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Moments Of Progressively Type-II Censored Order Statistics From Weibull-Geometric Distribution And Associated Inference*

Areej M. AL-Zaydi†, Bander Al-Zahrani‡

Abstract

In this article, we derive explicit expressions as well as recurrence relations for the single and product moments of progressively Type-II right censored order statistics from the Weibull-geometric distribution. Based on these moments, the best linear unbiased estimators for the scale and location-scale parameters of the Weibull-geometric distribution are obtained. The best linear unbiased predictors of future failure times are also discussed. Finally, an example is considered for illustrative purposes.

1 Introduction

A random variable X is said to have a Weibull-geometric (WG) distribution (Barreto-Souza et al. [18]) if its probability density function (PDF) and its cumulative distribution function (CDF) are

$$f(x) = \theta\beta^{\theta}(1-p)x^{\theta-1}e^{-(\beta x)^{\theta}}\left[1-p\ e^{-(\beta x)^{\theta}}\right]^{-2} \tag{1}$$

and

$$F(x) = \frac{1-e^{-(\beta x)^{\theta}}}{1-p\ e^{-(\beta x)^{\theta}}},\ x>0,\ \theta,\beta>0,\ p\in(0,1), \tag{2}$$

where β is the scale parameter, and p and θ are the shape parameters. The WG distribution contains the extended exponential-geometric (EEG), exponential-geometric (EG), and Weibull distributions as a special sub-models. When $p=0$ the WG distribution is the Weibull distribution; for $\theta=1$ and $0<p<1$, the WG distribution becomes the EG distribution; when $\theta=1$ for any $p<1$ the WG distribution is the EEG distribution.

It is easy to see that $f(x)$ and $F(x)$ satisfy the following relationship:

$$x^{-\theta+1}f(x) = \theta\beta^{\theta}\left[(1-F(x)) + \frac{p}{1-p}(1-F(x))^2\right]. \tag{3}$$

The k^{th} moments of the WG distribution in Equation (1) is given as:

$$E(X^k) = (1-p)\beta^{-k}\Gamma(k/\theta+1)\Phi(p,k/\theta,1), \tag{4}$$

where $\Phi(z,s,a) = \{\Gamma(s)\}^{-1}\int_0^{\infty} t^{s-1}e^{-at}(1-ze^{-t})^{-1}dt$, for $z<1, a, s>0$, is Lerch's transcendental function (see Erdelyi et al. [21]), which can be easily evaluated using Mathematica, for example. More details on this distribution and its properties can be found in Barreto-Souza et al. [18].

Hamedani and Ahsanullah [23] provided various characterizations of the WG distribution. Jodra and Jimnez-Gamero [26] derived explicit expressions for moments of order statistics from the half-logistic distribution, the WG distribution, and the long-term WG distribution. Athar et al. [4] derived some recurrence relations for single and product moments of generalized order statistics from WG distribution and presented some characterization results. Estimations and predictions of WG distribution have been studied by different

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†Department of Mathematics and Statistics, Taif University, Taif, Saudi Arabia (Corresponding author)
‡Department of Statistics, King Abdulaziz University, Jeddah, Saudi Arabia

authors; see for example, Elhag et al. [20] and Jaheen and Ali [24, 25]. Progressive censoring is a common censoring technique, especially in the reliability and survival analysis. It allows the experimenter to save time and cost, which is advantageous when the items being tested are costly. In progressive Type-II right censoring, suppose that n units are placed on a life-test with the progressive censoring scheme $(r_1, r_2, \cdots, r_m)$. At the first failure time X_1, r_1 units are randomly removed from the remaining $n-1$ surviving units. At the time of second failure X_2, r_2 units from the remaining $n-2-r_1$ units are randomly removed. Finally, at the m^{th} failure X_m, all remaining surviving units $r_m = n - m - \sum_{\nu=1}^{m-1} r_\nu$ are removed from the experiment. The values $(m; r_1, r_2, \cdots, r_m)$ are previously fixed and $m \le n$. The m ordered observed failure times denoted by $X_{1:m:n}^{(r_1,r_2,\cdots,r_m)}$, $X_{2:m:n}^{(r_1,r_2,\cdots,r_m)}$, $\cdots$, $X_{m:m:n}^{(r_1,r_2,\cdots,r_m)}$ are called the progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(r_1, r_2, \cdots, r_m)$. The joint PDF of this censored sample is given by (Balakrishnan and Sandhu [14] and Balakrishnan [5]):

$$f_{1,2,\cdots,m:m:n}(x_1, x_2, \cdots, x_m) = A \prod_{\nu=1}^{m} f(x_\nu)\,[1-F(x_\nu)]^{r_\nu} \quad \text{for } -\infty < x_1 < \cdots < x_m < \infty, \tag{5}$$

where $A \equiv A(n, m-1) = n(n-r_1-1)(n-r_1-r_2-2)...(n-r_1-r_2-...-r_{m-1}-m+1)$, with $A(n,0) = n$. It is noted that, if $r_1 = r_2 = ... = r_m = 0$, then $m = n$, the progressive Type-II right censoring reduces to usual order statistics. For more details on progressive censoring schemes, refer to the book by Balakrishnan and Aggarwala [6].

The marginal PDF of $X_{s:m:n}^{(r_1,\cdots,r_m)}$, $1 \le s \le m \le n$ is given by (Kamps and Cramer [27])

$$f_{X_{s:m:n}}(x) = f(x)\left(\prod_{i=1}^{s}\gamma_i\right)\sum_{j=1}^{s} a_{j,s}\,[1-F(x)]^{\gamma_j - 1}, \quad x \in \Re, \tag{6}$$

and the joint PDF of $X_{s:m:n}^{(r_1,\cdots,r_m)}$ and $X_{t:m:n}^{(r_1,\cdots,r_m)}$ $(1 \le s < t \le m \le n)$ is given by

$$f_{X_{s:m:n},X_{t:m:n}}(x,y) = f(x)f(y)\left(\prod_{i=1}^{t}\gamma_i\right)\sum_{j=1}^{s}\sum_{i=s+1}^{t} a_{j,s}\,a_{i,t}^{(s)}\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i - 1}(1-F(x))^{\gamma_j - 2}, \quad x < y, \tag{7}$$

where $\gamma_i = \sum_{\nu=i}^{m}(r_\nu + 1)$, $a_{j,s} = \prod_{\substack{i=1\\ i\neq j}}^{s}\frac{1}{\gamma_i - \gamma_j}$, $1 \le j \le s \le m$, and $a_{j,t}^{(s)} = \prod_{\substack{i=s+1\\ i\neq j}}^{t}\frac{1}{\gamma_i - \gamma_j}$, $s+1 \le j \le t$.

Over the past few decades, the moments of ordered random variables and the recurrence relations between them have garnered a lot of attention in the statistical literature. Several researchers have studied the moments based on order statistics, progressively Type-II censored order statistics, and generalized order statistics for various distributions. For example, see Balakrishnan and Aggarwala [6], Balakrishnan et al. [7], Balakrishnan and Saleh [10, 11, 12, 13], Balakrishnan et al. [16, 17], Athar and Akhter [3], Kumar and Dey [29], Dey et al. [19], Singh and Khan [36], Kumar et al. [30, 31, 32], Liu and Balakrishnan [33], Khan et al. [28], Shrahili et al. [35], Al-Zahrani and AL-Zaydi [1], among others.

In this paper, in Section 2, we present explicit expressions and some recurrence relations for single and product moments of progressively Type-II censored order statistics from the WG distribution. We provide Thomas-Wilson's mixture formula for moments in Section 3. In Section 4, we describe the recursive computational algorithm. In Section 5, we introduce the best linear unbiased estimators (BLUEs) of the scale and location-scale parameters. In Section 6, we present the best linear unbiased predictors (BLUPs) of the censored failure times. An illustrative example is given in Section 7. Finally, a conclusion is provided in Section 8.

2 Relations for Single and Product Moments of Progressively Type-II Censored Order Statistics

In this section, we obtain explicit expressions and some recurrence relations for single and product moments of progressively Type-II censored order statistics from the WG distribution.

