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Turán Type Inequalities For The Polar Derivative Of A Polynomial *

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Abstract

In this paper, some lower bound estimates for the maximum modulus of the polar derivative $D_\alpha \mathscr{P}(z)$ of a polynomial $\mathscr{P}(z)$ of degree n not vanishing in the region $|z| > k$, $k \geq 1$ are established. The obtained results produce inequalities that are sharper than the ones previously known.

1 Introduction

Polynomial inequalities have pertinent applications in all those mathematical models whose solutions lead to problems of evaluating how large or small a derivative of an algebraic polynomial can be in terms of a maximum modulus of a polynomial and bounds of such problems are of some practical importance. since there are no closed formulae for meticulous evaluation of these bounds, whatever is available in literature is in the form of approximations. For practical importance, Mathematicians only designate methods for obtaining approximate bounds. When computed efficiently, these approximate bounds are quite acceptable for the needs of investigators and scientists. There is a desire to look for better and improved bounds. This inclination of getting improved bounds influences our work. In this paper, we refine and generalise some Turán type inequalities for polynomials. To begin with, let $\mathscr{P}(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n and $\mathscr{P}'(z)$ be its derivative, then it was shown by Turán [14] that if $\mathscr{P}(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |\mathscr{P}'(z)| \geq \frac{n}{2} \max_{|z|=1} |\mathscr{P}(z)|. \tag{1}$$

For a polynomial $\mathscr{P}(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, Govil [6] proved that

$$\max_{|z|=1} |\mathscr{P}'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |\mathscr{P}(z)|. \tag{2}$$

The result is best possible and equality in (2) holds for $\mathscr{P}(z) = z^n + k^n$. In literature there exists several extensions and generalizations of inequalities (1) and (2) (for reference see ([1]- [3], [13]). Dubinin [5] obtained the refinement of inequality (1) by introducing some of the coefficients of $\mathscr{P}(z)$. In fact, he proved that if $\mathscr{P}(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |\mathscr{P}'(z)| \geq \frac{1}{2}\left(n + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right) \max_{|z|=1} |\mathscr{P}(z)|. \tag{3}$$

For a polynomial $\mathscr{P}(z)$ of degree n, the polar derivative $D_\alpha \mathscr{P}(z)$ of $\mathscr{P}(z)$ with respect to a complex number α is defined as

$$D_\alpha \mathscr{P}(z) := n\mathscr{P}(z) + (\alpha - z)\mathscr{P}'(z).$$

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$D_\alpha \mathscr{P}(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha \mathscr{P}(z)}{\alpha} = \mathscr{P}'(z). \tag{4}$$

Bernstein type inequalities on complex polynomials have been extended from ordinary derivative to polar derivative of complex polynomials. For reference, see([2], [4], [7], [8]). P. Kumar [9] proved a polar derivative inequality for the class of polynomials having all its zeros in $|z| \le k$, $k \ge 1$, by including the coefficients and considering the modulus of each individual zero of the underlying polynomial. Kumar, in fact proved that if $\mathscr{P}(z) = \sum\limits_{\nu=0}^{n} c_\nu z^\nu = c_n \prod\limits_{\nu=1}^{n} (z - z_\nu)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex number α with $|\alpha| \ge k$

$$\max_{|z|=k} |D_\alpha \mathscr{P}(z)| \ge \left[\frac{2(|\alpha|-k)}{1+k^n} + (|\alpha|-k) \frac{(|c_n|k^n - |c_0|)(k-1)}{(1+k^n)(|c_n|k^n + k|c_0|)} \right] \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} \max_{|z|=1} |\mathscr{P}(z)|.$$

Since $k \ge 1$, therefore $\frac{k}{k+|z_\nu|} \ge \frac{1}{2}$, for $1 \le \nu \le n$, the above inequality in particular gives the following result.

Theorem 1 *If $\mathscr{P}(z) = \sum\limits_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha \mathscr{P}(z)| \ge n \left(\frac{|\alpha|-k}{1+k^n} \right) \left(1 + \frac{(|c_n|k^n - |c_0|)(k-1)}{2(|c_n|k^n + k|c_0|)} \right) \max_{|z|=1} |\mathscr{P}(z)|. \tag{5}$$

Very recently, A. Mir et al. [10] proved the following result which provides an improvement over Theorem 1.

