf{b}_1, \alpha'''\rangle}{k_1}$. We also have $k_1' = \langle \mathbf{n}, \alpha'''\rangle$ and $k_n' = \langle \mathbf{N}, \alpha'''\rangle$.

Furthermore, k_2 cannot be zero. Otherwise, the third derivative of α can be written as a linear combination of α' and α'' which contradicts with the fact that α is a Frenet curve.

3.2.3 Third Curvature (k_3)

Similarly, we need to find the fourth derivative of the line of curvature to obtain its third curvature. If we take the derivative of α''', we may write

$$\begin{aligned} \alpha^{(4)} =& \big[-2k_1k_1' - k_g^1(k_g^1)' - k_nk_n'\big]\mathbf{T} + \big[(k_g^1)'' - k_1^2k_g^1 - k_g^1(k_g^2)^2\big]\mathbf{E} \\ &+ \big[2(k_g^1)'k_g^2 + k_g^1(k_g^2)'\big]\mathbf{D} + \big(k_n'' - k_1^2k_n\big)\mathbf{N}. \end{aligned}$$

On the other hand, for the fourth derivative, from (4) we also have

$$\alpha^{(4)} = \mathbf{R}_1u_1^{(4)} + \mathbf{R}_2u_2^{(4)} + \mathbf{R}_3u_3^{(4)} + \Omega_3,$$

where

$$\Omega_3 = 4\sum_{i,j=1}^{3}\mathbf{R}_{ij}u_i'''u_j' + 3\sum_{i,j=1}^{3}\mathbf{R}_{ij}u_i''u_j'' + 6\sum_{i,j,k=1}^{3}\mathbf{R}_{ijk}u_i''u_j'u_k' + \sum_{i,j,k,\ell=1}^{3}\mathbf{R}_{ijk\ell}u_i'u_j'u_k'u_\ell'.$$

Hence, we may write

$$\begin{aligned} \alpha^{(4)} &= \big[-2k_1k_1' - k_g^1(k_g^1)' - k_nk_n'\big]\mathbf{T} + \big[(k_g^1)'' - k_1^2k_g^1 - k_g^1(k_g^2)^2\big]\mathbf{E} \\ &\quad + \big[2(k_g^1)'k_g^2 + k_g^1(k_g^2)'\big]\mathbf{D} + \big(k_n'' - k_1^2k_n\big)\mathbf{N} \\ &= \mathbf{R}_1u_1^{(4)} + \mathbf{R}_2u_2^{(4)} + \mathbf{R}_3u_3^{(4)} + \Omega_3. \end{aligned} \tag{27}$$

If we take the inner product of both sides of (27) with $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$, respectively, we get the following equations

$$\begin{aligned} & g_{11}u_1^{(4)} + g_{12}u_2^{(4)} + g_{13}u_3^{(4)} - \langle \mathbf{E}, \mathbf{R}_1\rangle(k_g^1)'' - k_g^1(k_g^2)'\langle \mathbf{D}, \mathbf{R}_1\rangle \\ =\; & -\langle \Omega_3, \mathbf{R}_1\rangle - [2k_1k_1' + k_g^1(k_g^1)' + k_nk_n']\langle \mathbf{T}, \mathbf{R}_1\rangle - \big[k_1^2k_g^1 + k_g^1(k_g^2)^2\big]\langle \mathbf{E}, \mathbf{R}_1\rangle + 2(k_g^1)'k_g^2\langle \mathbf{D}, \mathbf{R}_1\rangle, \end{aligned} \tag{28}$$

$$\begin{aligned}
&g_{12}u_1^{(4)}+g_{22}u_2^{(4)}+g_{23}u_3^{(4)}-\langle \mathbf{E},\mathbf{R}_2\rangle(k_g^1)''-k_g^1(k_g^2)'\langle \mathbf{D},\mathbf{R}_2\rangle\\
=\;&-\langle \Omega_3,\mathbf{R}_2\rangle-\big[2k_1k_1'+k_g^1(k_g^1)'+k_nk_n'\big]\langle \mathbf{T},\mathbf{R}_2\rangle-\big[k_1^2k_g^1+k_g^1(k_g^2)^2\big]\langle \mathbf{E},\mathbf{R}_2\rangle+2(k_g^1)'k_g^2\langle \mathbf{D},\mathbf{R}_2\rangle,
\end{aligned}$$