2.1 Single Moments

Theorem 1 *For $1 \le s \le m \le n$ and $k > 0$,*

$$\alpha_{s:m:n}^{(r_1,\cdots,r_m)^{(k)}} = \frac{\Gamma\left(\frac{k}{\theta}+1\right)}{\beta^k}\left(\prod_{i=1}^{s}\gamma_i\right)\sum_{j=1}^{s}\sum_{\nu=0}^{\infty} a_{j,s}\frac{(1-p)^{\gamma_j}p^{\nu}}{(\gamma_j+\nu)^{\frac{k}{\theta}+1}}\binom{\gamma_j+\nu}{\nu}. \tag{8}$$

Proof. From Equation (6), we have

$$\alpha_{s:m:n}^{(r_1,\cdots,r_m)^{(k)}} = E\left[X_{s:m:n}^{(r_1,\cdots,r_m)^{(k)}}\right] = \left(\prod_{i=1}^{s}\gamma_i\right)\sum_{j=1}^{s} a_{j,s}\int_0^{\infty} x^k f(x)\,[1-F(x)]^{\gamma_j-1}\,dx. \tag{9}$$

Using Equations (1) and (2) in Equation (9), we obtain

$$\begin{aligned}\alpha_{s:m:n}^{(r_1,\cdots,r_m)^{(k)}} &= \left(\prod_{i=1}^{s}\gamma_i\right)\sum_{j=1}^{s} a_{j,s}\,\theta\,\beta^{\theta}(1-p)^{\gamma_j}\int_0^{\infty} x^{k+\theta-1}e^{-\gamma_j(\beta x)^{\theta}}\left(1-p\,e^{-(\beta x)^{\theta}}\right)^{-(\gamma_j+1)}dx \\ &= \left(\prod_{i=1}^{s}\gamma_i\right)\sum_{j=1}^{s}\sum_{\nu=0}^{\infty} a_{j,s}\,\theta\,\beta^{\theta}(1-p)^{\gamma_j}p^{\nu}\binom{\gamma_j+\nu}{\nu}I(x),\end{aligned} \tag{10}$$

where

$$I(x) = \int_0^{\infty} x^{k+\theta-1}e^{-(\gamma_j+\nu)(\beta x)^{\theta}}dx = \frac{1}{\theta\,\beta^{k+\theta}(\gamma_j+\nu)^{\frac{k}{\theta}+1}}\Gamma\left(\frac{k}{\theta}+1\right). \tag{11}$$

Upon substituting Equation (11) into Equation (10), and simplifying the resulting expression, we get the result of (8). ■

Theorem 2 *For $2 \le m \le n$ and $k \ge \theta - 1$,*

$$\begin{aligned}\alpha_{1:m:n+1}^{(r_1+1,\cdots,r_m)^{(k+1)}} &= \frac{(n+1)(1-p)}{np(r_1+2)}\left\{\frac{(k+1)}{\theta\beta^{\theta}}\alpha_{1:m:n}^{(r_1,\cdots,r_m)^{(k-\theta+1)}} - (n-r_1-1)\alpha_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+1)}}\right. \\ &\quad \left. -(r_1+1)\alpha_{1:m:n}^{(r_1,\cdots,r_m)^{(k+1)}}\right\} - \frac{(n-r_1-1)}{(r_1+2)}\alpha_{1:m-1:n+1}^{(r_1+r_2+2,\cdots,r_m)^{(k+1)}}.\end{aligned} \tag{12}$$

Proof. From Equation (5), we have

$$\begin{aligned}\alpha_{1:m:n}^{(r_1,r_2,\cdots,r_m)^{(k-\theta+1)}} &= E\left[X_{1:m:n}^{(r_1,\cdots,r_m)^{(k-\theta+1)}}\right] \\ &= A\underset{0<x_2<\cdots<x_m<\infty}{\int\int\cdots\int} I(x_2)\prod_{\nu=2}^{m} f(x_\nu)\,[1-F(x_\nu)]^{r_\nu}\,dx_\nu,\end{aligned} \tag{13}$$

where

$$I(x_2) = \int_0^{x_2} x_1^{k-\theta+1}f(x_1)\,[1-F(x_1)]^{r_1}\,dx_1.$$

Using Equation (3), we get

$$I(x_2) = \theta\beta^{\theta}\left\{\frac{p}{(1-p)}\int_0^{x_2} x_1^k\,[1-F(x_1)]^{r_1+2}\,dx_1 + \int_0^{x_2} x_1^k\,[1-F(x_1)]^{r_1+1}\,dx_1\right\}. \tag{14}$$

By integrating (14) by parts and then substituting into Equation (13), we have

$$\begin{aligned}\frac{1}{\theta\beta^{\theta}}\alpha_{1:m:n}^{(r_1,\cdots,r_m)^{(k-\theta+1)}} &= \frac{np}{(n+1)(1-p)(k+1)}\Bigg\{(n-r_1-1)\alpha_{1:m-1:n+1}^{(r_1+r_2+2,\cdots,r_m)^{(k+1)}} \\ &\quad +(r_1+2)\alpha_{1:m:n+1}^{(r_1+1,\cdots,r_m)^{(k+1)}}\Bigg\}+\frac{1}{(k+1)}\Bigg\{(n-r_1-1) \\ &\quad \times\alpha_{1:m-1:n}^{(r_1+r_2+1,\cdots,r_m)^{(k+1)}}+(r_1+1)\alpha_{1:m:n}^{(r_1,\cdots,r_m)^{(k+1)}}\Bigg\}. \qquad (15)\end{aligned}$$

Upon rearranging the above equation, we get the required result. ■

Theorem 3 *For $m=1, n=1,2,\cdots,$ and $k\geq\theta-1$,*

$$\alpha_{1:1:n+1}^{(n)^{(k+1)}}=\frac{(1-p)}{p}\Bigg\{\frac{(k+1)}{\theta\beta^{\theta}n}\alpha_{1:1:n}^{(n-1)^{(k-\theta+1)}}-\alpha_{1:1:n}^{(n-1)^{(k+1)}}\Bigg\}. \qquad (16)$$

Proof. From Equation (5), we have

$$\alpha_{1:1:n}^{(r_1)^{(k-\theta+1)}}=A(n,0)\int_0^{\infty}x_1^{k-\theta+1}f(x_1)\left[1-F(x_1)\right]^{r_1}dx_1,$$

where $r_1=n-1$. Using Equation (3), we get

$$\alpha_{1:1:n}^{(r_1)^{(k-\theta+1)}}=\theta\beta^{\theta}n\Bigg\{\frac{p}{(1-p)}\int_0^{\infty}x_1^{k}\left[1-F(x_1)\right]^{r_1+2}dx_1+\int_0^{\infty}x_1^{k}\left[1-F(x_1)\right]^{r_1+1}dx_1\Bigg\}. \qquad (17)$$

Integrating Equation (17) by parts yields,

$$\begin{aligned}\alpha_{1:1:n}^{(r_1)^{(k-\theta+1)}} &= \frac{\theta\beta^{\theta}n}{(k+1)}\Bigg\{\frac{p(r_1+2)}{(1-p)}\int_0^{\infty}x_1^{k+1}f(x_1)\left[1-F(x_1)\right]^{r_1+1}dx_1 \\ &\quad +(r_1+1)\int_0^{\infty}x_1^{k+1}f(x_1)\left[1-F(x_1)\right]^{r_1}dx_1\Bigg\} \\ &= \frac{\theta\beta^{\theta}n}{(k+1)}\Bigg\{\frac{p}{(1-p)}\alpha_{1:1:n+1}^{(r_1+1)^{(k+1)}}+\alpha_{1:1:n}^{(r_1)^{(k+1)}}\Bigg\}.\end{aligned}$$

which leads to (16). ■

Proceeding on similar lines, the following relations can easily be derived.

