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Relations For Moments Of Log-Kumaraswamy Distribution Based On Generalized Order Statistics And Associated Inferences*

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Abstract

This article addresses the problem of exact and explicit expressions for single and product moments for log-Kumaraswamy distribution based on generalized order statistics. Results are further deduced for order statistics and records, and then the distribution is characterized through truncated moments. Finally, the maximum likelihood estimator (MLE), mean square error and biasedness for the parameters of log-Kumaraswamy distribution is computed through simulation study.

1 Introduction

The concept of generalized order statistics (*gos*) was introduced by [1] as a general framework for models of order statistics such as order statistics, k-th upper record values, upper record values, progressively Type-II censored order statistics, sequential order statistics and Pfeifer's records. Let $X_1, X_2, ...$ be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with distribution function (*df*) $F(x)$ and probability density function (*pdf*) $f(x)$. Assuming that $k > 0$, $n \in N$ and $m \in \mathbb{R}$. The random variables $X(1, n, m, k)$, $X(2, n, m, k), \ldots, X(n, n, m, k)$, are said to be generalized order statistics (*gos*) from continuous population with the *df* $F(x)$, if their joint *pdf* can be written as

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n),$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$ of $\mathbb{R}^n$. Choosing the parameters appropriately, various important models related to ordered random variables like k-th record values ($m_1 = m_2 = \cdots = m_{n-1} = -1$, $k \in N$, i.e. $\gamma_i = k$), Pfeifer's record values ($\gamma_i = \beta_i$, β_1, β_2, ..., $\beta_n > 0$), order statistics ($m = 0$, $k = 1$, i.e. $\gamma_i = n - i + 1$), order statistics with non-integral sample size ($\gamma_i = (\beta - i + 1)$; $\beta > 0$), progressively type-II censored order statistics

$$m_i = R_i, \quad n = m_0 + \sum_{j=1}^{m_0} R_j, R_j \in N_0 \quad \text{and} \quad \gamma_i = n - \sum_{v=1}^{j-1} R_v - j + 1, \quad 1 \leq j \leq m_0,$$

where m_0 is fixed number of failure units to be observed and sequential order statistics ($\gamma_i = (n-i+1)\beta_i$, β_1, $\beta_2, ..., \beta_n > 0$) can be obtained as particular cases of *gos*.

When $m_1 = m_2 = \cdots = m_{n-1} = m$, in this case, the *pdf* of r-th *gos* $X(r, n, m, k)$ is given by [1]

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)), \quad -\infty \leq x \leq \infty \tag{1}$$

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and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given by

$$f_{X(r,n,m,k).X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} \times[\bar{F}(y)]^{\gamma_s-1} f(x)f(y), \quad -\infty \le x < y \le \infty, \tag{2}$$

where

$$\bar{F}(x) = 1 - F(x),$$

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad r = 1,2,\ldots,n,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & , \quad m \ne -1, \\ -\ln(1-x), & , \quad m = -1, \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \quad x \in [0,1).$$

The moment based on *gos* has always been the topic of interest among researchers. Various approaches are available in literature. Several researchers have utilized the concept of *gos* to obtain the moments of various continuous distributions. References have been made to several works, including [2], [5], [7], [8], [14], [15], [17], [18], [19], [20], [23], [24], and among others. Recently, in [16], it is established the relation of extended exponential distribution based on *gos*. In [13], the authors derived some useful moment expressions and investigated the resulting estimation properties of Lindley distribution based on *gos*. In [21], authors established the relations for moments of generalized inverse Lindley distribution based on *gos*. Additionally, they demonstrated various estimation properties. In [12], authors have highlighted the moments and entropy of the Kumaraswamy power function distribution based on *gos*.

