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On λ-Statistical Convergence Of Sequences In Gradual Normed Linear Spaces*

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Abstract

In this paper by using the notion of λ-density and (V, λ) -summability, we introduce the notion of λ-statistical convergence of sequences in gradual normed linear spaces. Based on this concept, we introduce a new sequence space $S_\lambda(G)$ and investigate some of its properties. Also, we find its relations with $S(G)$ and $[V, \lambda]_G$-summability. Finally, we introduce and investigate the concept of gradual λ-statistical Cauchy sequences.

1 Introduction

The idea of fuzzy sets [32] was first introduced by Zadeh in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The term "fuzzy number" plays a crucial role in the study of fuzzy set theory. Fuzzy numbers were basically the generalization of intervals, not numbers. Even fuzzy numbers do not obey a few algebraic properties of the classical numbers. So the term "fuzzy number":is debatable to many authors due to its different behavior. The term "fuzzy intervals":is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et al. [16] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function which is defined in the interval $(0, 1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have used in computation and optimization problems.

In 2011, Sadeqi and Azari [25] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological points of view. Further progress in this direction has occurred due to Ettefagh et al. [13, 14], Choudhury and Debnath [8], and many others. For an extensive study on gradual real numbers, one may refer to [1, 10, 22, 29] where many more references can be found.

On the other hand, in 1951 Fast [15] and Steinhaus [28] introduced the idea of statistical convergence independently using the idea of natural density [17]. Later on, it was further investigated and generalized from the sequence space point of view by Fridy [18, 19], Salat [26], Rath and Tripathy [24], Tripathy [30, 31] and many mathematicians [2, 4, 5, 6, 7, 20] across the globe.

In 2000, statistical convergence was extended to λ-statistical convergence by Mursaleen [23] as follows: Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} - \lambda_n \le 1, \quad \lambda_1 = 1.$$

The generalized de la Valée-Pousin mean is defined by

$$t_n((x_k)) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

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where $I_n = [n - \lambda_n + 1,\ n]$. A sequence (x_k) is said to be (V, λ)-summable to a number L (see [21] for details) if,

$$t_n((x_k)) \to L \text{ as } n \to \infty.$$

We write

$$[V, \lambda] = \left\{ (x_k) : \lim_{n\to\infty} \left(\frac{1}{\lambda_n} \sum_{k\in I_n} |x_k - L| \right) = 0 \ \text{ for some } L \right\}$$

for the sets of sequences (x_k), which are strongly (V, λ)-summable to L, i.e., $x_k \to L[V, \lambda]$.

A sequence (x_k) is said to be λ-statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n :| x_k - L |\geq \varepsilon\}| = 0.$$

In this case, L is called the λ-statistical limit of the sequence (x_k) and we write $S_\lambda - \lim x_k = L$ or $x_k \to L(S_\lambda)$. Here S_λ denotes the set of all λ-statistically convergent sequences. It is obvious that if $\lambda_n = n$, then S_λ is coincident with S, where S is the set of all statistical convergent sequences (for more details one may see [3, 9, 11, 12]).

2 Preliminaries

Definition 1 ([16]) *A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative if for every $\xi \in (0, 1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.*

In [16], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2 *Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi), : \forall \xi \in (0, 1]$. In particular, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined as follows:*

$$A_{\tilde{r}_1+\tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi) \quad \text{and} \quad A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi), \quad \forall \xi \in (0, 1].$$

Definition 3 ([25]) *Let X be a real vector space. The function $||\cdot||_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\xi \in (0, 1]$, following conditions are true for any $x, y \in X$:*

(G_1) $A_{||x||_G}(\xi) = A_{\tilde{0}}(\xi)$ iff $x = 0$;

(G_2) $A_{||\alpha x||_G}(\xi) = |\alpha| A_{||x||_G}(\xi)$ for any $\alpha \in \mathbb{R}$;

(G_3) $A_{||x+y||_G}(\xi) \leq A_{||x||_G}(\xi) + A_{||y||_G}(\xi)$.