Theorem 4 *For $2\leq i\leq m-1, m\leq n$ and $k\geq\theta-1$,*

$$\begin{aligned}\alpha_{i:m:n+1}^{(r_1,\cdots,r_i+1,\cdots,r_m)^{(k+1)}} &= \frac{A(n+1,i-1)(1-p)}{(r_i+2)A(n,i-1)p}\Bigg\{\frac{(k+1)}{\theta\beta^{\theta}}\alpha_{i:m:n}^{(r_1,\cdots,r_m)^{(k-\theta+1)}} \\ &\quad -\frac{p}{(1-p)}\Bigg(\frac{A(n,i)}{A(n+1,i-1)}\,\alpha_{i:m-1:n+1}^{(r_1,\cdots,r_{i-1},r_i+r_{i+1}+2,r_{i+2},\cdots,r_m)^{(k+1)}} \\ &\quad -\frac{A(n,i-1)}{A(n+1,i-2)}\,\alpha_{i-1:m-1:n+1}^{(r_1,\cdots,r_{i-1}+r_i+2,r_{i+1},\cdots,r_m)^{(k+1)}}\Bigg) \\ &\quad -(n-r_1-\cdots-r_i-i)\,\alpha_{i:m-1:n}^{(r_1,\cdots,r_{i-1},r_i+r_{i+1}+1,r_{i+2},\cdots,r_m)^{(k+1)}} \\ &\quad +(n-r_1-\cdots-r_{i-1}-i+1)\alpha_{i-1:m-1:n}^{(r_1,\cdots,r_{i-1}+r_i+1,r_{i+1},\cdots,r_m)^{(k+1)}} \\ &\quad -(r_i+1)\alpha_{i:m:n}^{(r_1,\cdots,r_m)^{(k+1)}}\Bigg\}. \qquad (18)\end{aligned}$$

Theorem 5 *For $2 \leq m \leq n$ and $k \geq \theta - 1$,*

$$\begin{aligned} \alpha_{m:m:n+1}^{(r_1,\cdots,r_m+1)^{(k+1)}} &= \frac{A(n+1,m-1)}{(r_m+2)A(n,m-1)}\Bigg\{\frac{(1-p)(k+1)}{\theta\beta^\theta p}\alpha_{m:m:n}^{(r_1,\cdots,r_m)^{(k-\theta+1)}} \\ &+\frac{A(n,m-1)}{A(n+1,m-2)}\alpha_{m-1:m-1:n+1}^{(r_1,\cdots,r_{m-1}+r_m+2)^{(k+1)}} + \frac{(1-p)(r_m+1)}{p} \\ &\times\left(\alpha_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+1)}} - \alpha_{m:m:n}^{(r_1,\cdots,r_m)^{(k+1)}}\right)\Bigg\}. \end{aligned} \tag{19}$$

2.2 Product Moments

Theorem 6 *For $1 \leq s < t \leq m \leq n$ and $k, l > 0$,*

$$\begin{aligned} \alpha_{s,t:m:n}^{(r_1,\cdots,r_m)^{(k,l)}} &= \left(\prod_{i=1}^{t}\gamma_i\right)\sum_{j=1}^{s}\sum_{i=s+1}^{t}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} a_{j,s}\, a_{i,t}^{(s)}(1-p)^{\gamma_j}p^{\nu_1+\nu_2} \\ &\times\binom{\gamma_j-\gamma_i+\nu_1}{\nu_1}\binom{\gamma_i+\nu_2}{\nu_2}\frac{\Gamma(\frac{k+l}{\theta}+2)}{\beta^{k+l}(\frac{k}{\theta}+1)(\gamma_j+\nu_1+\nu_2)^{\frac{k+l}{\theta}+2}} \\ &\times{}_2F_1\left(1, \frac{k+l}{\theta}+2;\, \frac{k}{\theta}+2;\, \frac{\gamma_j-\gamma_i+\nu_1}{\gamma_j+\nu_1+\nu_2}\right), \end{aligned} \tag{20}$$

where ${}_2F_1(a,\, b;\, c;\, x)$ denotes the Gauss hypergeometric function defined by

$${}_2F_1(a,\, b;\, c;\, x) = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{x^j}{j!},$$

where $(e)_j = e(e+1)\cdots(e+j-1)$ denotes the ascending factorial.

Proof. From Equation (7), we have

$$\begin{aligned} &\alpha_{s,t:m:n}^{(r_1,\cdots,r_m)^{(k,l)}} \\ &= \left(\prod_{i=1}^{t}\gamma_i\right)\sum_{j=1}^{s}\sum_{i=s+1}^{t} a_{j,s}\, a_{i,t}^{(s)}\int_0^\infty\int_x^\infty x^k y^l f(x)f(y)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i-1}(1-F(x))^{\gamma_j-2}\,dydx. \end{aligned} \tag{21}$$

Using Equations (1) and (2) in Equation (21), we get

$$\begin{aligned} \alpha_{s,t:m:n}^{(r_1,\cdots,r_m)^{(k,l)}} &= \left(\prod_{i=1}^{t}\gamma_i\right)\sum_{j=1}^{s}\sum_{i=s+1}^{t}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} a_{j,s}\, a_{i,t}^{(s)}\theta^2\beta^{2\theta} \\ &\times(1-p)^{\gamma_j}p^{\nu_1+\nu_2}\binom{\gamma_j-\gamma_i+\nu_1}{\nu_1}\binom{\gamma_i+\nu_2}{\nu_2} \\ &\times\int_0^\infty x^{k+\theta-1}\, e^{-(\gamma_j-\gamma_i+\nu_1)(\beta x)^\theta} I(x)dx, \end{aligned} \tag{22}$$

where

$$I(x) = \int_x^\infty y^{l+\theta-1}\, e^{-(\gamma_i+\nu_2)(\beta y)^\theta}\,dy = \frac{\Gamma\left(\frac{l}{\theta}+1, (\gamma_i+\nu_2)(\beta x)^\theta\right)}{\theta\beta^{l+\theta}(\gamma_i+\nu_2)^{\frac{l}{\theta}+1}}.$$

On substituting the above expression of $I(x)$ in Equation (22), we find that

$$\alpha_{s,t:m:n}^{(r_1,\cdots,r_m)^{(k,l)}} = \left(\prod_{i=1}^{t}\gamma_i\right)\sum_{j=1}^{s}\sum_{i=s+1}^{t}\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} a_{j,s}\, a_{i,t}^{(s)} \frac{(1-p)^{\gamma_j} p^{\nu_1+\nu_2}}{\beta^{k+l}(\gamma_i+\nu_2)^{\frac{l}{\theta}+1}} \times \binom{\gamma_j-\gamma_i+\nu_1}{\nu_1}\binom{\gamma_i+\nu_2}{\nu_2}\int_0^{\infty} w^{\frac{k}{\theta}}\, e^{-(\gamma_j-\gamma_i+\nu_1)w} \times \Gamma\left(\frac{l}{\theta}+1,(\gamma_i+\nu_2)w\right)dw, \tag{23}$$

where $w=(\beta x)^{\theta}$. Now, using Equation (6.455.1) in Gradshteyn and Ryzhik [22] to calculate the integral in Equation (23), we get the result of (20). ■

Theorem 7 *For $1 \le i \le m-2$, and $m \le n$,*

$$\alpha_{i,i+1:m:n+1}^{(r_1,\cdots,r_{i+1}+1,\cdots,r_m)^{(k,l+1)}} = \frac{(1-p)A(n+1,i)}{p(r_{i+1}+2)A(n,i)}\Bigg\{\frac{(l+1)}{\theta\beta^{\theta}}\alpha_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,l-\theta+1)}} -(n-r_1-\cdots-r_{i+1}-i-1)\alpha_{i,i+1:m-1:n}^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,l+1)}} -\frac{pA(n,i+1)}{(1-p)A(n+1,i)}\alpha_{i,i+1:m-1:n+1}^{(r_1,\cdots,r_{i+1}+r_{i+2}+2,\cdots,r_m)^{(k,l+1)}} +\frac{pA(n,i)}{(1-p)A(n+1,i-1)}\alpha_{i:m-1:n+1}^{(r_1,\cdots,r_i+r_{i+1}+2,\cdots,r_m)^{(k+l+1)}} +(n-r_1-\cdots-r_i-i)\alpha_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+l+1)}} -(r_{i+1}+1)\alpha_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,l+1)}}\Bigg\}. \tag{24}$$

Proof. From Equation (5), we have

$$\alpha_{i,i+1:m:n}^{(r_1,r_2,\cdots,r_m)^{(k,l-\theta+1)}} = E\left[X_{i:m:n}^{(r_1,r_2,\cdots,r_m)^k} X_{i+1:m:n}^{(r_1,r_2,\cdots,r_m)^{l-\theta+1}}\right] = A \underset{0<x_1<\cdots<x_i<x_{i+2}<x_m<\infty}{\int\int\cdots\int} x_i^k I(x_{i+1}) f(x_1)\left[1-F(x_1)\right]^{r_1} \times\cdots\times f(x_i)\left[1-F(x_i)\right]^{r_i} f(x_{i+2})\left[1-F(x_{i+2})\right]^{r_{i+2}} \times\cdots\times f(x_m)\left[1-F(x_m)\right]^{r_m} dx_1\cdots dx_i dx_{i+2}\cdots dx_m, \tag{25}$$

where

$$I(x_{i+1}) = \int_{x_i}^{x_{i+2}} x_{i+1}^{l-\theta+1} f(x_{i+1})\left[1-F(x_{i+1})\right]^{r_{i+1}} dx_{i+1}.$$