$$\begin{aligned}
&g_{13}u_1^{(4)}+g_{23}u_2^{(4)}+g_{33}u_3^{(4)}-\langle \mathbf{E},\mathbf{R}_3\rangle(k_g^1)''-k_g^1(k_g^2)'\langle \mathbf{D},\mathbf{R}_3\rangle\\
=\;&-\langle \Omega_3,\mathbf{R}_3\rangle-\big[2k_1k_1'+k_g^1(k_g^1)'+k_nk_n'\big]\langle \mathbf{T},\mathbf{R}_3\rangle-\big[k_1^2k_g^1+k_g^1(k_g^2)^2\big]\langle \mathbf{E},\mathbf{R}_3\rangle+2(k_g^1)'k_g^2\langle \mathbf{D},\mathbf{R}_3\rangle.
\end{aligned}$$

Differentiating (25) and (26), we get

$$\begin{aligned}
&(h_{11}-k_ng_{11})u_1^{(4)}+(h_{12}-k_ng_{12})u_2^{(4)}+(h_{13}-k_ng_{13})u_3^{(4)}=\rho_5,\\
&(h_{12}-k_ng_{12})u_1^{(4)}+(h_{22}-k_ng_{22})u_2^{(4)}+(h_{23}-k_ng_{23})u_3^{(4)}=\rho_6,
\end{aligned}\tag{29}$$

where

$$\begin{aligned}
\rho_5 =\;& -\Big(h_{11}'''-k_n'''g_{11}-3k_n''g_{11}'-3k_n'g_{11}''-k_ng_{11}'''\Big)u_1'-3\Big(h_{11}''-k_n''g_{11}-2k_n'g_{11}'-k_ng_{11}''\Big)u_1''\\
&-3\Big(h_{11}'-k_n'g_{11}-k_ng_{11}'\Big)u_1'''-\Big(h_{12}'''-k_n'''g_{12}-3k_n''g_{12}'-3k_n'g_{12}''-k_ng_{12}'''\Big)u_2'\\
&-3\Big(h_{12}''-k_n''g_{12}-2k_n'g_{12}'-k_ng_{12}''\Big)u_2''-3\Big(h_{12}'-k_n'g_{12}-k_ng_{12}'\Big)u_2'''\\
&-\Big(h_{13}'''-k_n'''g_{13}-3k_n''g_{13}'-3k_n'g_{13}''-k_ng_{13}'''\Big)u_3'-3\Big(h_{13}''-k_n''g_{13}-2k_n'g_{13}'-k_ng_{13}''\Big)u_3''\\
&-3\Big(h_{13}'-k_n'g_{13}-k_ng_{13}'\Big)u_3''',
\end{aligned}$$

$$\begin{aligned}
\rho_6 =\;& -\Big(h_{12}'''-k_n'''g_{12}-3k_n''g_{12}'-3k_n'g_{12}''-k_ng_{12}'''\Big)u_1'-3\Big(h_{12}''-k_n''g_{12}-2k_n'g_{12}'-k_ng_{12}''\Big)u_1''\\
&-3\Big(h_{12}'-k_n'g_{12}-k_ng_{12}'\Big)u_1'''-\Big(h_{22}'''-k_n'''g_{22}-3k_n''g_{22}'-3k_n'g_{22}''-k_ng_{22}'''\Big)u_2'\\
&-3\Big(h_{22}''-k_n''g_{22}-2k_n'g_{22}'-k_ng_{22}''\Big)u_2''-3\Big(h_{22}'-k_n'g_{22}-k_ng_{22}'\Big)u_2'''\\
&-\Big(h_{23}'''-k_n'''g_{23}-3k_n''g_{23}'-3k_n'g_{23}''-k_ng_{23}'''\Big)u_3'-3\Big(h_{23}''-k_n''g_{23}-2k_n'g_{23}'-k_ng_{23}''\Big)u_3''\\
&-3\Big(h_{23}'-k_n'g_{23}-k_ng_{23}'\Big)u_3'''.
\end{aligned}$$