The pair $(X, ||\cdot||_G)$ is called a gradual normed linear space (GNLS).

Example 1 ([25]) *Let $X = \mathbb{R}^m$ and for $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m, \xi \in (0, 1]$, define $||\cdot||_G$ by*

$$A_{||x||_G}(\xi) = e^{\xi} \sum_{i=1}^{m} |x_i|.$$

Then, $||\cdot||_G$ is a gradual norm on $\mathbb{R}^m$ and $(\mathbb{R}^m, ||\cdot||_G)$ is a GNLS.

Definition 4 ([25]) *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual convergent to $x \in X$, if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that*

$$A_{||x_k - x||_G}(\xi) < \varepsilon, \quad \forall k \geq N.$$

Symbolically, $x_k \xrightarrow{||\cdot||_G} x$.

Definition 5 (**[14]**) *Let $(X, ||\cdot||_G)$ be a GNLS. Then, a sequence (x_k) in X is said to be gradual bounded if for every $\xi \in (0,1]$, there exists $M = M(\xi) > 0$ such that $A_{||x_k||_G}(\xi) < M$, $\forall k \in \mathbb{N}$.*

Definition 6 (**[25]**) *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual Cauchy, if for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x_j||_G}(\xi) < \varepsilon$, $\forall k, j \geq N$.*

Theorem 1 (**[25, Theorem 3.6]**) *Let $(X, ||\cdot||_G)$ be a GNLS. Then every gradual convergent sequence in X is also a gradual Cauchy sequence.*

Definition 7 (**[27]**) *Let $E \subseteq \mathbb{N}$, the set of all natural numbers and let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} - \lambda_n \leq 1$, $\lambda_1 = 1$. Then, the number*

$$\delta_\lambda(E) = \lim_{n\to\infty} \frac{|\{k \in E : k \in I_n\}|}{\lambda_n}$$

is said to be the λ-density of E. If $\lambda_n = n$, $\forall\ n \in \mathbb{N}$, then λ-density coincides with natural density.

We observed the following results related to λ-density:

i) λ-density of a finite subset of $\mathbb{N}$ is zero.

ii) λ-density of the set of all natural numbers is 1.

iii) λ-density of the set of all even natural numbers is $\frac{1}{2}$, if $\lim\limits_n \frac{n}{\lambda_n}$ exists.

Throughout the paper, we use the following notation: If (x_k) is a sequence such that x_k satisfies a property P $\forall\ k$ except for a set of λ-density zero, then we say that x_k satisfies the property P for "almost all k":and we abbreviate this by "a.a.k."

Definition 8 *Let $(X, ||\cdot||_G)$ be any GNLS. We define the new sequence space $[V, \lambda]_G$ as follows:*

$$[V,\lambda]_G = \left\{ (x_k) : \lim_{n\to\infty} \frac{1}{\lambda_n} \left(\sum_{k\in I_n} A_{||x_k - x||_G}(\xi) \right) = 0 \ \text{ for some } x \in X \text{ and all } \xi \in (0,1] \right\}.$$

Definition 9 *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual statistically convergent to $x \in X$ if for every $\xi \in (0,1]$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{ k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon \right\}\right| = 0.$$

Symbolically, $x_k \xrightarrow{st-||\cdot||_G} x$. The set $S(G)$ denotes the set of all gradually statistical convergent sequences.

3 Main Results

Definition 10 *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradually λ-statistical convergent to $x \in X$ if for every $\xi \in (0,1]$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{ k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon \right\}\right| = 0.$$

Or equivalently, $A_{||x_k - x||_G}(\xi) < \varepsilon$ a.a.k. In this case, x is called the gradual λ-statistical limit of the sequence (x_k) and we write

$$S_\lambda - ||\cdot||_G \lim x_k = x \quad or \quad x_k \xrightarrow{S_\lambda - ||\cdot||_G} x.$$

We shall also use $S_\lambda(G)$ to denote the set of all gradually λ-statistical convergent sequences.