Theorem 2 *If $\mathscr{P}(z) = \sum\limits_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex number α with $|\alpha| \ge k$,*

$$\begin{aligned} \max_{|z|=1} |D_\alpha \mathscr{P}(z)| \quad &\ge \quad n \left(\frac{|\alpha|-k}{1+k^n} \right) \left(1 + \frac{k^n|c_n| - |c_0|}{n(k^n|c_n| + |c_0|)} \right) \\ &\times \left(1 + \frac{(|c_n|k^n - |c_0|)(k-1)}{2(|c_n|k^n + k|c_0|)} \right) \max_{|z|=1} |\mathscr{P}(z)|. \end{aligned} \tag{6}$$

Dividing both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, inequality (6) gives

$$\begin{aligned} \max_{|z|=1} |\mathscr{P}'(z)| \quad &\ge \quad \left(\frac{n}{1+k^n} \right) \left(1 + \frac{k^n|c_n| - |c_0|}{n(k^n|c_n| + |c_0|)} \right) \\ &\times \left(1 + \frac{(|c_n|k^n - |c_0|)(k-1)}{2(|c_n|k^n + k|c_0|)} \right) \max_{|z|=1} |\mathscr{P}(z)|. \end{aligned} \tag{7}$$

2 Main Results

In this paper, we first present the following Turán type inequality for a polynomial providing a refinement as well as generalization of Theorem 2. More precisely, we prove.

Theorem 3 *If $\mathscr{P}(z) = z^s(c_0 + c_1 z + ... + c_{n-s} z^{n-s})$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex number α with $|\alpha| \ge k$,*

$$\begin{aligned} \max_{|z|=1} |D_\alpha \mathscr{P}(z)| \quad &\ge \quad \left(\frac{|\alpha|-k}{1+k^{n-s}} \right) \left(n + s + \frac{k^{n-s}|c_{n-s}| - |c_0|}{k^{n-s}|c_{n-s}| + |c_0|} \right) \\ &\times \left(1 + \frac{(|c_{n-s}|k^{n-s} - |c_0|)(k-1)}{2(|c_{n-s}|k^{n-s} + k|c_0|)} \right) \max_{|z|=1} |\mathscr{P}(z)|. \end{aligned} \tag{8}$$

In view of (4), the result is sharp in limiting case when $|\alpha| \to \infty$ as shown by the polynomial $\mathscr{P}(z) = z^n + k^n$.

Remark 1 *Taking $s = 0$ in Theorem 3, it reduces to Theorem 2 and dividing inequality (8) both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following refinement as well as generalization of inequality (7).*

Corollary 1 *If $\mathscr{P}(z) = z^s(c_0 + c_1 z + ... + c_{n-s} z^{n-s})$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then*

$$\begin{aligned}\max_{|z|=1} |\mathscr{P}'(z)| \quad \ge \quad & \left(\frac{1}{1+k^{n-s}}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right) \\ & \times \left(1+\frac{(|c_{n-s}|k^{n-s}-|c_0|)(k-1)}{2(|c_{n-s}|k^{n-s}+k|c_0|)}\right)\max_{|z|=1}|\mathscr{P}(z)|\,.\end{aligned}$$

The result is sharp and equality holds for $\mathscr{P}(z) = z^n + k^n$.

Theorem 3 gives much better bound than the bound obtained from Theorem 2. We show this with the help of following example.

Example 1 *Consider $\mathscr{P}(z) = z^2(z^2 - 2)$. Here we take $k = 2$ and $\alpha = 3$, then clearly $\mathscr{P}(z)$ is a polynomial of degree 4 having all its zeros in $|z| \le 2$ with 2-fold zeros at origin. Using Theorem 2, we see that*

$$\max_{|z|=1} |D_\alpha \mathscr{P}(z)| \ge 1.32,$$

where as Theorem 3 gives

$$\max_{|z|=1} |D_\alpha \mathscr{P}(z)| \ge 4.2,$$

which is better than the bound obtained from Theorem 2.