Thus, the equations (28) through (29) constitute a system of linear equations with the unknowns $u_1^{(4)}$, $u_2^{(4)}$, $u_3^{(4)}$, $(k_g^1)''$ and $(k_g^2)'$. This system is given in matrix notation by $\mathcal{Q}Y=\mathcal{W}$, where

$$Y=\begin{pmatrix} u_1^{(4)}\\ u_2^{(4)}\\ u_3^{(4)}\\ (k_g^1)''\\ (k_g^2)'\end{pmatrix},\quad \mathcal{W}=\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \rho_5\\ \rho_6\end{pmatrix},$$

$$\begin{aligned}
\tau_1&=-\langle \Omega_3,\mathbf{R}_1\rangle-[2k_1k_1'+k_g^1(k_g^1)'+k_nk_n']\langle \mathbf{T},\mathbf{R}_1\rangle-\big[k_1^2k_g^1+k_g^1(k_g^2)^2\big]\langle \mathbf{E},\mathbf{R}_1\rangle+2(k_g^1)'k_g^2\langle \mathbf{D},\mathbf{R}_1\rangle,\\
\tau_2&=-\langle \Omega_3,\mathbf{R}_2\rangle-[2k_1k_1'+k_g^1(k_g^1)'+k_nk_n']\langle \mathbf{T},\mathbf{R}_2\rangle-\big[k_1^2k_g^1+k_g^1(k_g^2)^2\big]\langle \mathbf{E},\mathbf{R}_2\rangle+2(k_g^1)'k_g^2\langle \mathbf{D},\mathbf{R}_2\rangle,\\
\tau_3&=-\langle \Omega_3,\mathbf{R}_3\rangle-[2k_1k_1'+k_g^1(k_g^1)'+k_nk_n']\langle \mathbf{T},\mathbf{R}_3\rangle-\big[k_1^2k_g^1+k_g^1(k_g^2)^2\big]\langle \mathbf{E},\mathbf{R}_3\rangle+2(k_g^1)'k_g^2\langle \mathbf{D},\mathbf{R}_3\rangle.
\end{aligned}$$

Since $\det\mathcal{Q}\neq 0$, the unknowns in Y can be computed. Thus, the third curvature can be obtained by $k_3=\frac{\langle \mathbf{b}_2,\alpha^{(4)}\rangle}{k_1k_2}$.

4 A Special Ruled Hypersurface in $\mathbb{E}^4$

In this section, we consider two special vector fields along a Frenet curve in $\mathbb{E}^4$, and we construct a developable ruled hypersurface associated with these new vector fields. We show that the base curve is always a line of curvature on the obtained hypersurface.

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ with non-zero curvatures k_1, k_2, k_3, where $\frac{k_2}{k_3}$ is constant, and let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ denote its Frenet frame. Let us now introduce the following unit vector fields defined along β:

$$\mathcal{H}_1(s) = \mathbf{b_1}(s), \quad \mathcal{H}_2(s) = \frac{1}{\sqrt{k_2^2(s) + k_3^2(s)}} \left\{ k_2(s)\mathbf{n}(s) - k_3(s)\mathbf{b}_2(s) \right\}.$$

Since $\{\mathcal{H}_1, \mathcal{H}_2\}$ is orthonormal, we define the ruled hypersurface

$$\psi(s, u, v) = \beta(s) + u\mathcal{H}_1(s) + v\mathcal{H}_2(s), \quad s \in I, \quad u, v \in \mathbb{R},$$

and call it the $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface of $\beta(s)$.

Theorem 1 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with nonzero curvatures k_1, k_2, k_3, where $\frac{k_2}{k_3}$ is constant. Then*

i) (s_0, u_0, v_0) is a singular point of the $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface of $\beta(s)$ if and only if

$$v_0 = \frac{\sqrt{k_2^2 + k_3^2}}{k_1 k_2}(s_0).$$

ii) β is a line of curvature on the $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface of $\beta(s)$.

Proof. i) Since the partial derivatives of $\psi(s, u, v)$ are

$$\psi_s = \left(1 - \frac{vk_1k_2}{\sqrt{k_2^2 + k_3^2}}\right)\mathbf{t}(s) - uk_2\mathbf{n}(s) + v\sqrt{k_2^2 + k_3^2}\mathbf{b}_1(s) + uk_3\mathbf{b}_2(s),$$

$$\psi_u = \mathbf{b}_1(s), \quad \psi_v = \frac{1}{\sqrt{k_2^2(s) + k_3^2(s)}} \left\{ k_2(s)\mathbf{n}(s) - k_3(s)\mathbf{b}_2(s) \right\},$$

we have

$$\psi_s \otimes \psi_u \otimes \psi_v = \frac{k_3}{\sqrt{k_2^2 + k_3^2}} \left(1 - \frac{vk_1k_2}{\sqrt{k_2^2 + k_3^2}}\right)\mathbf{n}(s) + \frac{k_2}{\sqrt{k_2^2 + k_3^2}} \left(1 - \frac{vk_1k_2}{\sqrt{k_2^2 + k_3^2}}\right)\mathbf{b}_2(s).$$