Using Equation (3), we get

$$I(x_{i+1}) = \theta\beta^{\theta}\Bigg\{\frac{p}{(1-p)}\int_{x_i}^{x_{i+2}} x_{i+1}^{l}\left[1-F(x_{i+1})\right]^{r_{i+1}+2} dx_{i+1} + \int_{x_i}^{x_{i+2}} x_{i+1}^{l}\left[1-F(x_{i+1})\right]^{r_{i+1}+1} dx_{i+1}\Bigg\}. \tag{26}$$

Integrating Equation (26) by parts and then substituting into Equation (25), we find that

$$\begin{aligned}\alpha_{i,i+1:m:n}^{(r_1,r_2,\cdots,r_m)^{(k,l-\theta+1)}} &= \frac{\theta\beta^\theta}{(l+1)}\Bigg\{\frac{p(r_{i+1}+2)A(n,i)}{(1-p)A(n+1,i)}\alpha_{i,i+1:m:n+1}^{(r_1,\cdots,r_{i+1}+1,\cdots,r_m)^{(k,l+1)}}\\ &+(n-r_1-\cdots-r_{i+1}-i-1)\alpha_{i,i+1:m-1:n}^{(r_1,\cdots,r_{i+1}+r_{i+2}+1,\cdots,r_m)^{(k,l+1)}}\\ &-(n-r_1-\cdots-r_i-i)\alpha_{i:m-1:n}^{(r_1,\cdots,r_i+r_{i+1}+1,\cdots,r_m)^{(k+l+1)}}\\ &-\frac{pA(n,i)}{(1-p)A(n+1,i-1)}\alpha_{i:m-1:n+1}^{(r_1,\cdots,r_i+r_{i+1}+2,\cdots,r_m)^{(k+l+1)}}\\ &+\frac{pA(n,i+1)}{(1-p)A(n+1,i)}\alpha_{i,i+1:m-1:n+1}^{(r_1,\cdots,r_{i+1}+r_{i+2}+2,\cdots,r_m)^{(k,l+1)}}\\ &+(r_{i+1}+1)\alpha_{i,i+1:m:n}^{(r_1,\cdots,r_m)^{(k,l+1)}}\Bigg\}.\end{aligned}$$

Rearranging the terms, we obtain the required recurrence relation in (24). ■

Similarly, by using the relation in (3) and proceeding as in the proof of Theorem 7, we obtain the following recurrence relations.

Theorem 8 *For $1 \le i < j \le m-1$, $j-i \ge 2$, and $m \le n$,*

$$\begin{aligned}\alpha_{i,j:m:n+1}^{(r_1,\cdots,r_j+1,\cdots,r_m)^{(k,l+1)}} &= \frac{(1-p)A(n+1,j-1)}{p(r_j+2)A(n,j-1)}\Bigg\{\frac{(l+1)}{\theta\beta^\theta}\alpha_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,l-\theta+1)}}\\ &+(n-r_1-\cdots-r_{j-1}-j+1)\alpha_{i,j-1:m-1:n}^{(r_1,\cdots,r_{j-1}+r_j+1,\cdots,r_m)^{(k,l+1)}}\\ &+\frac{pA(n,j-1)}{(1-p)A(n+1,j-2)}\alpha_{i,j-1:m-1:n+1}^{(r_1,\cdots,r_{j-1}+r_j+2,\cdots,r_m)^{(k,l+1)}}\\ &-\frac{pA(n,j)}{(1-p)A(n+1,j-1)}\alpha_{i,j:m-1:n+1}^{(r_1,\cdots,r_j+r_{j+1}+2,\cdots,r_m)^{(k,l+1)}}\\ &-(n-r_1-\cdots-r_j-j)\alpha_{i,j:m-1:n}^{(r_1,\cdots,r_j+r_{j+1}+1,\cdots,r_m)^{(k,l+1)}}\\ &-(r_j+1)\alpha_{i,j:m:n}^{(r_1,\cdots,r_m)^{(k,l+1)}}\Bigg\}.\end{aligned}$$

Theorem 9 *For $1 \le i \le m-2$, and $m \le n$,*

$$\begin{aligned}\alpha_{i,m:m:n+1}^{(r_1,\cdots,r_m+1)^{(k,l+1)}} &= \frac{(1-p)A(n+1,m-1)}{p(r_m+2)A(n,m-1)}\Bigg\{\frac{(l+1)}{\theta\beta^\theta}\alpha_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,l-\theta+1)}}\\ &+(r_m+1)\left(\alpha_{i,m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k,l+1)}}-\alpha_{i,m:m:n}^{(r_1,\cdots,r_m)^{(k,l+1)}}\right)\\ &+\frac{pA(n,m-1)}{(1-p)A(n+1,m-2)}\alpha_{i,m-1:m-1:n+1}^{(r_1,\cdots,r_{m-1}+r_m+2)^{(k,l+1)}}\Bigg\}.\end{aligned}$$

Theorem 10 *For $2 \le m \le n$,*

$$\begin{aligned}\alpha_{m-1,m:m:n+1}^{(r_1,\cdots,r_m+1)^{(k,l+1)}} &= \frac{(1-p)A(n+1,m-1)}{p(r_m+2)A(n,m-1)}\Bigg\{\frac{(l+1)}{\theta\beta^\theta}\alpha_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,l-\theta+1)}}\\ &+(r_m+1)\left(\alpha_{m-1:m-1:n}^{(r_1,\cdots,r_{m-1}+r_m+1)^{(k+l+1)}}-\alpha_{m-1,m:m:n}^{(r_1,\cdots,r_m)^{(k,l+1)}}\right)\\ &+\frac{pA(n,m-1)}{(1-p)A(n+1,m-2)}\alpha_{m-1:m-1:n+1}^{(r_1,\cdots,r_{m-1}+r_m+2)^{(k+l+1)}}\Bigg\}.\end{aligned}$$

Remark 1 *Setting $r_i = 0, i = 1, 2, \cdots, m$, the relations given in Section 2 reduce to the corresponding recurrence relations based on the usual order statistics.*

Remark 2 *For $\theta = 1, \beta = 1$, we get recurrence relations for single and product moments of progressively Type-II censored order statistics from the EG distribution as obtained by Al-Zahrani and AL-Zaydi [1] and at $\theta = 2$, we get the relations for the Rayleigh-geometric distribution.*

Remark 3 *By letting $\theta = 1$ and $r_i = 0, i = 1, 2, \cdots, m$ in the relations given in Section 2, we deduce the recurrence relations given in Balakrishnan et al. [16] for the single and product moments of the usual order statistics from the EG distribution.*

3 Thomas-Wilson's Mixture Formula for Moments

Let $\mathbf{X}_u = (X_{1:n}, \cdots, X_{n:n})^{'}$ denote the $n \times 1$ vector of order statistics from a sample of size n and $\mathbf{X}_{ps} = (X_{1:m:n}, \cdots, X_{m:m:n})^{'}$ be the $m \times 1$ vector of progressively Type-II right censored order statistics with censoring scheme $(r_1, \cdots, r_m)$. Let K_j be the rank of progressively Type-II right censored order statistics, that is, $X_{j:m:n} = X_{K_j}$, where K_j can take on the values $K_{j-1}+1$, $K_{j-1}+2, \cdots, j+\sum_{\nu=1}^{j-1} r_\nu$, $j = 2, 3, \cdots, m$ and $K_1 = 1$. The joint probability function of the rank vector $(K_1, \cdots, K_m)$ is given by Thomas and Wilson [38]

$$P(K_1, \cdots, K_m) = P(K_1) \prod_{j=2}^{m} P(K_j|K_1, \cdots, K_{j-1}),$$

with $P(K_1) = 1$ and

$$P(K_j|K_1, \cdots, K_{j-1}) = \frac{\binom{n-K_j}{\sum_{\nu=1}^{j-1}(r_\nu + 1) - K_j + 1}}{\binom{n-K_{j-1}}{\sum_{\nu=1}^{j-1}(r_\nu + 1) - K_{j-1}}}, j = 2, \cdots, m.$$