In this paper, we mainly focus on study of *gos* arising from the log-Kumaraswamy distribution. The motivation of the paper is in two folds: first is to derive the exact expression for single moments, product moments, and characterization using the technique of truncated moment, from the log-Kumaraswamy distribution, and second is to compute the maximum likelihood estimate, mean square error and biaseness from the log-Kumaraswamy distribution based on order statistics. [10] introduced log-exponentiated Kumaraswamy (log-EK) distribution. Note that the two parameter log-Kumaraswamy distribution is a particular member of the log-exponentiated Kumaraswamy (log-EK) distribution. Two parameter log-Kumaraswamy distribution has the probability density function (*pdf*) and corresponding cumulative distribution function (*cdf*) are given here

$$f(x) = \alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}\left[1-\left(1-e^{-x}\right)^{\alpha}\right]^{\beta-1}, \quad x>0,\ \alpha>0,\ \beta>0, \tag{3}$$

with the corresponding distribution function (*df*)

$$F(x) = 1 - \left[1-\left(1-e^{-x}\right)^{\alpha}\right]^{\beta}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{4}$$

The hazard rate function is given by

$$h(x) = \frac{\alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}}{[1-(1-e^{-x})^{\alpha}]}.$$

Note that $f(x)$ and $F(x)$ satisfy the relation

$$\alpha\beta\bar{F}(x) = [e^x(1-e^{-x})^{1-\alpha} - e^x(1-e^{-x})]f(x).$$

The basic tools for studying the reliability and ageing characteristics of the system are hazard rate and mean residual lifetime. Both deals with the residual lifetime of the system. The hazard rate gives the rate

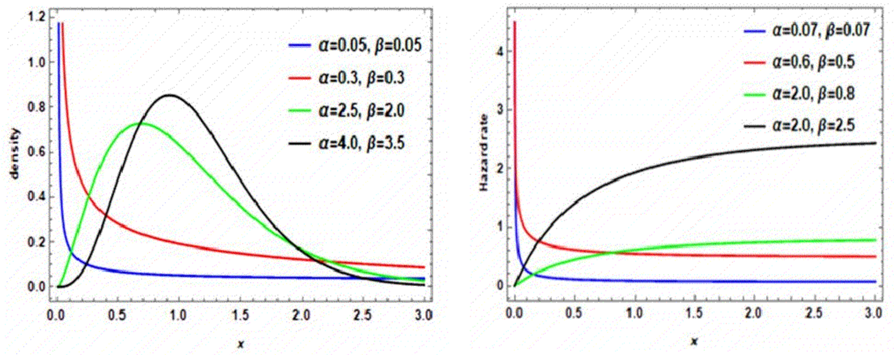

Figure 1: PDF and HF of the log-kumaraswamy distribution.

of failure of the system immediately after time x and mean residual lifetime measure the expected value of the remaining lifetime of the system. Here the behavior of the *pdf* and hazard rate function for some combination of the values of the model parameter are shown in given figures respectively.

Gauss hypergeometric function is defined as

$$ {}_2F_1(a,b;c,x) = \frac{\sum_{k=0}^{\infty}(a)_k(b)_k x^k}{(c)_k k!}, $$

where, $c \neq 0, -1, -2, \ldots$. It converges if one of the following conditions hold:

- $|x| < 1$;
- $|x| = 1, \ \mathrm{Re}(c-a-b) > 0$.

2 Single Moments of Log-Kumaraswamy Distribution

In this section, we obtain exact and explicit expressions for single moment of *gos* from log-Kumaraswamy distribution in the form of incomplete beta function. Further, by putting specific values of m and k, we obtained the moments for order statistics and upper records. We shall first establish the exact expression for $E[X^j(r-1,n,m,k)]$.

For $m \neq -1$ from (2), we have

$$ \mu^j(r,n,m,k) = \frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^j\big(\bar{F}(x)\big)^{\gamma_r-1} f(x) g_m^{r-1}\big(F(x)\big)dx. \tag{5} $$

On expanding $g_m^{r-1}\big(F(x)\big) = \big(\frac{1}{m+1}(1-\bar{F}(x)^{m+1})\big)^{r-1}$ in (5), we get

$$ \mu^j(r,n,m,k) = A\int_0^{\infty} x^j\left[\bar{F}(x)\right]^{\gamma_{r-u}-1} f(x)dx, $$

$$ \mu^j(r,n,m,k) = A\beta\int_0^{1}[-\ln(1-(1-t)^{1/\alpha})]^j t^{\beta\gamma_{r-u}-1}dt, \tag{6} $$

$$ [-\ln(1-t)]^j = \left(\sum_{w=1}^{\infty}\frac{t^w}{w}\right)^j = \sum_{w=0}^{\infty} Z_w(j)t^{j+w}, \ |t|<1, \tag{7} $$

where

$$A = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}, \quad t = \left[\bar{F}(x)\right]^{1/\beta}$$

and $Z_w(j)$ is the coefficient of t^{j+w} in the above expansion (see [3, p-44]).