Example 2 *Let $X = \mathbb{R}^m$ and $||\cdot||_G$ be the norm defined in Example 1. Consider the sequence (λ_n) defined by*

$$\lambda_n = \begin{cases} 1, & n = 1, \\ \frac{n}{2}, & n \geq 2. \end{cases}$$

Then, the sequence (x_k) in $\mathbb{R}^m$ defined as

$$x_k = \begin{cases} (0,0,...,0,m), & \text{if } k = p^2 \text{ and } p \in \mathbb{N}, \\ (0,0,.....,0,0), & \text{otherwise}, \end{cases}$$

is gradually λ-statistical convergent to $\mathbf{0}$ in $\mathbb{R}^m$ where $\mathbf{0}$ denotes the m-tuple $(0,0,....0,0)$.

Justification. We have

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\right\}\right| \\ = \;& 2 \lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in [\frac{n}{2}+1, n] : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\right\}\right| \\ \leq \;& 2 \lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\right\}\right| \\ \leq \;& 2 \lim_{n\to\infty} \frac{[\sqrt{n}]}{n} \\ = \;& 0 \end{aligned}$$

where $[x]$ denotes the largest integer less than or equal to x. Hence, we conclude that $x_k \xrightarrow{S_\lambda - ||\cdot||_G} \mathbf{0}$.

Example 3 *Let $X = \mathbb{R}$ and for any $x \in \mathbb{R}$, let $||\cdot||_G$ be the norm defined as*

$$A_{||x||_G} = e^{\xi}|x|.$$

Consider the sequence (λ_n) defined in Example 2. Then, the sequence (x_k) in X defined as $x_k = k^2$ is not gradually λ-statistical convergent.

Justification. For any $x \in \mathbb{R}$, we have $x \leq 0$ or $x > 0$. Then, for each of the following cases, (x_k) will not gradually λ-statistical convergent to x.

Case-I: If $x \leq 0$, we choose $\varepsilon = \frac{1}{2}e^{\xi}$. Then, we have

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\right\}\right| \\ = \;& \lim_{n\to\infty} \frac{2}{n} \left|\left\{k \in [\frac{n}{2}+1, n] : A_{||k^2 - x||_G}(\xi) \geq \frac{1}{2}e^{\xi}\right\}\right| \\ = \;& \begin{cases} \lim_{n\to\infty} \frac{2}{n}(\frac{n}{2} - 1) & \text{when } n \text{ is even}, \\ \lim_{n\to\infty} \frac{2}{n}(\frac{n+1}{2} - 1) & \text{when } n \text{ is odd}, \end{cases} \\ = \;& 1 \\ \neq \;& 0. \end{aligned}$$

Case-II: If $x > 0$, then there exists $k_0 \in \mathbb{N}$ such that $x_{k_0-1} \leq x \leq x_{k_0}$.

Subcase-I: If $0 < x < 1$, then choose $\varepsilon = \frac{e^{\xi}}{2}\min\{x, 1-x\}$. Then, it is easy to verify that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\right\}\right| = 1 \neq 0.$$

Subcase-II: If $x \geq 1$, then choose $\varepsilon = \frac{e^{\xi}}{2}\min\{x - x_{k_0-1}, x_{k_0} - x\}$. Then, it is easy to verify that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\right\}\right| = 1 \neq 0.$$

From the above case study, we can conclude that (x_k) is not gradually λ-statistical convergent.