Then, one can see that $\psi_s \otimes \psi_u \otimes \psi_v$ vanishes if and only if $1 - \frac{vk_1k_2}{\sqrt{k_2^2+k_3^2}} = 0$, i.e. (s_0, u_0, v_0) is a singular point of the $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface if and only if

$$v_0 = \frac{\sqrt{k_2^2 + k_3^2}}{k_1 k_2}(s_0).$$

ii) We have $u = v = 0$ for the points of β. Thus, $\beta(s)$ is a regular point of $\psi(s, u, v)$ for all $s \in I$. Then, the unit normal vector field of the hypersurface $\psi(s, u, v)$ restricted to the curve β is

$$\mathbf{N}_\beta(s) = \frac{k_3}{\sqrt{k_2^2 + k_3^2}}\mathbf{n}(s) + \frac{k_2}{\sqrt{k_2^2 + k_3^2}}\mathbf{b}_2(s).$$

Differentiating the unit normal vector field $\mathbf{N}_\beta$ with respect to s, we obtain

$$\mathbf{N}'_\beta(s) = -\frac{k_1k_3}{\sqrt{k_2^2 + k_3^2}}\mathbf{t}(s)$$

which shows that β is a line of curvature on the $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface. ■

Theorem 2 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with arc-length parameter s. The $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface associated with β is developable.*

Proof. We have

$$\operatorname{rank}[\mathbf{t}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1', \mathcal{H}_2'] = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{k_2}{\sqrt{k_2^2+k_3^2}} & 0 & -\frac{k_3}{\sqrt{k_2^2+k_3^2}} \\ 0 & -k_2 & 0 & k_3 \\ \frac{-k_1k_2}{\sqrt{k_2^2+k_3^2}} & 0 & \sqrt{k_2^2+k_3^2} & 0 \end{bmatrix} = 3.$$

Then, according to (6), the $\mathcal{H}_1\mathcal{H}_2$-ruled hypersurface associated with β is developable. ■

5 An Illustrative Example

Let us consider the hypersurface M with its parametric equation

$$\begin{aligned} \mathbf{R}(u_1, u_2, u_3) = & \left(\left(1 - \frac{\sqrt{2}}{3}u_2\right)\cos\left(\frac{\sqrt{2}}{\sqrt{3}}u_1\right) + \frac{1}{\sqrt{3}}u_3 \sin\left(\frac{\sqrt{2}}{\sqrt{3}}u_1\right),\right. \\ & \left(1 - \frac{\sqrt{2}}{3}u_2\right)\sin\left(\frac{\sqrt{2}}{\sqrt{3}}u_1\right) - \frac{1}{\sqrt{3}}u_3 \cos\left(\frac{\sqrt{2}}{\sqrt{3}}u_1\right), \\ & \left(1 + \frac{\sqrt{2}}{3}u_2\right)\cos\left(\frac{1}{\sqrt{3}}u_1\right) - \frac{\sqrt{2}}{\sqrt{3}}u_3 \sin\left(\frac{1}{\sqrt{3}}u_1\right), \\ & \left.\left(1 + \frac{\sqrt{2}}{3}u_2\right)\sin\left(\frac{1}{\sqrt{3}}u_1\right) + \frac{\sqrt{2}}{\sqrt{3}}u_3 \cos\left(\frac{1}{\sqrt{3}}u_1\right)\right), \end{aligned} \tag{30}$$

defined over

$$\mathcal{B} = \left\{(u_1, u_2, u_3) \in \mathbb{E}^3 \,|\, u_1, u_3 \in \mathbb{R}, u_2 \neq \frac{9}{\sqrt{2}}\right\}.$$

Then the unit normal vector field of M is given by

$$\mathbf{N} = \frac{\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3}{\|\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3\|} = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\sqrt{2}}{\sqrt{3}}u_1\right), \sin\left(\frac{\sqrt{2}}{\sqrt{3}}u_1\right), \cos\left(\frac{1}{\sqrt{3}}u_1\right), \sin\left(\frac{1}{\sqrt{3}}u_1\right)\right).$$