On using the logarithmic expansion (7) in (6), we have

$$\mu^j(r,n,m,k) = A\beta \sum_{p=0}^{\infty} \alpha_p(j) \int_0^1 t^{\beta\gamma_{r-u}-1}(1-t)^{\frac{(p+j)}{\alpha}} dt.$$

Now using the relation $\int_0^1 x^{a-1}(1-x)^{b-1}dx = Beta(a,b)$, in the above equation, we have

$$\mu^j(r,n,m,k) = \frac{\beta C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{p=0}^{\infty}\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \alpha_p(j) Beta(\beta\gamma_{r-u}, (p+j)/\alpha + 1). \tag{8}$$

In case of upper records, the r-th single moment from log-Kumaraswamy distribution is given by

$$\mu^j(r,n,-1,k) = \frac{\alpha(\beta k)^r}{(r-1)!} \int_0^{\infty} x^j[-ln(1-(1-e^{-x})^{\alpha})]^{r-1}[1-(1-e^{-x})^{\alpha}]^{\beta k-1} e^{-x}(1-e^{-x})^{\alpha-1}dx. \tag{9}$$

Let $w = (1-e^{-x})^{\alpha}$ in (9). We get

$$\mu^j(r,n,-1,k) = \frac{(\beta k)^r}{(r-1)!} \int_0^1 [-ln(1-w^{1/\alpha})]^j[-ln(1-w)]^{n-1}(1-w)^{\beta k-1}dw.$$

Now using (7) in the above equation and simplifying appropriately, we get

$$\mu^j(r,n,-1,k) = \frac{(\beta k)^r}{(r-1)!} \sum_{p=0}^{\infty} \alpha_p(j)(-1)^{r-1} \int_0^1 w^{(p+j)/\alpha}[ln(1-w)]^{r-1}(1-w)^{\beta k-1}dw.$$

Using the relation

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}[ln^r(a-x)]dx = a^{\alpha-1}\frac{d^r}{d\beta^r}[a^{\beta}\beta(\alpha,\beta)]$$

([22, p-502]), we obtained the result given in (10) and the computation of $\frac{d^r}{d\beta^r}[a^{\beta}\beta(\alpha,\beta)]$ can further be simplified by using the recurrence relation

$$\frac{d^r}{d\beta^r}\beta(\alpha,\beta) = \sum_{k=0}^{r-1}[\psi^{(r-k-1)}(\beta) - \psi^{(r-k-1)}(\alpha+\beta)]\frac{d^k}{d\beta^k}\beta(\alpha,\beta),$$

where $\psi^{(r-k-1)}(x)$ is the k-th derivative of diagamma function given by

$$\psi(x) = \frac{dlog\Gamma x}{dx} = \frac{\Gamma' x}{\Gamma x}, \quad x > 0$$

where $\Gamma(.)$ is the gamma function.

$$\mu^j(r,n,-1,k) = \frac{(\beta k)^r}{(r-1)!} \sum_{p=0}^{\infty} \alpha_p(j)(-1)^{r-1}\frac{d^{r-1}}{d\beta^{r-1}}\beta((j+p)/\alpha+1, \beta k). \tag{10}$$

Remark 1 *When $m = 0$, $k = 1$ in (8), the single moment of order statistics from log-Kumaraswamy distribution is given by*

$$\mu^j_{r:n} = \beta C_{r:n} \sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\alpha_p(j)Beta(\beta(n-r+u+1), (p+j)/\alpha+1).$$

Remark 2 *Putting $k = 1$ in (10), we deduce the explicit expression for the upper record from log-Kumaraswamy distribution in the form*

$$\mu^j(r,n,-1,1) = \frac{\beta^r}{(r-1)!}\sum_{p=0}^{\infty}\alpha_p(j)(-1)^{r-1}\frac{d^{r-1}}{d\beta^{r-1}}\beta((j+p)/\alpha+1,\beta).$$

3 Numerical Computation

We have computed the first four moments of aforesaid distribution based on order statistics for different arbitrary values of parameters.