Lemma 1 *Let $p \geq 2$ be a fixed natural number and $B = \{n \in \mathbb{N} : n^{\frac{1}{p}} \in \mathbb{N}\}$. Then, $\delta_\lambda(B) = 0$ if $\lim_{n\to\infty} \frac{n^{\frac{1}{p}}}{\lambda_n}$ exists.*

Proof. Let

$$B_n = \{k \in B : k \in I_n\} \text{ and } \lim_n \frac{n^{\frac{1}{p}}}{\lambda_n} = l.$$

Then, it is easy to show that

$$|B_n| = [n^{\frac{1}{p}}] - [(n - \lambda_n + \frac{1-(-1)^n}{2})^{\frac{1}{p}}],$$

where $[x]$ denotes the largest integer $\leq x$.

Case-I: Assume that n is even. We have

$$n^{\frac{1}{p}} - 1 \leq [n^{\frac{1}{p}}] \leq n^{\frac{1}{p}}.$$

Then

$$\frac{n^{\frac{1}{p}} - 1}{\lambda_n} \leq \frac{[n^{\frac{1}{p}}]}{\lambda_n} \leq \frac{n^{\frac{1}{p}}}{\lambda_n}.$$

It follows that

$$\lim_{n\to\infty} \frac{[n^{\frac{1}{p}}]}{\lambda_n} = l.$$

Also,

$$(n-\lambda_n)^{\frac{1}{p}} - 1 \leq [(n-\lambda_n)^{\frac{1}{p}}] \leq (n-\lambda_n)^{\frac{1}{p}}.$$

Then

$$\frac{(n-\lambda_n)^{\frac{1}{p}} - 1}{\lambda_n} \leq \frac{[(n-\lambda_n)^{\frac{1}{p}}]}{\lambda_n} \leq \frac{(n-\lambda_n)^{\frac{1}{p}}}{\lambda_n}.$$

So

$$\frac{n^{\frac{1}{p}}}{\lambda_n}(1-\frac{\lambda_n}{n})^{\frac{1}{p}} - \frac{1}{\lambda_n} \leq \frac{[(n-\lambda_n)^{\frac{1}{p}}]}{\lambda_n} \leq \frac{n^{\frac{1}{p}}}{\lambda_n}(1-\frac{\lambda_n}{n})^{\frac{1}{p}}.$$

If $\frac{\lambda_n}{n} < 1$, then from above,

$$\frac{n^{\frac{1}{p}}}{\lambda_n} - O(\frac{\lambda_n}{n}) - \frac{1}{\lambda_n} \leq \frac{[(n-\lambda_n)^{\frac{1}{p}}]}{\lambda_n} \leq \frac{n^{\frac{1}{p}}}{\lambda_n} - O(\frac{\lambda_n}{n}).$$

Therefore,

$$\lim_{n\to\infty} \frac{[(n-\lambda_n)^{\frac{1}{p}}]}{\lambda_n} = l.$$

If $\frac{\lambda_n}{n} = 1$, then $\lim_{n\to\infty} \frac{[(n-\lambda_n)^{\frac{1}{p}}]}{\lambda_n} = l$ is trivial. Therefore,

$$\lim_{n\to\infty} \frac{[n^{\frac{1}{p}}]}{\lambda_n} - \lim_{n\to\infty} \frac{[(n-\lambda_n)^{\frac{1}{p}}]}{\lambda_n} = l - l = 0.$$

Hence, if n is even then $\lim_{n\to\infty} |B_n|/\lambda_n = 0$.

Case-II: If n is odd, using a similar technique it can be easily shown that $\lim_{n\to\infty} |B_n|/\lambda_n = 0$.

Hence, from the above two cases we can conclude that $\delta_\lambda(B) = 0$. ■

Theorem 2 *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$. Then, x is unique.*

Proof. The proof is easy, so omitted. ■

Theorem 3 *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$ and $y_k \xrightarrow{S_\lambda - ||\cdot||_G} y$. Then*

(i) $x_k + y_k \xrightarrow{S_\lambda - ||\cdot||_G} x + y$; *and*

(ii) $cx_k \xrightarrow{S_\lambda - ||\cdot||_G} cx$ *for any* $c \in \mathbb{R}$.