Thus, if N is the number of all possible rank vectors of $(K_1, \cdots, K_m)$, then for each rank vector, we can define an $m \times n$ indicator matrix $\mathbf{D}_\ell$, $\ell = 1, \cdots, N$, whose $(i, j)^{th}$ element is one if $j = K_i$ and zero otherwise, so that $\mathbf{X}_{ps} = \mathbf{D}_\ell \mathbf{X}_u$ for some ℓ. Therefore, by denoting the mean vector of X_u by $\boldsymbol{\alpha}_u$ and the variance-covariance matrix of X_u by $\boldsymbol{\Sigma}_u$, the mean vector and variance-covariance matrix of the progressively Type-II right censored order statistics are obtained as:

$$\boldsymbol{\alpha} = EE(\mathbf{X}_{ps}|\mathbf{D}_\ell) = E(\mathbf{D}_\ell \boldsymbol{\alpha}_u) = \left(\sum_{\ell=1}^{N} \mathbf{D}_\ell \mathbf{P}_\ell \right) \boldsymbol{\alpha}_u, \tag{27}$$

$$\boldsymbol{\Sigma} = Var(\mathbf{X}_{ps}) = \sum_{\ell=1}^{N} \mathbf{D}_\ell \left(\boldsymbol{\Sigma}_u + \boldsymbol{\alpha}_u \boldsymbol{\alpha}_u^{'} \right) \mathbf{D}_\ell^{'} \mathbf{P}_\ell - \boldsymbol{\alpha}\boldsymbol{\alpha}^{'}, \tag{28}$$

where $\mathbf{P}_\ell$ is the probability function of the rank vector corresponding to $\mathbf{D}_\ell$ (for $\ell = 1, \cdots, N$). For further details of the Thomas-Wilson's mixture formula, see [6] and [9].

4 Computational Algorithm

Using the recurrence relations obtained in Section 2 in a recursive manner, along with the mixture formula for missing moments, one can calculate the moments of progressive Type II right censored order statistics from the WG distribution for all sample sizes and all censoring schemes. The algorithm for the case $\theta = 2$ is as follows: Setting $r_1 = r_2 = \cdots = r_m = 0$ in Equations (12) and (18), we obtain

$$\alpha_{1:n+1}^{(k+1)} = \frac{(1-p)}{p}\left\{\frac{(k+1)}{\theta\beta^{\theta}n}\alpha_{1:n}^{(k-\theta+1)} - \alpha_{1:n}^{(k+1)}\right\}, \quad k \geq \theta - 1, \ n = 1, 2, \cdots,$$

$$\begin{aligned}\alpha_{i:n+1}^{(k+1)} &= \frac{(1-p)(n+1)}{p(n-i+2)}\left\{\frac{(k+1)}{\theta\beta^{\theta}(n-i+1)}\alpha_{i:n}^{(k-\theta+1)} - \alpha_{i:n}^{(k+1)} + \alpha_{i-1:n}^{(k+1)}\right\} \\ &\quad + \alpha_{i-1:n+1}^{(k+1)}, \ k \geq \theta - 1, \ i = 1, 2, \cdots, n, \end{aligned} \tag{29}$$

where $\alpha_{1:1}^{(k+1)} = E(X^{k+1})$ is given by Equation (4) and $\alpha_{1:n}^{(0)} = 1$. The values of $\alpha_{i:n+1}$ (for all i and n) can be computed based on Equation (8). All the moments of the form $\alpha_{1:n+1}^{(2)}$ (for $n = 1, 2, \cdots$) can be determined by using Equation (29), and then $\alpha_{i:n+1}^{(2)}$ (for $i = 2, 3, \cdots n$) can be computed from Equation (29). The values of $\alpha_{n+1:n+1}^{(2)}$, $n = 1, 2, \cdots$, can be obtained by using the following relation (Balakrishnan and Sultan [15])

$$\alpha_{i:n+1}^{(k+1)} = \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1}\binom{j-1}{n-i}\binom{n}{j}\alpha_{1:j}^{(k+1)}.$$

Thus, all the first and second moments of usual order statistics for all sample sizes n will be obtained.

Now starting with $\alpha_{1:1:1}^{(0)} = \alpha_{1:1}^{(1)} = E(X)$ and $\alpha_{1:1:1}^{(0)^{(2)}} = \alpha_{1:1}^{(2)} = E(X^2)$, $\alpha_{1:1:n+1}^{(n)}$ can be computed by using Equation (8) and $\alpha_{1:1:n+1}^{(n)^{(2)}}$ can be determined by using Equation (15) for all sample sizes n. When $n = m = 2$, we have $\alpha_{i:2:2}^{(0,0)} = \alpha_{i:2}$ and $\alpha_{i:2:2}^{(0,0)^{(2)}} = \alpha_{i:2}^{(2)}$, $i = 1, 2$ and these values can be obtained by using the results of usual order statistics. When $n = 3$ and $m = 2$, we have two censoring schemes as $(r_1, r_2) = (1, 0)$ or $(0, 1)$. When $(r_1, r_2) = (1, 0)$, $\alpha_{1:2:3}^{(1,0)^{(2)}}$ can be determined using Equation (12), and $\alpha_{1:2:3}^{(1,0)}$, $\alpha_{2:2:3}^{(1,0)}$, $\alpha_{2:2:3}^{(1,0)^{(2)}}$ can be obtained from the mixture formula. When $(r_1, r_2) = (0, 1)$, mixture formula can be used to compute the moments of the form $\alpha_{1:2:3}^{(0,1)}$, $\alpha_{1:2:3}^{(0,1)^{(2)}}$, $\alpha_{2:2:3}^{(0,1)}$, and Equation (19) can be used to obtain $\alpha_{2:2:3}^{(0,1)^{(2)}}$. When $n = 3$ and $m = 3$, we have only one censoring scheme as $(r_1, r_2, r_3) = (0, 0, 0)$. In this situation, we have $\alpha_{i:3:3}^{(0,0,0)} = \alpha_{i:3}$ and $\alpha_{i:3:3}^{(0,0,0)^{(2)}} = \alpha_{i:3}^{(2)}$, $i = 1, 2, 3$ and these values can be evaluated by using the results of usual order statistics. When $n = 4$ and $m = 2$, we have three censoring schemes as $(r_1, r_2) = (2, 0)$, $(1, 1)$ or (0, 2). From Equation (12), we can obtain the moments of the form $\alpha_{1:2:4}^{(2,0)^{(2)}}$ for $r_1 = 2$ and $r_2 = 0$. Also, the moments of the form $\alpha_{1:2:4}^{(2,0)}$, $\alpha_{2:2:4}^{(2,0)}$, $\alpha_{2:2:4}^{(2,0)^{(2)}}$ can be obtained from the mixture formula. When $(r_1, r_2) = (1, 1)$, mixture formula can be used to determine the moments of the form $\alpha_{2:2:4}^{(1,1)}$, $\alpha_{1:2:4}^{(1,1)}$, Equation (12) can be used to compute $\alpha_{1:2:4}^{(1,1)^{(2)}}$ and Equation (19) can be used to determine $\alpha_{2:2:4}^{(1,1)^{(2)}}$. When $(r_1, r_2) = (0, 2)$, mixture formula can be used to obtain the moments of the form $\alpha_{1:2:4}^{(0,2)}$, $\alpha_{1:2:4}^{(0,2)^{(2)}}$, $\alpha_{2:2:4}^{(0,2)}$, and Equation (19) can be used to determine $\alpha_{2:2:4}^{(0,2)^{(2)}}$. When $n = 4$ and $m = 3$, we have three censoring schemes as $(r_1, r_2, r_3) = (1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$. When $(r_1, r_2, r_3) = (1, 0, 0)$, Equation (12) can be used to obtain $\alpha_{1:3:4}^{(1,0,0)^{(2)}}$ and mixture formula can be used to determine the moments of the forms $\alpha_{i:3:4}^{(1,0,0)}$ and $\alpha_{j:3:4}^{(1,0,0)^{(2)}}$, $i = 1, 2, 3$, $j = 2, 3$. When $(r_1, r_2, r_3) = (0, 1, 0)$, mixture formula can be used to compute the moments of the form $\alpha_{i:3:4}^{(0,1,0)}$, $\alpha_{j:3:4}^{(0,1,0)^{(2)}}$, $i = 1, 2, 3$, $j = 1, 3$ and Equation (18) can be used to compute the moments of the form $\alpha_{2:3:4}^{(0,1,0)^{(2)}}$. For $r_1 = r_2 = 0$ and $r_3 = 1$, $\alpha_{i:3:4}^{(0,0,1)}$ and $\alpha_{j:3:4}^{(0,0,1)^{(2)}}$, $i = 1, 2, 3$, $j = 1, 2$ can be determined from mixture formula and Equation (19) can be used to obtain $\alpha_{3:3:4}^{(0,0,1)^{(2)}}$. When $n = m = 4$, we have $\alpha_{i:4:4}^{(0,0,0,0)} = \alpha_{i:4}$ and $\alpha_{i:4:4}^{(0,0,0,0)^{(2)}} = \alpha_{i:4}^{(2)}$, $i = 1, ..., 4$ and these values can be determined by using the results of usual order statistics. This process may be followed similarly to obtain all the desired first and second order moments for all sample sizes and all censoring schemes.