In this case, the first fundamental form coefficients of M are

$$g_{11} = 1 - \frac{2\sqrt{2}}{9}u_2 + \frac{2}{9}u_2^2 + \frac{4}{9}u_3^2, \quad g_{12} = -\frac{4}{9}u_3, \quad g_{13} = \frac{4}{9}u_2, \quad g_{22} = \frac{4}{9}, \quad g_{23} = 0, \quad g_{33} = 1,$$

and the second fundamental form coefficients of M are

$$h_{11} = \frac{u_2}{9} - \frac{1}{\sqrt{2}}, \quad h_{12} = h_{13} = h_{22} = h_{23} = h_{33} = 0.$$

Thus, the normal curvature in the direction (λ, μ) is obtained from (7) as

$$k_n(\lambda, \mu) = \frac{\frac{u_2}{9} - \frac{1}{\sqrt{2}}}{1 - \frac{2\sqrt{2}}{9}u_2 + \frac{2}{9}u_2^2 + \frac{4}{9}u_3^2 - \frac{8}{9}u_3\lambda + \frac{8}{9}u_2\mu + \frac{4}{9}\lambda^2 + \mu^2}. \tag{31}$$

By taking the partial derivatives of k_n with respect to λ and μ, for the extremal values of k_n we obtain $\lambda = u_3$ and $\mu = -\frac{4}{9}u_2$. Substituting these results into (31) yields

$$k_n(\lambda,\mu) = \frac{1}{2\left(\frac{u_2}{9} - \frac{1}{\sqrt{2}}\right)} = \frac{1}{2\omega},$$

where $\omega = \frac{u_2}{9} - \frac{1}{\sqrt{2}}$. Then the coefficient matrix of the system (8) becomes

$$\mathcal{A} = \begin{pmatrix} \frac{-1}{\omega}\left(\frac{8u_2^2}{81} + \frac{2u_3^2}{9}\right) & \frac{2}{9\omega}u_3 & -\frac{2}{9\omega}u_2 \\ \frac{2}{9\omega}u_3 & -\frac{2}{9\omega} & 0 \\ -\frac{2}{9\omega}u_2 & 0 & -\frac{1}{2\omega} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ \frac{2}{9\omega}u_3 & -\frac{2}{9\omega} & 0 \\ -\frac{2}{9\omega}u_2 & 0 & -\frac{1}{2\omega} \end{pmatrix}$$

which means $rank(\mathcal{A}) = 2$. We have

$$\left(h_{22} - k_n g_{22}\right)\left(h_{33} - k_n g_{33}\right) - \left(h_{23} - k_n g_{23}\right)^2 = \frac{1}{9\omega^2} \neq 0.$$

If we denote

$$a_4 = \left(h_{22} - k_n g_{22}\right)\left(h_{33} - k_n g_{33}\right) - \left(h_{23} - k_n g_{23}\right)^2 = \frac{1}{9\omega^2},$$

$$a_5 = \left(h_{13} - k_n g_{13}\right)\left(h_{23} - k_n g_{23}\right) - \left(h_{12} - k_n g_{12}\right)\left(h_{33} - k_n g_{33}\right) = \frac{u_3}{9\omega^2},$$

and consider

$$a_1 = \left(h_{12} - k_n g_{12}\right)\left(h_{23} - k_n g_{23}\right) - \left(h_{22} - k_n g_{22}\right)\left(h_{13} - k_n g_{13}\right) = -\frac{4u_2}{81\omega^2},$$

we obtain the system corresponding to (13) as

$$\begin{cases} u_1' = \mp\frac{a_4}{\sqrt{g_{11}a_4^2+2g_{12}a_4a_5+2g_{13}a_1a_4+2g_{23}a_1a_5+g_{22}a_5^2+g_{33}a_1^2}} = \mp\frac{1}{\sqrt{2}\,\omega}, \\ u_2' = \mp\frac{a_5}{\sqrt{g_{11}a_4^2+2g_{12}a_4a_5+2g_{13}a_1a_4+2g_{23}a_1a_5+g_{22}a_5^2+g_{33}a_1^2}} = \mp\frac{u_3}{\sqrt{2}\,\omega}, \\ u_3' = \mp\frac{a_1}{\sqrt{g_{11}a_4^2+2g_{12}a_4a_5+2g_{13}a_1a_4+2g_{23}a_1a_5+g_{22}a_5^2+g_{33}a_1^2}} = \pm\frac{4u_2}{9\sqrt{2}\,\omega}. \end{cases} \tag{32}$$