Next, we present the following generalization of Theorem 3. Besides, the obtained result sharpens inequality (5) as well.

Theorem 4 *If $\mathscr{P}(z) = z^s(c_0 + c_1 z + ... + c_{n-s} z^{n-s})$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex number α with $|\alpha| \ge k$ and $0 \le t \le 1$,*

$$\begin{aligned}\max_{|z|=1} |D_\alpha \mathscr{P}(z)| \quad \ge \quad & \left(\frac{|\alpha|-k}{1+k^{n-s}}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)\left[(1+\frac{\varLambda_1}{2})\max_{|z|=1}|\mathscr{P}(z)|\right. \\ & \left.+\frac{1}{2k^n}(k^{n-s}-1-\varLambda_1)tm\right] - \left(\frac{|\alpha|-k}{2k^n}\right)\left(\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)tm \\ & +\frac{(|\alpha|(n-s)+k(n+s))}{2k^n}tm \end{aligned} \tag{9}$$

where

$$m = \min_{|z|=k} |\mathscr{P}(z)| \quad \text{and} \quad \varLambda_1 = \frac{(|c_{n-s}|k^{n-s}-|c_0|-tm)(k-1)}{(|c_{n-s}|k^{n-s}+k|c_0|-tm)}. \tag{10}$$

In view of (4), the result is sharp in limiting case when $|\alpha| \to \infty$ as shown by the polynomial $\mathscr{P}(z) = z^n + k^n$.

Remark 2 *Taking $t = 0$, Theorem 4 reduces to Theorem 3. Also by taking $k = 1$ and $s = 0$ in Theorem 3, we obtain*

Corollary 2 *If $\mathscr{P}(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for any complex number α with $|\alpha|\ge 1$ and $0\le t\le 1$*

$$\begin{aligned}\max_{|z|=1}|D_\alpha \mathscr{P}(z)| \;&\geq\; \frac{n}{2}\left\{(|\alpha|-1)\max_{|z|=1}|\mathscr{P}(z)|+(|\alpha|+1)tm\right\}\\ &\quad+\left(\frac{|\alpha|-1}{2}\right)\left(\frac{|c_n|-tm-|c_0|}{|c_n|-tm+|c_0|}\right)\left(\max_{|z|=1}|\mathscr{P}(z)|-tm\right). \qquad (11)\end{aligned}$$

In view of (4), the result is sharp in limiting case when $|\alpha|\to\infty$ as shown by the polynomial $\mathscr{P}(z)=z^n+1$.

Remark 3 *Dividing both sides of inequality (11) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the sharp refinement of inequality (3). By dividing both sides of inequality (9) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the following result.*

Corollary 3 *If $\mathscr{P}(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then for any complex number α with $|\alpha|\ge k$*

$$\begin{aligned}\max_{|z|=1}|\mathscr{P}'(z)| \;&\geq\; \left(\frac{1}{1+k^{n-s}}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)\Bigg[(1+\frac{\Lambda_1}{2})\max_{|z|=1}|\mathscr{P}(z)|\\ &\quad+\frac{1}{2k^n}(k^{n-s}-1-\Lambda_1)tm\Bigg]-\frac{1}{2k^n}\left(\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)tm\\ &\quad+\frac{(n-s)}{2k^n}tm,\end{aligned}$$

where Λ_1 is defined in (10).

Corollary 4 *The result is sharp and equality holds for $\mathscr{P}(z)=z^n+k^n$.*

Remark 4 *For $t=0$, Corollary 3 reduces to Corollary 1.*

Theorem 4 provides much better information than Theorem 1. We show this with the help of following example.