Similarly, using the findings of Subsection 2.2, one can obtain all product moments for all sample sizes and all censoring schemes in a recursive manner.

5 BLUEs of the Scale and Location-Scale Parameters

Let $(Y_{1:m:n}, Y_{2:m:n}, \cdots, Y_{m:m:n})$ be a progressively Type-II right censored sample from the scaled WG distribution with PDF

$$h(y;\sigma) = \frac{1}{\sigma} f(\frac{y}{\sigma}), \;\; y \geq 0, \, \sigma > 0, \tag{30}$$

where σ is the scale parameter and $f(.)$ is the standard density. Let us denote $X_{i:m:n} = Y_{i:m:n}/\sigma$, $E(X_{i:m:n}) = \alpha_i$ and $Cov(X_{i:m:n}, X_{j:m:n}) = \sigma_{ij}$, $1 \leq i, \; j \leq m$. Then, the BLUE of σ and its variance are (see [2] and [8])

$$\sigma^* = \frac{\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1}}{\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha}} \mathbf{Y} = \sum_{i=1}^{m} a_i Y_{i:m:n}, \;\; \text{and} \;\; Var(\sigma^*) = \frac{\sigma^2}{\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha}},$$

where $\mathbf{Y}' = (Y_{1:m:n}, Y_{2:m:n}, \cdots, Y_{m:m:n})$, $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\boldsymbol{\Sigma} = ((\sigma_{ij}))$, $1 \leq i, \; j \leq m$. Table 1 presents the coefficients of the BLUE σ^* and the values of $\mathrm{Var}(\sigma^*)/\sigma^2$ for sample size $n = 10$ with $\beta = 0.5$, $\theta = 2$, $p = 0.10$, 0.25, 0.50 and 0.75 and different progressive censoring schemes.

Now, let $(Y_{1:m:n}, Y_{2:m:n}, \cdots, Y_{m:m:n})$ be a progressively Type-II right censored sample from the location-scale WG distribution with PDF

$$h(y;\mu,\sigma) = \frac{1}{\sigma} f(\frac{y-\mu}{\sigma}), \; y \geq \mu, \, \sigma > 0, \tag{31}$$

where μ and σ are the location and scale parameters, respectively. Let us denote $X_{i:m:n} = (Y_{i:m:n} - \mu)/\sigma$, $E(X_{i:m:n}) = \alpha_i$ and $Cov(X_{i:m:n}, X_{j:m:n}) = \sigma_{ij}$, $1 \leq i, \; j \leq m$. Then, the BLUEs of μ and σ are (see [2])

$$\mu^* = \left\{ \frac{\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha} \, \mathbf{1}' \, \boldsymbol{\Sigma}^{-1} - \boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1} \, \boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha})(\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}) - (\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1})^2} \right\} \mathbf{Y} = \sum_{i=1}^{m} b_i Y_{i:m:n},$$

$$\sigma^* = \left\{ \frac{\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1} \, \boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} - \mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha} \, \mathbf{1}' \, \boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha})(\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}) - (\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1})^2} \right\} \mathbf{Y} = \sum_{i=1}^{m} c_i Y_{i:m:n},$$

and the variances and covariance of these BLUEs are

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha}}{(\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha})(\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}) - (\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1})^2} \right\},$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}}{(\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha})(\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}) - (\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1})^2} \right\},$$

and

$$Cov(\mu^*, \sigma^*) = \sigma^2 \left\{ \frac{-\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}}{(\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\alpha})(\mathbf{1}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1}) - (\boldsymbol{\alpha}' \, \boldsymbol{\Sigma}^{-1} \, \mathbf{1})^2} \right\},$$

where $\mathbf{Y}' = (Y_{1:m:n}, Y_{2:m:n}, \cdots, Y_{m:m:n})$, $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, $\boldsymbol{\Sigma} = ((\sigma_{ij}))$, $1 \leq i, \; j \leq m$ and $\mathbf{1}' = (1, 1, \cdots, 1)_{m \times 1}$. Table 2 displays the coefficients of BLUEs for μ and σ and the values $\mathrm{Var}(\mu^*)/\sigma^2$, $\mathrm{Var}(\sigma^*)/\sigma^2$ and $\mathrm{Cov}(\mu^*, \sigma^*)/\sigma^2$ for sample size $n = 10$ with $\beta = 0.5$, $\theta = 2$, $p = 0.10$, 0.25, 0.50 and 0.75 and different progressive censoring schemes. Using the conditions $\sum_{i=1}^{m} b_i = 1$ and $\sum_{i=1}^{m} c_i = 0$, the coefficients of the BLUEs μ^* and σ^* in Table 2 are verified.

Table 1: Coefficients of the BLUE σ^* and the values of $\text{Var}(\sigma^*)/\sigma^2$ for some selected progressive censoring schemes for the scaled WG distribution.

p	m, n	Scheme	Coefficients(a_i)				$\frac{Var(\sigma^*)}{\sigma^2}$
0.10	2, 10	8, 0	0.8152,	0.3055			0.1384
0.10	2, 10	0, 8	0.0824,	1.1599			0.1342
0.25	2, 10	8, 0	0.9264,	0.3118			0.1455
0.25	2, 10	0, 8	0.1016,	1.2512			0.1382
0.50	2, 10	8, 0	1.2286,	0.3268			0.1610
0.50	2, 10	0, 8	0.1515,	1.4895			0.1456
0.75	2, 10	8, 0	1.9657,	0.3560			0.1853
0.75	2, 10	0, 8	0.2617,	2.0379			0.1547
0.10	3, 10	7, 0, 0	0.4747,	0.1392,	0.2388		0.0912
0.10	3, 10	0, 0, 7	0.0567,	0.0809,	0.8487		0.0888
0.25	3, 10	7, 0, 0	0.5496,	0.1512,	0.2417		0.0973
0.25	3, 10	0, 0, 7	0.0707,	0.0970,	0.9058		0.0925
0.50	3, 10	7, 0, 0	0.7582,	0.1840,	0.2475		0.1112
0.50	3, 10	0, 0, 7	0.1077,	0.1400,	1.0546		0.0999
0.75	3, 10	7, 0, 0	1.2857,	0.2644,	0.2555		0.1345
0.75	3, 10	0, 0, 7	0.1912,	0.2401,	1.3977		0.1093
0.10	4, 10	6, 3*0	0.3077,	0.0917,	0.1244,	0.1951	0.0679
0.10	4, 10	3*0, 6	0.0433,	0.0619,	0.0757,	0.6570	0.0665
0.25	4, 10	6, 3*0	0.3601,	0.1044,	0.1299,	0.1976	0.0730
0.25	4, 10	3*0, 6	0.0546,	0.0751,	0.0887,	0.6933	0.0701
0.50	4, 10	6, 3*0	0.5083,	0.1403,	0.1449,	0.2019	0.0849
0.50	4, 10	3*0, 6	0.0849,	0.1107,	0.1229,	0.7880	0.0774
0.75	4, 10	6, 3*0	0.8911,	0.2342,	0.1819,	0.2053	0.1055
0.75	4, 10	3*0, 6	0.1549,	0.1955,	0.2008,	1.0079	0.0872
0.10	5, 10	5, 4*0	0.2091, 0.1650	0.0673,	0.0868,	0.1107,	0.0540
0.10	5, 10	4*0, 5	0.0351, 0.5182	0.0502,	0.0618,	0.0724,	0.0532
0.25	5, 10	5, 4*0	0.2467, 0.1677	0.0789,	0.0945,	0.1132,	0.0584
0.25	5, 10	4*0, 5	0.0446, 0.5420	0.0615,	0.0728,	0.0813,	0.0567
0.50	5, 10	5, 4*0	0.3541, 0.1720	0.1124,	0.1163,	0.1202,	0.0686
0.50	5, 10	4*0, 5	0.0708, 0.6009	0.0924,	0.1029,	0.1072,	0.0640
0.75	5, 10	5, 4*0	0.6353, 0.1744	0.2024,	0.1733,	0.1379,	0.0868
0.75	5, 10	4*0, 5	0.1330, 0.7320	0.1684,	0.1728,	0.1745,	0.0744