Proof. The proof is easy, so omitted. ■

Theorem 4 *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$ if and only if there exists $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $\delta_\lambda(M) = 1$ and $(x_{m_k}) \xrightarrow{||\cdot||_G} x$.*

Proof. Firstly, we assume that there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ satisfying

$$\delta_\lambda(M) = 1 \quad \text{and} \quad (x_{m_k}) \xrightarrow{||\cdot||_G} x.$$

Then, for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that

$$A_{||x_{m_k} - x||_G}(\xi) < \varepsilon, \quad \forall k \geq N.$$

Let $B(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$. Then, the inclusion

$$B(\xi, \varepsilon) \subset \mathbb{N} \setminus \{m_{N+1}, m_{N+2}, ...\}$$

holds and as a consequence we have $\delta_\lambda(B(\xi, \varepsilon)) = 0$. Hence, $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$.

For the converse part, assume that $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$ holds. Then, for every $\xi \in (0,1]$ and $j \in \mathbb{N}$, $\delta_\lambda(M_j) = 1$, where

$$M_j = \left\{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \frac{1}{j}\right\}.$$

From the construction of M_j's, it is clear that

$$M_1 \supset M_2 \supset ... \supset M_j \supset M_{j+1} \supset .... \tag{1}$$

Let us choose $v_1 \in M_1$ to be an arbitrary element. Then, there exists $v_2 \in M_2$ such that

$$\frac{1}{\lambda_n} |\{k \in I_n : k \in M_2\}| > \frac{1}{2} \quad \text{for all } n \geq v_2.$$

In a similar way, there exists $v_3 \in M_3$ such that

$$\frac{1}{\lambda_n} |\{k \in I_n : k \in M_3\}| > \frac{2}{3} \quad \text{for all } n \geq v_3.$$

Proceeding like this, we can construct an increasing sequence (v_j) of positive integers such that $v_j \in M_j$ and

$$\frac{1}{\lambda_n} |\{k \in I_n : k \in M_j\}| > 1 - \frac{1}{j} \quad \text{for all } n \geq v_j. \tag{2}$$

Let us construct M as follows: each natural number of the interval $[1, v_1]$ belongs to M and any natural number of the interval $[v_j, v_{j+1}]$ belongs to M if and only if it belongs to $M_j : (j \in \mathbb{N})$.

From (1) and (2), we have for each $v_j \leq n < v_{j+1}$,

$$\frac{|\{k \in I_n : k \in M_j\}|}{\lambda_n} \geq \frac{|\{k \in I_n : k \in M_j\}|}{\lambda_n} > 1 - \frac{1}{j}.$$

Consequently, $\delta_\lambda(M) = 1$. Let $\varepsilon > 0$ be given. By Archimedean property, choose $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. Furthermore, let $k \in M$ be such that $k \geq v_j$. Then, there exists $t \geq j$ such that $v_t \leq k \leq v_{t+1}$. But by the definition of M, $k \in M_t$. Therefore,

$$A_{||x_k - x||_G}(\xi) < \frac{1}{t} \leq \frac{1}{j} < \varepsilon.$$

Hence, $(x_{m_k}) \xrightarrow{||\cdot||_G} x$ holds and the proof is complete. ■

Remark 1 *Every subsequence of a gradually λ-statistical convergent sequence is not necessarily gradually λ-statistical convergent.*

Example 4 *Let $X = \mathbb{R}$ and $||\cdot||_G$ be the norm defined in Example 3. Consider the sequence (λ_n) defined by*

$$\lambda_n = \begin{cases} 1, & n = 1, \\ \frac{n}{2}, & n \geq 2. \end{cases}$$

Let

$$x_k = \begin{cases} k, & k = p^2,\ p \in \mathbb{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

By Lemma 1 with p = 2, then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \left|\left\{k \in I_n : A_{||x_k - 0||_G}(\xi) \geq \varepsilon\right\}\right| = \delta_\lambda(B) = 0,$$

where $B = \{n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}\}$. Therefore, $x_k \xrightarrow{S_\lambda - ||\cdot||_G} 0$. But the sequence considered in Example 3 is not gradually λ-statistical convergent although it is a subsequence of the above sequence.