Table 1. First moments of order statistics

n	r	$\alpha=1.5,\ \beta=3$				$\alpha=1.5,\ \beta=4$			
		$E(X_{r:n})$	$E(X^2_{r:n})$	$E(X^3_{r:n})$	$E(X^4_{r:n})$	$E(X_{r:n})$	$E(X^2_{r:n})$	$E(X^3_{r:n})$	$E(X^4_{r:n})$
1	1	0.52942	0.44941	0.52245	0.77061	0.42472	0.28564	0.26068	0.299645
2	1	0.31328	0.15307	0.10032	0.08206	0.25341	0.09923	0.05176	0.03350
	2	0.74557	0.74575	0.94457	1.45916	0.59603	0.47205	0.46959	0.56578
3	1	0.23253	0.08327	0.03962	0.02333	0.18882	0.05449	0.02076	0.00975
	2	0.47476	0.29265	0.22174	0.19952	0.38260	0.18873	0.11376	0.08101
	3	0.88097	0.9723	1.30599	2.08899	0.70274	0.61371	0.64751	0.80817
4	1	0.18883	0.05449	0.02076	0.00975	0.15369	0.03586	0.01099	0.004135
	2	0.36366	0.16961	0.09616	0.06408	0.294235	0.11038	0.05009	0.02660
	3	0.58587	0.41570	0.34733	0.33496	0.47097	0.26708	0.17743	0.13542
	4	0.97934	1.15783	1.62554	2.67366	0.78000	0.72925	0.80419	1.03243
5	1	0.16092	0.03937	0.01267	0.00500	0.13119	0.02601	0.00675	0.00214
	2	0.30043	0.11497	0.05316	0.02873	0.24369	0.07526	0.02796	0.01209
	3	0.45851	0.25157	0.16066	0.11710	0.37005	0.16307	0.08329	0.04836
	4	0.67078	0.52512	0.47178	0.48020	0.53824	0.33642	0.24019	0.19345
	5	1.05648	1.31601	1.91398	3.22202	0.84045	0.82746	0.94520	1.24217
n	r	$\alpha=2.5,\ \beta=3.5$				$\alpha=2.5,\ \beta=4$			
		$E(X_{r:n})$	$E(X^2_{r:n})$	$E(X^3_{r:n})$	$E(X^4_{r:n})$	$E(X_{r:n})$	$E(X^2_{r:n})$	$E(X^3_{r:n})$	$E(X^4_{r:n})$
1	1	0.77742	0.80234	1.02629	1.56358	0.72175	0.68645	0.80420	1.1162
2	1	0.53520	0.36782	0.30471	0.29348	0.49964	0.31893	0.24435	0.21684
	2	1.01966	1.23686	1.74786	2.83369	0.94385	1.05396	1.36405	2.01556
3	1	0.43554	0.24007	0.15759	0.11900	2.01556	0.20943	0.12769	0.08929
	2	0.73451	0.62330	0.59895	0.64243	0.68365	0.53793	0.47766	0.47192
	3	1.16223	1.54364	2.32232	3.92932	1.07396	1.31198	1.80724	2.78738
4	1	0.37798	0.17927	0.10053	0.06445	0.35431	0.15695	0.08196	0.04880
	2	0.60820	0.42250	0.32879	0.28268	0.56760	0.36685	0.26488	0.21075
	3	0.86082	0.82410	0.86911	1.00219	0.79970	0.70902	0.69045	0.73309
	4	1.2627	1.78349	2.80672	4.90503	1.16538	1.51297	2.17951	3.47215
5	1	0.33938	0.14368	0.07157	0.04058	0.31847	0.12611	0.05860	0.03093
	2	0.53238	0.32162	0.21635	0.15988	0.49769	0.28032	0.17540	0.12029
	3	0.72193	0.57381	0.49744	0.46687	0.67248	0.49665	0.39910	0.34644
	4	0.95342	0.99096	1.11689	1.35907	0.88451	0.85059	0.88468	0.99086
	5	1.34002	1.98162	3.22918	5.79151	1.23559	1.67856	2.50322	4.09247

4 Product Moments of log-Kumaraswamy Distribution

In this section, we obtain the exact and explicit expression for product moments of generalized order statistics from log-Kumaraswamy distribution in form of hypergeometric function.