Table 1: (continued)

p	m, n	Scheme	Coefficients(a_i)				$\frac{Var(\sigma^*)}{\sigma^2}$
0.10	6, 10	4, 5*0	0.1443,	0.0524,	0.0660,	0.0806,	0.0448
			0.0992,	0.1433			
0.10	6, 10	5*0, 4	0.0295,	0.0422,	0.0521,	0.0608,	0.0444
			0.0682,	0.4105			
0.25	6, 10	4, 5*0	0.1715,	0.0627,	0.0741,	0.0853,	0.0486
			0.1004,	0.1460			
0.25	6, 10	5*0, 4	0.0378,	0.0521,	0.0618,	0.0692,	0.0478
			0.0753,	0.4247			
0.50	6, 10	4, 5*0	0.2497,	0.0927,	0.0977,	0.0986,	0.0576
			0.1036,	0.1507			
0.50	6, 10	5*0, 4	0.0611,	0.0799,	0.0890,	0.0929,	0.0550
			0.0938,	0.4597			
0.75	6, 10	4, 5*0	0.4568,	0.1745,	0.1609,	0.1339,	0.0737
			0.1119,	0.1532			
0.75	6, 10	5*0, 4	0.1182,	0.1500,	0.1536,	0.1560,	0.0659
			0.1378,	0.5351			
0.10	7, 10	3, 6*0	0.0986,	0.0425,	0.0527,	0.0631,	0.0383
			0.0746,	0.0897,	0.1268		
0.10	7, 10	6*0, 3	0.0254,	0.0364,	0.0450,	0.0526,	0.0382
			0.0594,	0.0669,	0.3194		
0.25	7, 10	3, 6*0	0.1181,	0.0515,	0.0607,	0.0687	0.0416
			0.0774,	0.0904,	0.1297		
0.25	7, 10	6*0, 3	0.0328,	0.0453,	0.0538,	0.0603,	0.0413
			0.0657,	0.0709,	0.3284		
0.50	7, 10	3, 6*0	0.1746,	0.0782,	0.0838,	0.0850	0.0496
			0.0857,	0.0918,	0.1346		
0.50	7, 10	6*0, 3	0.0539,	0.0705,	0.0787,	0.0822,	0.0484
			0.0832,	0.0829,	0.3480		
0.75	7, 10	3, 6*0	0.3256,	0.1515,	0.1467,	0.1292,	0.0640
			0.1080,	0.0951,	0.1374		
0.75	7, 10	6*0, 3	0.1074,	0.1364,	0.1394,	0.1423,	0.0598
			0.1252,	0.1157,	0.3837		

Table 2: Coefficients of BLUEs for μ and σ and the values of $\mathrm{Var}(\mu^*)/\sigma^2$, $\mathrm{Var}(\sigma^*)/\sigma^2$ and $\mathrm{Cov}(\mu^*, \sigma^*)/\sigma^2$ for some selected progressive censoring schemes for the location-scale WG distribution.

p	m, n	Scheme	Coefficients (b_i)	Coefficients (c_i)	$\frac{Var(\mu^*)}{\sigma^2}$	$\frac{Var(\sigma^*)}{\sigma^2}$	$\frac{Cov(\mu^*,\sigma^*)}{\sigma^2}$
0.10	2, 10	$8, 0$	1.4054, -0.4054	-0.7598, 0.7598	0.2425	0.4429	-0.2717
0.10	2, 10	$0, 8$	2.8358, -1.8358	-3.4405, 3.4405	0.3553	0.6826	-0.4414
0.25	2, 10	$8, 0$	1.3880, -0.3880	-0.7923, 0.7923	0.2090	0.4660	-0.2588
0.25	2, 10	$0, 8$	2.8159, -1.8159	-3.7079, 3.7079	0.2999	0.6870	-0.4057
0.50	2, 10	$8, 0$	1.3543, -0.3543	-0.8779, 0.8779	0.1502	0.5244	-0.2337
0.50	2, 10	$0, 8$	2.7807, -1.7807	-4.4117, 4.4117	0.2045	0.6963	-0.3356
0.75	2, 10	$8, 0$	1.3103, -0.3103	-1.0765, 1.0765	0.0857	0.6471	-0.1989
0.75	2, 10	$0, 8$	2.7423, -1.7423	-6.0444, 6.0444	0.1049	0.7091	-0.2411
0.10	3, 10	$7, 0, 0$	1.3212, -0.0442, -0.2770	-0.6519, 0.1769, 0.4750	0.1749	0.2184	-0.1491
0.10	3, 10	$0, 0, 7$	1.9089, 0.2146, -1.1235	-1.8260, -0.1307, 1.9568	0.2349	0.3173	-0.2317
0.25	3, 10	$7, 0, 0$	1.3114, -0.0524, -0.2590	-0.6865, 0.2006, 0.4859	0.1497	0.2304	-0.1411
0.25	3, 10	$0, 0, 7$	1.8990, 0.2001, -1.0990	-1.9679, -0.1178, 2.0857	0.1982	0.3210	-0.2128
0.50	3, 10	$7, 0, 0$	1.2940, -0.0696, -0.2243	-0.7812, 0.2668, 0.5144	0.1058	0.2610	-0.1259
0.50	3, 10	$0, 0, 7$	1.8820, 0.1729, -1.0550	-2.3432, -0.0852, 2.4284	0.1352	0.3291	-0.1760
0.75	3, 10	$7, 0, 0$	1.2762, -0.0977, -0.1784	-1.0186, 0.4409, 0.5777	0.0585	0.3253	-0.1057
0.75	3, 10	$0, 0, 7$	1.8648, 0.1406, -1.0054	-3.2194, -0.0170, 3.2364	0.0694	0.3413	-0.1269
0.10	4, 10	$6, 3*0$	1.2627, 0.0392, -0.0795, -0.2224	-0.6000, 0.0635, 0.1815, 0.3550	0.1521	0.1465	-0.1094
0.10	4, 10	$3*0, 6$	1.5666, 0.1983, 0.0686, -0.8334	-1.2693, -0.1043, 0.0182, 1.3554	0.1907	0.2004	-0.1598
0.25	4, 10	$6, 3*0$	1.2569, 0.0294, -0.0808, -0.2055	-0.6353, 0.0811, 0.1938, 0.3603	0.1297	0.1543	-0.1027
0.25	4, 10	$3*0, 6$	1.5608, 0.1879, 0.0564, -0.8051	-1.3683, -0.0962, 0.0372, 1.4273	0.1610	0.2039	-0.1467
0.50	4, 10	$6, 3*0$	1.2477, 0.0090, -0.0834, -0.1733	-0.7336, 0.1313, 0.2279, 0.3744	0.0908	0.1749	-0.0904
0.50	4, 10	$3*0, 6$	1.5511, 0.1686, 0.0346, -0.7543	-1.6312, -0.0759, 0.0846, 1.6226	0.1098	0.2118	-0.1215
0.75	4, 10	$6, 3*0$	1.2430, -0.0234, -0.0882, -0.1313	-0.9889, 0.2696, 0.3153, 0.4039	0.0493	0.2183	-0.0746
0.75	4, 10	$3*0, 6$	1.5425, 0.1452, 0.0082, -0.6959	-2.2500, -0.0309, 0.1881, 2.0928	0.0564	0.2244	-0.0880
0.10	5, 10	$5, 4*0$	1.2116, 0.0808, -0.0154, -0.0851, -0.1919	-0.5649, 0.0156, 0.0967, 0.1650, 0.2876	0.1403	0.1113	-0.0896
0.10	5, 10	$4*0, 5$	1.3789, 0.1879, 0.0774, 0.0096, -0.6538	-0.9822, -0.0884, 0.0047, 0.0653, 1.0005	0.1666	0.1439	-0.1229
0.25	5, 10	$5, 4*0$	1.2084, 0.0721, -0.0218, -0.0828, -0.1759	-0.6003, 0.0283, 0.1098, 0.1712, 0.2910	0.1193	0.1170	-0.0837
0.25	5, 10	$4*0, 5$	1.3753, 0.1798, 0.0676, 0.0014, -0.6242	-1.0585, -0.0828, 0.0186, 0.0801, 1.0426	0.1407	0.1472	-0.1128
0.50	5, 10	$5, 4*0$	1.2043, 0.0544, -0.0344, -0.0784, -0.1459	-0.6996, 0.0648, 0.1464, 0.1888, 0.2997	0.0831	0.1322	-0.0727
0.50	5, 10	$4*0, 5$	1.3702, 0.1648, 0.0493, -0.0149, -0.5693	-1.2641, -0.0681, 0.0549, 0.1218, 1.1555	0.0961	0.1552	-0.0936