We should note that when the minus(plus) sign is used in u_1' and u_2', the plus(minus) sign must be used in u_3'. Thus, solving (32) with the initial conditions $u_1(0) = 0$, $u_2(0) = 0$, $u_3(0) = 0$ and substituting the solutions into (30) yields a line of curvature β on M passing through the initial point $\beta(0) = \mathbf{R}\left(u_1(0), u_2(0), u_3(0)\right) = (1,0,1,0) = P$. If we choose the signs as $-$, $-$, $+$ in (32), respectively, we have $u_1' = 1$, $u_2' = u_3' = 0$ at P, i.e. the tangent vector at P of β is obtained as

$$\mathbf{T} = \mathbf{R}_1 = \left(0, \frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right).$$

We also have $\Omega_1 = \left(-\frac{2}{3}, 0, -\frac{1}{3}, 0\right)$, $\rho_1 = \rho_2 = 0$ at P. If we solve (32) with the given initial point via the ode45 function of MATLAB R2014a and substitute the results into (30), we obtain the line of curvature. This line of curvature has been projected into the hyperplane $w = 0$ and its projection is displayed in Figure 1 together with the projections of the parameter surfaces of the given hypersurface M.

First curvature

To obtain the first curvature of the line of curvature above, we need to substitute the known results into (22). However, since

$$\left(h_{22} - k_n g_{22}\right)\left(h_{33} - k_n g_{33}\right) - \left(h_{23} - k_n g_{23}\right)^2 = \frac{1}{9\omega^2} \neq 0,$$

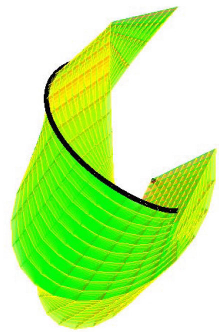

Figure 1: Line of curvature of the hypersurface M passing through the initial point $P = (1, 0, 1, 0)$

we must use the equations obtained by differentiating the second and third equations of (8) instead of the fourth and fifth equations of (22). In this case, the system corresponding to (22) at P is obtained as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{9} & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & \frac{2\sqrt{2}}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1'' \\ u_2'' \\ u_3'' \\ k_g^1 \langle \mathbf{E}, \mathbf{R}_2 \rangle \\ k_g^1 \langle \mathbf{E}, \mathbf{R}_3 \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\sqrt{2}}{9} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the solution of this system is obtained as

$$u_1'' = u_2'' = u_3'' = 0, \quad k_g^1 \langle \mathbf{E}, \mathbf{R}_2 \rangle = \frac{\sqrt{2}}{9}, \quad k_g^1 \langle \mathbf{E}, \mathbf{R}_3 \rangle = 0$$

which yields

$$\beta''(0) = \Omega_1 = \mathbf{R}_{11} = \left(-\frac{2}{3}, 0, -\frac{1}{3}, 0\right), \quad \mathbf{n}(0) = \frac{\beta''(0)}{\|\beta''(0)\|} = (\frac{-2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}, 0),$$

$$\Omega_2 = \mathbf{R}_{111} = \left(0, -\frac{2\sqrt{6}}{9}, 0, -\frac{\sqrt{3}}{9}\right), \quad \rho_3 = \rho_4 = 0.$$

Since $\mathbf{N}(P) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$, we find

$$\mathbf{E}(P) = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \quad \mathbf{D}(P) = \left(0, \frac{\sqrt{3}}{3}, 0, -\frac{\sqrt{6}}{3}\right).$$

Thus, we obtain

$$k_g^1(0) = \frac{\sqrt{2}}{6}, \quad k_1(0) = \frac{\sqrt{5}}{3}.$$

Second curvature

Similarly, for the second curvature of β at P, we obtain the system corresponding to the system $\mathcal{Q}X = \mathcal{S}$ as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{9} & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{\sqrt{2}}{6} \\ 0 & \frac{2\sqrt{2}}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1''' \\ u_2''' \\ u_3''' \\ \left(k_g^1\right)' \\ k_g^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{2}}{9} \\ 0 \\ 0 \end{pmatrix}$$

whose solution is

$$u_1''' = u_2''' = u_3''' = 0, \quad \left(k_g^1\right)'(0) = 0, \quad k_g^2(0) = -\frac{2}{3}.$$