Theorem 5 *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then,*

(i) $x_k \xrightarrow{[V,\lambda]_G} x$ implies $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$ but the converse is not true;

(ii) If (x_k) is gradually bounded and $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$, then, $x_k \xrightarrow{[V,\lambda]_G} x$.

Proof. (i) Let $\varepsilon > 0$ be arbitrary and $x_k \xrightarrow{[V,\lambda]_G} x$. Then, the proof follows directly from the following fact:

$$\sum_{k\in I_n} A_{||x_k - x||_G}(\xi) \geq \sum_{\substack{k\in I_n \\ A_{||x_k - x||_G}(\xi)\geq\varepsilon}} A_{||x_k - x||_G}(\xi) \geq \varepsilon \left|\left\{k \in I_n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\right\}\right|.$$

For the converse part, we construct a counterexample by considering the gradual normed space $(\mathbb{R}, ||\cdot||_G)$, where $||\cdot||_G$ is the norm defined in Example 3. Consider the sequence (λ_n) be defined in Example 2. Define a sequence (x_k) by

$$x_k = \begin{cases} k, & n - [\sqrt{\frac{n}{2}}] + 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $\varepsilon > 0$ with $0 < \varepsilon e^\xi \leq 1$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : A_{||x_k - 0||_G}(\xi) \geq \varepsilon\right\}\right| = \lim_{n\to\infty} \frac{2}{n}[\sqrt{\frac{n}{2}}] = 0.$$

Hence, $x_k \xrightarrow{S_\lambda - ||\cdot||_G} 0$ holds. On the other hand,

$$\begin{aligned} & \frac{1}{\lambda_n}\sum_{k\in I_n} A_{||x_k-0||_G}(\xi) \\ = \ & \frac{2}{n}\cdot \sum_{k\in[\frac{n}{2}+1,n]} A_{||x_k||_G}(\xi) \\ = \ & \frac{2e^\xi}{n}\cdot\{(n-[\sqrt{\frac{n}{2}}]+1)+(n-[\sqrt{\frac{n}{2}}]+2)+\cdots+(n-[\sqrt{\frac{n}{2}}]+[\sqrt{\frac{n}{2}}])\} \\ \to \ & \infty \ \text{ as } n\to\infty, \end{aligned}$$

i.e., (x_k) does not converge to 0 in $[V,\lambda]_G$.

(ii) Let $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$ and (x_k) be gradually bounded, say $A_{||x_k-x||_G}(\xi)\le M,: \forall k\in\mathbb{N}$. Then, for any $\varepsilon>0$, we have

$$\begin{aligned} \frac{1}{\lambda_n}\sum_{k\in I_n} A_{||x_k-x||_G}(\xi) &= \sum_{\substack{k\in I_n \\ A_{||x_k-x||_G}(\xi)\ge\varepsilon}} A_{||x_k-x||_G}(\xi) + \sum_{\substack{k\in I_n \\ A_{||x_k-x||_G}(\xi)<\varepsilon}} A_{||x_k-x||_G}(\xi) \\ &\le \frac{M}{\lambda_n}\left|\left\{k\in I_n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right| + \varepsilon; \end{aligned}$$

which consequently implies that $x_k \xrightarrow{[V,\lambda]_G} x$. ■

Theorem 6 *$S_\lambda(G)\supseteq S(G)$ if $\liminf_n \frac{\lambda_n}{n}>0$.*

Proof. Suppose $\liminf_n \frac{\lambda_n}{n}>0$ and $x_k \xrightarrow{st-||\cdot||_G} x$. Then, for sufficiently large n, there exists $\delta>0$ such that $\frac{\lambda_n}{n}>\delta$. Now for any $\varepsilon>0$ and $\xi\in(0,1]$, we have