Theorem 1 *For the distribution given in (4) and for $1 \le r < s \le n$, $k = 1, 2, ..., m \ne -1$,*

$$\mu^{i,j}_{r,s,n,m,k} = \frac{\beta C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}\gamma_{s-v}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{\lambda=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{u+v}\alpha_\lambda(j)\alpha_p(i)$$
$$\times \binom{r-1}{u}\binom{s-r-1}{v} B(\beta\gamma_{r-u}, ((i+p)/\alpha)+1)$$
$$\times {}_3F_2(\beta\gamma_{s-v}, -(j+\lambda)/\alpha; \beta\gamma_{r-u}, \beta\gamma_{s-v}+1; \beta\gamma_{r-u}+((i+p)/\alpha)+1; 1). \quad (11)$$

Proof. In view (2) and (4), the product moment for r-th and s-th *gos* is given by

$$\mu^{i,j}_{r,s,n,m,k} = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!} \int_0^\infty \int_x^\infty x^i y^j [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y) dy dx. \quad (12)$$

On expanding $[h_m(F(y)) - h_m(F(x))]^{s-r-1}$ and $g_m^{r-1}(F(x))$ binomially in (12), we get

$$\mu^{i,j}_{r,s,n,m,k} = \frac{C_{s-1}}{(r-1)!\ (s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \int_0^\infty \int_x^\infty x^i y^j [\bar{F}(x)]^{(s-r-v+u)(m+1)-1} f(x) [\bar{F}(y)]^{\gamma_{s-v}-1} f(y)\, dy dx. \quad (13)$$

Consider

$$I(x) = \int_x^\infty y^j\ [\bar{F}(y)]^{\gamma_{s-v}-1} f(y)\, dy.$$

Let $w = [\bar{F}(y)]^{1/\beta}$ in $I(x)$ and simplifying accordingly. We have

$$I(x) = \beta \int_0^{[\bar{F}(x)]^{1/\beta}} [-ln(1-(1-w)^{1/\alpha})]^j w^{\beta\gamma_{s-v}-1} dw.$$

After using the relation given in (7), we have

$$I(x) = \beta \sum_{\lambda=0}^{\infty} \alpha_\lambda(j) \int_0^{[\bar{F}(x)]^{1/\beta}} w^{\beta\gamma_{s-v}-1}(1-w)^{(j+\lambda)/\alpha} dw.$$

Now using incomplete Beta function

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1} dt = \frac{x^p}{p} {}_2F_1(p, 1-q; p+1; x),$$

we have

$$I(x) = \frac{\beta \sum_{\lambda=0}^{\infty} \alpha_\lambda(j) \left[\bar{F}(x)\right]^{\gamma_{s-v}}}{\beta\gamma_{s-v}} \left[{}_2F_1(\beta\gamma_{s-v}, 1-(j+\lambda)/\alpha - 1; \beta\gamma_{s-v}+1; [\bar{F}(x)]^{1/\beta}) \right]. \quad (14)$$

Using (14), (13) can be written

$$\mu^{i,j}_{r,s,n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}\gamma_{s-v}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{\lambda=0}^{\infty}(-1)^{u+v}\alpha_\lambda(j)\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times {}_2F_1(\beta\gamma_{s-v}, (1-(j+\lambda)/\alpha - 1); \beta\gamma_{s-v}+1; [\bar{F}(x)]^{1/\beta}) \int_0^\infty x^i\ [\bar{F}(x)]^{\gamma_{r-u}-1} f(x)\, dx.$$