Example 2 *Consider $\mathscr{P}(z)=z^2(z+\frac{5}{4})$. Here we take $k=2$ and $\alpha=3$, then clearly $\mathscr{P}(z)$ is a polynomial of degree 3 having all its zeros in $|z|\le 2$ with 2-fold zeros at origin. Also*

$$\max_{|z|=1}|\mathscr{P}(z)|=\frac{9}{4}\quad \text{and}\quad \min_{|z|=2}|\mathscr{P}(z)|=3.$$

Using Theorem 1, we see that

$$\max_{|z|=1}|D_\alpha\mathscr{P}(z)|\ge 1.12,$$

where as Theorem 4 for $t=1$ gives

$$\max_{|z|=1}|D_\alpha\mathscr{P}(z)|\ge 3.76,$$

which is better than the bound obtained from (5).

3 Lemmas

For the proof of the main results, we need the following lemmas. The first lemma is a simple deduction from Maximum Modulus Principle (see [12]).

Lemma 1 *If $\mathscr{P}(z)$ is a polynomial of degree at most n, then for $R \geq 1$*

$$\max_{|z|=R} |\mathscr{P}(z)| \leq R^n \max_{|z|=1} |\mathscr{P}(z)| .$$

The following lemma is due to A. Mir [11].

Lemma 2 *If $\mathscr{P}(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$, is a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$ and $0 \leq t \leq 1$,*

$$\max_{|z|=k} |\mathscr{P}(z)| \leq \frac{(1+R^n)(|c_0| + R|c_n| - tm_1)}{(1+R)(|c_0| + |c_n| - tm_1)} \max_{|z|=1} |\mathscr{P}(z)| - \left(\frac{(1+R^n)(|c_0| + R|c_n| - tm_1)}{(1+R)(|c_0| + |c_n| - tm_1)} - 1 \right) tm_1,$$

where $m_1 = \min_{|z|=1} |\mathscr{P}(z)|$.

Next lemma is due to Kumar [9].

Lemma 3 *If $\mathscr{P}(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=k} |\mathscr{P}(z)| \geq \left(\frac{2k^n}{1+k^n} + \frac{k^n(|c_n|k^n - |c_0|)(k-1)}{(1+k^n)(|c_n|k^n + k|c_0|)} \right) \max_{|z|=1} |\mathscr{P}(z)| .$$

Lemma 4 *If $\mathscr{P}(z) = z^s(c_0 + c_1 z + ... + c_{n-s} z^{n-s})$, $\quad 0 \leq s \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for any complex number α with $|\alpha| \geq 1$ and $|z| = 1$,*

$$|D_\alpha \mathscr{P}(z)| \geq (|\alpha| - 1) \left\{ \frac{n+s}{2} + \frac{|c_{n-s}| - |a_0|}{2(|a_{n-s}| + |a_0|)} \right\} |\mathscr{P}(z)| .$$

The above lemma is due to Govil and Kumar [8].

4 Proofs of the Theorems

Proof of Theorem 3. Since $\mathscr{P}(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, therefore all the zeros $g(z) = \mathscr{P}(kz)$ lie in $|z| \leq 1$. Hence by applying Lemma 4 to the polynomial $g(z)$ and noting that $\frac{|\alpha|}{k} \geq 1$, we obtain

$$\max_{|z|=1} |D_{\frac{\alpha}{k}} g(z)| \geq \left(\frac{|\alpha|}{k} - 1 \right) \left\{ \frac{n+s}{2} + \frac{(k^{n-s}|c_{n-s}| - |c_0|)}{2(k^{n-s}|c_{n-s}| + |c_0|)} \right\} \max_{|z|=1} |g(z)|,$$

or equivalently,

$$\max_{|z|=1} \left| n\mathscr{P}(kz) + \left(\frac{\alpha}{k} - z \right) k\mathscr{P}'(kz) \right| \geq \left(\frac{|\alpha| - k}{k} \right) \left\{ \frac{n+s}{2} + \frac{(k^{n-s}|c_{n-s}| - |c_0|)}{2(k^{n-s}|c_{n-s}| + |c_0|)} \right\} \max_{|z|=1} |\mathscr{P}(kz)| .$$

This gives

$$\max_{|z|=k} |D_\alpha \mathscr{P}(z)| \geq \left(\frac{|\alpha| - k}{k} \right) \left\{ \frac{n+s}{2} + \frac{(k^{n-s}|c_{n-s}| - |c_0|)}{2(k^{n-s}|c_{n-s}| + |c_0|)} \right\} \max_{|z|=k} |\mathscr{P}(z)| . \tag{12}$$