$$\begin{aligned} \frac{1}{n}\left|\left\{k\le n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right| &\ge \frac{1}{n}\left|\left\{k\in I_n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right| \\ &\ge \delta\frac{1}{\lambda_n}\left|\left\{k\in I_n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right|, \end{aligned}$$

which yields $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$. ■

Theorem 7 *$S(G)\supseteq S_\lambda(G)$ if $\liminf_n \frac{\lambda_n}{n}=1$.*

Proof. Suppose $\liminf_n \frac{\lambda_n}{n}=1$ and $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$. Then, for any $\varepsilon>0$ and $\xi\in(0,1]$, we have

$$\begin{aligned} \frac{1}{n}\left|\left\{k\le n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right| &\le \frac{1}{n}\left|\left\{k\le n-\lambda_n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right| + \frac{1}{n}\left|\left\{k\in I_n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right| \\ &\le \frac{n-\lambda_n}{n} + \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{k\in I_n : A_{||x_k-x||_G}(\xi)\ge\varepsilon\right\}\right|, \end{aligned}$$

which yields $x_k \xrightarrow{st-||\cdot||_G} x$. ■

Definition 11 *Let $(X,||\cdot||_G)$ be a GNLS. A sequence (x_k) in X is said to be gradual λ-statistical Cauchy if for every $\varepsilon>0$ and $\xi\in(0,1]$, there exists $N\in\mathbb{N}$ such that*

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n : A_{||x_k-x_N||_G}(\xi)\ge\varepsilon\right\}\right| = 0$$

or equivalently, $A_{||x_k-x_N||_G}(\xi)<\varepsilon$ a.a.k.

Theorem 8 *Let $(X, ||\cdot||_G)$ be a GNLS. Then, every gradually λ-statistical convergent sequence in X is gradual λ-statistical Cauchy.*

Proof. Let $x_k \xrightarrow{S_\lambda - ||\cdot||_G} x$. Then, for any $\varepsilon > 0$ and $\xi \in (0, 1]$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2} \right\} \right| = 0.$$

This implies that

$$A_{||x_k - x||_G}(\xi) < \frac{\varepsilon}{2} \quad \text{a.a.k},$$

i.e.

$$\delta_\lambda(\left\{ k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2} \right\}) = 0,$$

i.e.

$$\delta_\lambda(\left\{ k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \frac{\varepsilon}{2} \right\}) \neq 0.$$

Therefore, the set

$$\left\{ k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \frac{\varepsilon}{2} \right\} \neq \varnothing$$

Choose $N \in \mathbb{N}$ such that

$$N \in \left\{ k \in \mathbb{N} : A_{||x_k - x||_G} < \frac{\varepsilon}{2} \right\}.$$

Then we have,

$$\begin{aligned} A_{||x_k - x_N||_G}(\xi) &= A_{||x_k - x + x - x_N||_G}(\xi) \\ &\leq A_{||x_k - x_N||_G}(\xi) + A_{||x_N - x||_G}(\xi) \\ &< \varepsilon \quad a.a.k. \end{aligned}$$

Hence, (x_k) is gradual λ-statistical Cauchy. ■

Conclusion

In this paper, we have investigated a few fundamental properties of λ-statistical convergence in the gradual normed linear spaces. We also introduced $(V, \lambda)-$summability in the gradual normed linear spaces and established Theorem 5 to reveal the interrelationship between the notions. Finally, we have introduced the concept of λ-statistical Cauchy sequences in the gradual normed space and established the interrelationship between gradual λ-statistical convergent and gradual λ-statistical Cauchy sequences.

Summability theory and the convergence of sequences have wide applications in various branches of mathematics particularly, in mathematical analysis. Research in this direction based on gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The obtained results may be useful for future researchers to explore various notions of convergences in the gradual normed linear spaces in more detail.

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