Let $[\bar{F}(x)]^{1/\beta} = z$ in the above equation and from (7). We have

$$\mu^{i,j}_{r,s,n,m,k} = \frac{\beta C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v}$$
$$\times \frac{\sum_{\lambda=0}^{\infty} \sum_{p=0}^{\infty} \alpha_\lambda(j)\alpha_p(i)}{\gamma_{s-v}} {}_2F_1(\beta\gamma_{s-v}, -(j+\lambda)/\alpha; \beta\gamma_{s-v}+1; z)$$
$$\times \int_0^{\infty} z^{\beta\gamma_{r-u}-1}(1-z)^{(i+p)/\alpha} dz. \tag{15}$$

Now using the relation [4],

$$\int_0^1 u^{a-1}(1-u)^{b-1} {}_2F_1(c,d;e;u)du = B(a,b) {}_3F_2(c,d,a;e;a+b;1).$$

Using above result we get the explicit expression for product moments based on *gos* given in (11). ■

Remark 3 *If $m = 0$ and $k = 1$ in (11), we get the exact expression for the product moment of order statistics from log-Kumaraswamy distribution*

$$\mu^{i,j}_{r,s:n} = \frac{\beta C_{r,s:n}}{(n-s+v+1)} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} \sum_{\lambda=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \alpha_\lambda(j)\alpha_p(i)$$
$$\times B(\beta(n-r+u+1), ((i+p)/\alpha)+1)) \times {}_3F_2(\tau_1, \tau_2, \tau_3; \tau_4; \tau_5; 1).$$

where

$$\tau_1 = \beta(n-s+v+1), \quad \tau_2 = -(j+\lambda)/\alpha, \quad \tau_3 = \beta(n-r+u+1),$$
$$\tau_4 = \beta(n-s+v+1)+1 \quad \text{and} \quad \tau_5 = \beta(n-r+u+1)+(i+p)/\alpha+1.$$

The exact expression for product moment from log-Kumaraswamy distribution based on upper records could not be obtained.

5 Characterization Based on Truncated Moments

Theorem 2 *Suppose an absolutely continuous (with respect to Lebesque measure) random variable X has the df $F(x)$ and pdf $f(x)$ for $0 < x < \frac{1}{\beta\lambda}$. Then*

$$E(X|X \le x) = g(x)\eta(x),$$

where

$$\eta(x) = \frac{f(x)}{F(x)},$$

and

$$g(x) = \frac{-x[1-(1-e^{-x})^\alpha]}{\alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}} - \frac{\int_0^x [1-(1-e^{-x})^\alpha]^\beta du}{\alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}[1-(1-e^{-x})^\alpha]^{\beta-1}},$$

if and only if

$$f(x) = \alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}\left[1-\left(1-e^{-x}\right)^\alpha\right]^{\beta-1}, \quad x > 0, \ \alpha > 0, \ \beta > 0.$$

Proof. In view of [6] and (3), we have

$$E(X|X \leq x) = \frac{\alpha\beta}{F(x)} \int_0^x ue^{-u}(1-e^{-u})^{\alpha-1}\Big[1-\Big(1-e^{-u}\Big)^{\alpha}\Big]^{\beta-1} du. \tag{16}$$

Integrating (16) by parts, treating

$$e^{-u}(1-e^{-u})^{\alpha-1}\Big[1-\Big(1-e^{-u}\Big)^{\alpha}\Big]^{\beta-1}$$

as the part to be integrated and rest of the integrand for differentiation, we get

$$E(X|X \leq x) = \frac{1}{F(x)}\Big[-x\Big[1-\Big(1-e^{-x}\Big)^{\alpha}\Big]^{\beta} + \int_0^x \Big[1-\Big(1-e^{-u}\Big)^{\alpha}\Big]^{\beta} du\Big]. \tag{17}$$

After multiplying and dividing by $f(x)$ in (17), we have the result given (15).