Now by using Lemma 1 to the polynomial $D_\alpha \mathscr{P}(z)$ which is of degree $n-1$, we conclude that

$$\max_{|z|=k} |D_\alpha \mathscr{P}(z)| \leq k^{n-1} \max_{|z|=1} |D_\alpha \mathscr{P}(z)| . \tag{13}$$

Using inequality (13) in inequality (12), we obtain for $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha \mathscr{P}(z)| \geq \left(\frac{|\alpha| - k}{k^n} \right) \left\{ \frac{n+s}{2} + \frac{(k^{n-s}|c_{n-s}| - |c_0|)}{2(k^{n-s}|c_{n-s}| + |c_0|)} \right\} \max_{|z|=k} |\mathscr{P}(z)| . \tag{14}$$

By hypothesis $\mathscr{P}(z)=z^s h(z)$, where $h(z)=(c_0+c_1z+...+c_{n-s}z^{n-s})$ is a polynomial of degree $n-s$ having all its zeros in $|z|\leq k$, $k\geq 1$, therefore, by applying Lemma 3 to the polynomial $h(z)$, we get

$$\max_{|z|=k}|h(z)|\geq\left\{\frac{2k^{n-s}}{1+k^{n-s}}+\frac{k^{n-s}(|c_{n-s}|k^{n-s}-|c_0|)(k-1)}{(1+k^{n-s})(|c_{n-s}|k^{n-s}+k|c_0|)}\right\}\max_{|z|=1}|h(z)|.$$

Also

$$\max_{|z|=k}|h(z)|=\frac{1}{k^s}\max_{|z|=k}|\mathscr{P}(z)|$$

and

$$\max_{|z|=1}|h(z)|=\max_{|z|=1}|\mathscr{P}(z)|.$$

Replacing these in above inequality, we get

$$\max_{|z|=k}|\mathscr{P}(z)|\geq\left\{\frac{2k^{n}}{1+k^{n-s}}+\frac{k^{n}(|c_{n-s}|k^{n-s}-|c_0|)(k-1)}{(1+k^{n-s})(|c_{n-s}|k^{n-s}+k|c_0|)}\right\}\max_{|z|=1}|\mathscr{P}(z)|. \tag{15}$$

Inequality (15) in conjunction with inequality (14) yields for $|z|=1$ and $|\alpha|\geq k$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha\mathscr{P}(z)| \quad\geq\quad & \left(\frac{|\alpha|-k}{k^n}\right)\left\{\frac{n+s}{2}+\frac{(k^{n-s}|c_{n-s}|-|c_0|)}{2(k^{n-s}|c_{n-s}|+|c_0|)}\right\}\\ &\times\left\{\frac{2k^{n}}{1+k^{n-s}}+\frac{k^{n}(|c_{n-s}|k^{n-s}-|c_0|)(k-1)}{(1+k^{n-s})(|c_{n-s}|k^{n-s}+k|c_0|)}\right\}\max_{|z|=1}|\mathscr{P}(z)|,\end{aligned}$$

which on simplification gives for $|z|=1$ and $|\alpha|\geq k$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha\mathscr{P}(z)| \quad\geq\quad & \left(\frac{|\alpha|-k}{1+k^{n-s}}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right)\\ &\times\left(1+\frac{(|c_{n-s}|k^{n-s}-|c_0|)(k-1)}{2(|c_{n-s}|k^{n-s}+k|c_0|)}\right)\max_{|z|=1}|\mathscr{P}(z)|.\end{aligned}$$