Thus, we obtain

$$\beta'''(0) = \Omega_2 = \left(0, -\frac{2\sqrt{6}}{9}, 0, -\frac{\sqrt{3}}{9}\right), \quad \Omega_3 = \mathbf{R}_{1111} = \left(\frac{4}{9}, 0, \frac{1}{9}, 0\right).$$

Also, we find

$$\mathbf{b}_2(0) = \left(-\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}, 0\right), \quad \mathbf{b}_1(0) = \left(0, \frac{1}{\sqrt{3}}, 0, -\frac{\sqrt{6}}{3}\right), \quad k_1'(0) = 0, \quad k_n'(0) = 0.$$

Hence, the second curvature of β at P is obtained as

$$k_2(0) = \frac{\langle \mathbf{b}_1(0), \beta'''(0) \rangle}{k_1(0)} = -\frac{1}{3}\sqrt{\frac{2}{5}}.$$

Third curvature

Similarly, for the third curvature of β at P, we need to solve the following system which corresponds to the system $\mathcal{Q}Y = \mathcal{W}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{9} & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{\sqrt{2}}{6} \\ 0 & \frac{2\sqrt{2}}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1^{(4)} \\ u_2^{(4)} \\ u_3^{(4)} \\ \left(k_g^1\right)'' \\ \left(k_g^2\right)' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we solve this system, we obtain $u_1^{(4)} = u_2^{(4)} = u_3^{(4)} = 0$, $\left(k_g^2\right)' = \left(k_g^1\right)'' = 0$, and $\beta^{(4)}(0) = \Omega_3 = \left(\frac{4}{9}, 0, \frac{1}{9}, 0\right)$. Then, we find the third curvature of β as

$$k_3(0) = \frac{\left\langle \mathbf{b}_2(0), \beta^{(4)}(0) \right\rangle}{k_1(0)k_2(0)} = \frac{\sqrt{10}}{5}.$$

Remark 1 *Let us reconsider the system (32) by choosing the signs $-$, $-$, $+$, respectively:*

$$\begin{cases} u_1' = -\frac{1}{\sqrt{2}\,\omega}, \\ u_2' = -\frac{u_3}{\sqrt{2}\,\omega}, \\ u_3' = \frac{4u_2}{9\sqrt{2}\,\omega}. \end{cases} \tag{33}$$

If we multiply the second and third equations of (33) by $\frac{4}{9}u_2$ and u_3, respectively, and sum both sides of those equations, we have

$$\frac{4}{9}u_2u_2' + u_3u_3' = 0.$$

Integration of the above equation yields $\frac{4}{9}u_2^2+u_3^2=c$*, where* c *is a nonnegative constant. The initial conditions* $u_1(0)=0$*,* $u_2(0)=0$*,* $u_3(0)=0$ *ensure that* $c=0$ *which gives us* $u_2(s)=0$*,* $u_3(s)=0$*. If we substitute these results into the first equation of (33), we get* $u_1'=1$*, i.e.* $u_1(s)=s$*. Thus, the line of curvature* β *on* M *passing through the initial point*

$$\beta(0)=\mathbf{R}\left(u_1(0),u_2(0),u_3(0)\right)=(1,0,1,0)$$

is given by

$$\beta(s)=\mathbf{R}\left(s,0,0\right)=\left(\cos\left(\frac{\sqrt{2}}{\sqrt{3}}s\right),\sin\left(\frac{\sqrt{2}}{\sqrt{3}}s\right),\cos\left(\frac{1}{\sqrt{3}}s\right),\sin\left(\frac{1}{\sqrt{3}}s\right)\right).$$

The curvatures and the Frenet vector fields of β *were given in [6]. Comparison of the results reveals that the present method is accurate enough. The minus sign arising in the second curvature is due to our first binormal vector which is obtained in the opposite direction to the one given in [6].*

6 Conclusion

We developed a method to obtain the lines of curvature on parametric hypersurfaces in Euclidean 4-space. We showed that, even if such lines cannot be obtained analytically in general, it is possible to compute all curvatures of a Frenet line of curvature by using the extended Darboux frame along the curve. We also constructed a developable ruled hypersurface whose base curve is always a line of curvature. We verified the applicability of our techique by providing an example.

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