To prove the sufficiency part, we have from (15),

$$\frac{1}{F(x)}\int_0^x uf(u)du = \frac{g(x)f(x)}{F(x)}$$

or

$$\int_0^x uf(u)du = g(x)f(x). \tag{18}$$

Differentiating (18) on both sides with respect to x, we find that

$$xf(x) = g'(x)f(x) + g(x)f'(x).$$

Therefore,

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{x-g'(x)}{g(x)} \\ &= \frac{-\alpha(\beta-1)e^{-x}(1-e^{-x})^{\alpha-1}}{\Big[1-\Big(1-e^{-x}\Big)^{\alpha}\Big]} + \frac{(\alpha-1)e^{-x}}{(1-e^{-x})} - 1, \end{aligned} \tag{19}$$

where

$$g'(x) = x + g(x)\left(\frac{\alpha(\beta-1)e^{-x}(1-e^{-x})^{\alpha-1}}{\Big(1-\Big(1-e^{-x}\Big)^{\alpha}\Big)} - \frac{(\alpha-1)e^{-x}}{(1-e^{-x})} + 1\right).$$

Integrating (19) both sides with respect to x, we get

$$f(x) = ce^{-x}(1-e^{-x})^{\alpha-1}\Big[1-\Big(1-e^{-x}\Big)^{\alpha}\Big]^{\beta-1}, \quad x>0,\ \alpha>0,\ \beta>0.$$

We know that

$$\int_0^\infty f(x)dx = 1.$$

Thus,

$$\frac{1}{c} = \int_0^\infty e^{-x}(1-e^{-x})^{\alpha-1}\Big[1-\Big(1-e^{-x}\Big)^{\alpha}\Big]^{\beta-1} dx = \frac{1}{\alpha\beta}.$$

which proves that

$$f(x) = \alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}\Big[1-\Big(1-e^{-x}\Big)^{\alpha}\Big]^{\beta-1} dx, \quad \alpha>0, \quad \beta>0.$$

■

6 MLE for the Parameters α and β

In this Section, we obtain the expression for MLE of the shape parameters α and β of the log-Kumaraswamy distribution based on *gos*. The problem of estimation of parameters through MLE based on *gos* and *dgos* has always been the topic of interest among researchers. Recently the estimation of parameters of type Burr III and Weibull distributions, Burr model based gos and dgos have been discussed by various authors like [9, 11] and so on.

The likelihood function for log-Kumaraswamy distribution based of *gos* is given by

$$L(\alpha,\beta) = k\alpha^n\beta^n\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n}e^{-x_i}\left(1-e^{x_i}\right)^{\alpha-1}\right)\left(\prod_{i=1}^{n-1}\left[1-\left(1-e^{-x_i}\right)^{\alpha}\right]^{\beta(m_i+1)-1}\right) \times\left[1-\left(1-e^{-x_n}\right)^{\alpha}\right]^{\beta k-1},$$

$$\begin{aligned}\ln L(\alpha,\beta) &= \ln k+n\ln\alpha+n\ln\beta+\sum_{j=1}^{n-1}\ln\gamma_i+\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-x_i}\right)\\ &+(\beta k-1)\ln\left(1-\left(1-e^{-x_n}\right)^{\alpha}\right)+\sum_{i=1}^{n}x_i\\ &+\sum_{i=1}^{n-1}\left[\beta\left(m_i+1\right)-1\right]\ln\left(1-\left(1-e^{-x_i}\right)^{\alpha}\right).\end{aligned} \tag{20}$$

MLE for the parameters α and β are the solutions of the system of linear equations obtained by equating first partial derivative of (20) with respect to α and β to zero. Thus, the MLE $\hat{\beta}$ for β is given by

$$\hat{\beta} = -\frac{n}{H(\alpha,x)},$$

where

$$H(\alpha,x) = k\ln\left(1-\left(1-e^{-x_n}\right)^{\alpha}\right)+\sum_{i=1}^{n-1}\left(m_i+1\right)\ln\left(1-\left(1-e^{-x_i}\right)^{\alpha}\right).$$

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-x_i}\right)-\frac{\alpha\left(\hat{\beta}k-1\right)\left(1-e^{-x_n}\right)^{\alpha-1}}{1-\left(1-e^{-x_n}\right)^{\alpha}}-\sum_{i=1}^{n-1}\frac{\alpha\left[\hat{\beta}\left(m_i+1\right)-1\right]}{1-\left(1-e^{-x_i}\right)^{\alpha}}=0. \tag{21}$$

Since (21) is a non-linear equation, the MLE $\hat{\alpha}$ for α cannot be obtained directly. Therefore, an iteration method (Newton-Raphson) is applied to obtain the MLE of α.