This completes the proof of Theorem 3. ■

Proof of Theorem 4. By hypothesis $\mathscr{P}(z)$ has all its zeros in $|z|\leq k$, $\quad k\geq 1$. If $\mathscr{P}(z)$ has a zero on $|z|=k$, then $m=\min_{|z|=k}|\mathscr{P}(z)|=0$ and result follows from Theorem 3 in this case. So, we assume that $p(z)$ has all its zeros in $|z|<k$, so that $m>0$. Now if $f(z)=\mathscr{P}(kz)$, then the polynomial $f(z)$ has all its zeros in $|z|<1$. Also $m=\min_{|z|=k}|p(z)|=\min_{|z|=1}|f(z)|$, this implies for every $\lambda\in\mathbb{C}$ with $|\lambda|<1$,

$$|m\lambda z^n|\leq|f(z)|,\quad \text{for}\quad |z|=1.$$

It follows by Rouche's Theorem that the polynomial $g(z)=f(z)-\lambda mz^n$ has all zeros in $|z|<1$. Applying Lemma 4 to the polynomial $g(z)=f(z)-\lambda mz^n$ and noting that $\frac{|\alpha|}{k}\geq 1$, we get

$$\left|D_{\frac{\alpha}{k}}(f(z)-\lambda mz^n)\right|\geq\left(\frac{|\alpha|}{k}-1\right)\left\{\frac{n+s}{2}+\frac{|k^{n-s}c_{n-s}-\lambda m|-|c_0|}{2(|k^{n-s}c_{n-s}-\lambda m|+|a_0|)}\right\}|f(z)-\lambda mz^n|.$$

Using the fact that $s(x)=\frac{x-|c|}{x+|c|}$ is non-decreasing function of x and $|k^{n-s}c_{n-s}-\lambda m|\geq k^{n-s}|c_{n-s}|-|\lambda m|$, we obtain for $|\alpha|/k\geq 1$, $\;|\lambda|<1$ and $|z|=1$,

$$\left|D_{\frac{\alpha}{k}}f(z)-\frac{nm\alpha\lambda}{k}z^{n-1}\right|\geq\left(\frac{|\alpha|-k}{2k}\right)\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|\lambda|m-|c_0|}{k^{n-s}|c_{n-s}|-|\lambda|m+|c_0|}\right\}|f(z)-\lambda mz^n|. \tag{16}$$

Since all the zeros of $f(z)-\lambda m z^n$ lie in $|z|<1$, therefore it follows by Laguerre's Theorem that all the zeros of $D_{\frac{\alpha}{k}} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1}$ lie in $|z|<1$. This implies

$$\left|D_{\frac{\alpha}{k}} f(z)\right| \geq \frac{nm\alpha\lambda}{k}|z|^{n-1} \quad \text{for } |z| \geq 1. \tag{17}$$

Choosing argument of λ in the left hand side of (16) such that

$$\left|D_{\frac{\alpha}{k}} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1}\right| = |D_{\frac{\alpha}{k}} f(z)| - \frac{nm|\alpha||\lambda|}{k} \quad \text{for } |z|=1,$$

which is possible by (17), we get for $0 \leq t \leq 1$, $|\alpha| \geq k$ and $|z|=1$

$$\left|D_{\frac{\alpha}{k}} f(z)\right| - \frac{nm|\alpha|t}{k} \geq \left(\frac{|\alpha|-k}{2k}\right)\left\{n+s+\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right\}(|f(z)|-tm).$$

This gives for $|\alpha| \geq k$ and $|z|=1$,

$$\begin{aligned}\max_{|z|=k}|D_\alpha \mathscr{P}(z)| \quad &\geq \quad \left(\frac{|\alpha|-k}{2k}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)\max_{|z|=k}|\mathscr{P}(z)| \\ &\quad -\left(\frac{|\alpha|-k}{2k}\right)\left(\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)tm+\frac{(|\alpha|(n-s)+k(n+s))}{2k}tm. \end{aligned} \tag{18}$$

Since $f(z)=\mathscr{P}(kz)$ has all its zeros in $|z| \leq k$, $k \geq 1$ with s-fold zeros at origin, therefore the polynomial $q(z)=z^n f(1/z)$ is a polynomial of degree $n-s$ having no zero in $|z|<1$. Hence by applying Lemma 2 we obtain for $k \geq 1$ and $0 \leq t \leq 1$,

$$\max_{|z|=k}|q(z)| \leq \frac{(1+k^{n-s})(k^n|c_{n-s}|+k^{s+1}|c_0|-tm')}{(1+k)(k^n|c_{n-s}|+k^s|c_0|-tm')}\max_{|z|=1}|q(z)| - \left(\frac{(1+k^{n-s})(k^n|c_{n-s}|+k^{s+1}|c_0|-tm')}{(1+k)(k^n|c_{n-s}|+k^s|c_0|-tm')}-1\right)$$

where $m'=\min\limits_{|z|=1}|q(z)|$.

Also

$$m'=\min_{|z|=1}|q(z)|=\min_{|z|=1}|z^n p(k/z)|=\min_{|z|=k}|\mathscr{P}(z)|=\min_{|z|=1}|f(z)|=m,$$

$$\max_{|z|=1}|q(z)|=\max_{|z|=1}|f(z)|=\max_{|z|=k}|\mathscr{P}(z)|,$$

and

$$\max_{|z|=k}|q(z)|=k^n\max_{|z|=1}|\mathscr{P}(z)|.$$

Replacing these in above inequality, we get

$$\begin{aligned}\max_{|z|=k}|\mathscr{P}(z)| \quad &\geq \quad \left(\frac{(1+k)(k^{n-s}|c_{n-s}|+|c_0|-tm)}{(1+k^{n-s})(k^{n-s}|c_{n-s}|+k|c_0|-tm)}\right)k^n\max_{|z|=1}|\mathscr{P}(z)| \\ &\quad +\left(1-\frac{(1+k)(k^{n-s}|c_{n-s}|+|c_0|-tm)}{(1+k^{n-s})(k^{n-s}|c_{n-s}|+k|c_0|-tm)}\right)tm \\ &= \quad \left[\frac{2k^n}{1+k^{n-s}}+\frac{k^n(|c_{n-s}|k^{n-s}-|c_0|-tm)(k-1)}{(1+k^{n-s})(|c_n|k^{n-s}+k|c_0|-tm)}\right]\max_{|z|=1}|\mathscr{P}(z)| \\ &\quad +\left[\frac{k^{n-s}-1}{k^{n-s}+1}-\frac{(|c_{n-s}|k^{n-s}-|c_0|-tm)(k-1)}{(1+k^{n-s})(|c_{n-s}|k^{n-s}+k|c_0|-tm)}\right]tm. \end{aligned} \tag{19}$$

Inequality (18) in conjunction with inequality (19) and Lemma 1 yields for $|\alpha| \geq k$ and $0 \leq t \leq 1$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha \mathscr{P}(z)| \geq & \left(\frac{|\alpha|-k}{2k^n}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)\left[\left\{\frac{2k^n}{1+k^{n-s}}+\frac{k^n \Lambda_1}{1+k^{n-s}}\right\}\max_{|z|=1}|\mathscr{P}(z)|\right. \\ & \left.+\left\{\frac{k^{n-s}-1}{k^{n-s}+1}-\frac{\Lambda_1}{1+k^{n-s}}\right\}tm\right]-\left(\frac{|\alpha|-k}{2k^n}\right)\left(\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)tm \\ & +\frac{(|\alpha|(n-s)+k(n+s))}{2k^n}tm,\end{aligned}$$

where Λ_1 is defined in (10).

The above inequality after a simplification gives for $|\alpha| \geq k$ and $0 \leq t \leq 1$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha \mathscr{P}(z)| \geq & \left(\frac{|\alpha|-k}{1+k^{n-s}}\right)\left(n+s+\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)\left[(1+\frac{\Lambda_1}{2})\max_{|z|=1}|\mathscr{P}(z)|\right. \\ & \left.+\frac{1}{2k^n}(k^{n-s}-1-\Lambda_1)tm\right]-\left(\frac{|\alpha|-k}{2k^n}\right)\left(\frac{k^{n-s}|c_{n-s}|-tm-|c_0|}{k^{n-s}|c_{n-s}|-tm+|c_0|}\right)tm \\ & +\frac{(|\alpha|(n-s)+k(n+s))}{2k^n}tm.\end{aligned}$$

This completes the proof of Theorem 4. ■

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