7 Simulation Study

In this section, we have estimated the parameters (α,β) of log-Kumaraswamy distribution by the method of maximum likelihood estimation. The performance of ML estimators is evaluated by bias and Mean Square Error (MSEs) for different sample sizes. All simulations are performed by R software with nlm function. In the simulation process, we simulated 10000 samples and the MLEs, bias, and mean square error (MSE) are calculated based on order statistics. In each replication, a random sample of size $n = 20$, 40, 60, 80 and 100 is drawn with different combinations of $(\alpha,\beta) = (0.5, 0.5)$, $(0.5, 1)$, $(1, 0.5)$, $(1, 1)$, $(2.5, 2.5)$ from the proposed distribution. In Table 2, we have shown the MLEs of parameters (α,β), their biases (Bias) and mean square errors (MSE) from log-Kumaraswamy distribution.

Table 2. MLE, Bias and MSEs of log-Kumaraswamy distribution based on order statistics

α, β	n	$MLE(\alpha)$	$Bias\alpha$	$MSE\alpha$	$MLE(\beta)$	$MSE(\beta)$	$Bias(\beta)$
0.5, 0.5	20	0.59721	0.00945	0.09721	0.61277	0.01272	0.11277
	40	0.57471	0.00558	0.07471	0.57582	0.00575	0.07582
	60	0.55129	0.00263	0.05129	0.56055	0.00367	0.06055
	80	0.54356	0.00189	0.04356	0.54723	0.00223	0.04723
	100	0.52305	0.00053	0.02305	0.52071	0.00043	0.02071
0.5, 1.0	20	0.58260	0.00682	0.08260	1.33975	0.11543	0.33975
	40	0.53213	0.00103	0.03213	1.20611	0.04248	0.20611
	60	0.51593	0.00025	0.01593	1.20583	0.04237	0.20583
	80	0.50625	0.00004	0.00625	1.08724	0.00761	0.08724
	100	0.50583	0.00003	0.00583	1.05060	0.00256	0.05060
1.0, 0.5	20	1.18715	0.03502	0.18715	0.57329	0.00537	0.07329
	40	1.13996	0.01959	0.13996	0.55130	0.00263	0.02549
	60	1.11214	0.01257	0.11214	0.51262	0.00016	0.01262
	80	1.10762	0.01158	0.10762	0.50459	0.00002	0.00459
	100	1.09902	0.00980	0.09902	0.50527	0.00003	0.00527
1.0, 1.0	20	0.18505	0.03424	0.18505	1.20204	0.04082	0.20204
	40	1.12999	0.016899	0.12999	1.12399	0.01537	0.12399
	60	1.09981	0.00996	0.09981	1.05202	0.00271	0.05201
	80	1.07405	0.00548	0.07405	1.01756	0.00031	0.01756
	100	1.04663	0.00217	0.04663	1.00317	0.00001	0.00317
2.5, 2.5	20	3.11422	0.37726	0.61422	2.57645	0.00584	0.07645
	40	2.79327	0.08601	0.29327	2.52466	0.00061	0.02466
	60	2.70809	0.04330	0.20809	2.50949	0.00009	0.00949
	80	2.55142	0.00264	0.05141	2.50484	0.00002	0.00484
	100	2.50572	0.00003	0.0057	2.50155	0.00002	0.00155

8 Conclusion

In this paper, the exact and explicit expressions for single and product moments based on generalized order statistics for aforesaid distribution are derived. These results are deduced into sub models like order statistics, upper records after putting the specific values of parameters. But in case of product moment, only exact moments of order statistics have been obtained; the upper record values could not be derived. Next, these expressions of order statistics are used to calculate the first four moments of aforementioned distribution for different combination of distribution's parameters. Further, by utilizing simulation study, we have estimated parameters of log-Kumaraswamy distribution based on order statistics and it is concluded that the average bias and MSEs for individual parameters of the given distribution fall close to zero when sample size increases, which furnishes the consistency of the estimators.

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