Table 2: (continued)

p	m, n	Scheme	Coefficients (b_i)				Coefficients (c_i)				$\frac{Var(\mu^*)}{\sigma^2}$	$\frac{Var(\sigma^*)}{\sigma^2}$	$\frac{Cov(\mu^*,\sigma^*)}{\sigma^2}$
0.75	5, 10	$5, 4*0$	1.2071,	0.0269,	-0.0535,	-0.0731,	-0.9622,	0.1668,	0.2441,	0.2347,	0.0446	0.1648	-0.0590
			-0.1074				0.3166						
0.75	5, 10	$4*0, 5$	1.3679,	0.1460,	0.0265,	-0.0358,	-1.7556,	-0.0332,	0.1362,	0.2240,	0.0495	0.1688	-0.0683
			-0.5046				1.4287						
0.10	6, 10	$4, 5*0$	1.1638,	0.1070,	0.0211,	-0.0349,	-0.5374,	-0.0103,	0.0537,	0.1011,	0.1328	0.0904	-0.0778
			-0.0848,	-0.1722			0.1488,	0.2442					
0.10	6, 10	$5*0, 4$	1.2556,	0.1803,	0.0824,	0.0235,	-0.8034,	-0.0774,	-0.0026,	0.0452,	0.1508	0.1108	-0.1001
			-0.0194,	-0.5224			0.0811,	0.7571					
0.25	6, 10	$4, 5*0$	1.1626,	0.0998,	0.0135,	-0.0383,	-0.5726,	-0.0011,	0.0655,	0.1098,	0.1128	0.0948	-0.0722
			-0.0805,	-0.1571			0.1519,	0.2466					
0.25	6, 10	$5*0, 4$	1.2541,	0.1737,	0.0739,	0.0152,	-0.8663,	-0.0731,	0.0085,	0.0582,	0.1274	0.1140	-0.0919
			-0.0256,	-0.4914			0.0938,	0.7789					
0.50	6, 10	$4, 5*0$	1.1621,	0.0852,	-0.0013,	-0.0447,	-0.6719,	0.0252,	0.0987,	0.1341,	0.0782	0.1067	-0.0620
			-0.0725,	-0.1288			0.1611,	0.2528					
0.50	6, 10	$5*0, 4$	1.2526,	0.1614,	0.0578,	-0.0000,	-1.0368,	-0.0616,	0.0383,	0.0930,	0.0872	0.1220	-0.0764
			-0.0365,	-0.4353			0.1258,	0.8413					
0.75	6, 10	$4, 5*0$	1.1698,	0.0632,	-0.0233,	-0.0542,	-0.9367,	0.0993,	0.1887,	0.1984,	0.0416	0.1327	-0.0496
			-0.0625,	-0.0929			0.1863,	0.2639					
0.75	6, 10	$5*0, 4$	1.2556,	0.1456,	0.0374,	-0.0199,	-1.4521,	-0.0322,	0.1068,	0.1809,	0.0451	0.1364	-0.0564
			-0.0505,	-0.3682			0.2009,	0.9956					
0.10	7, 10	$3, 6*0$	1.1181,	0.1256,	0.0456,	-0.0041,	-0.5141,	-0.0264,	0.0278,	0.0654,	0.1274	0.0766	-0.0698
			-0.0435,	-0.0831,	-0.1586		0.0984,	0.1353,	0.2137				
0.10	7, 10	$6*0, 3$	1.1664,	0.1742,	0.0854,	0.0323,	-0.6804,	-0.0690,	-0.0067,	0.0331,	0.1395	0.0892	-0.0844
			-0.0065,	-0.0386,	-0.4131		0.0633,	0.0903,	0.5694				
0.25	7, 10	$3, 6*0$	1.1184,	0.1196,	0.0381,	-0.0097,	-0.5490,	-0.0198,	0.0379,	0.0745,	0.1081	0.0801	-0.0645
			-0.0447,	-0.0777,	-0.1440		0.1041,	0.1367,	0.2156				
0.25	7, 10	$6*0, 3$	1.1663,	0.1688,	0.0779,	0.0246,	-0.7336,	-0.0656,	0.0026,	0.0441,	0.1179	0.0922	-0.0775
			-0.0125,	-0.0412,	-0.3838		0.0739,	0.0979,	0.5806				
0.50	7, 10	$3, 6*0$	1.1207,	0.1078,	0.0236,	-0.0206,	-0.6476,	-0.0009,	0.0665,	0.1001,	0.0747	0.0898	-0.0548
			-0.0470,	-0.0676,	-0.1170		0.1202,	0.1414,	0.2204				
0.50	7, 10	$6*0, 3$	1.1680,	0.1585,	0.0634,	0.0101,	-0.8798,	-0.0562,	0.02800,	0.0741,	0.0808	0.1001	-0.0646
			-0.0237,	-0.0465,	-0.3297		0.1021,	0.1201,	0.6117				
0.75	7, 10	$3, 6*0$	1.1318,	0.0904,	0.0025,	-0.0360,	-0.9121,	0.0526,	0.1440,	0.1686,	0.0395	0.1113	-0.0432
			-0.0504,	-0.0551,	-0.0832		0.1631,	0.1554,	0.2284				
0.75	7, 10	$6*0, 3$	1.1760,	0.1449,	0.0446,	-0.0090,	-1.2449,	-0.0302,	0.0881,	0.1526,	0.0419	0.1152	-0.0482
			-0.0383,	-0.0538,	-0.2643		0.1693,	0.1775,	0.6876				

6 Best Linear Unbiased Predictors

Let $Y_{1:m:n}^{(r_1,\dots,r_m)}, Y_{2:m:n}^{(r_1,\dots,r_m)}, \dots, Y_{m:m:n}^{(r_1,\dots,r_m)}$ is a progressively Type-II right censored sample observed from the scaled WG distribution in (30), and let $X_{i:m:n} = Y_{i:m:n}/\sigma$, $i = 1, 2, \dots, m$, be the corresponding progressively Type-II right censored order sample from the standard distribution. Let us denote the mean vector of

$$\mathbf{X}' = (X_{1:m+1:n}^{(r_1,\dots,r_{m-1},0,r_m-1)}, X_{2:m+1:n}^{(r_1,\dots,r_{m-1},0,r_m-1)}, \dots, X_{m:m+1:n}^{(r_1,\dots,r_{m-1},0,r_m-1)})$$

by $\boldsymbol{\alpha}' = (\alpha_{1:m+1:n}, \dots, \alpha_{m:m+1:n})$ and let us further denote the variance covariance matrix of $\mathbf{X}$ by $\boldsymbol{\Sigma} = ((\sigma_{i,j:m+1:n}))$, $i, j = 1, \dots, m$, where $\alpha_{i:m+1:n}$ and $\sigma_{i,j:m+1:n}$ denoting the means and covariances of the progressively Type-II right censored order statistics from the corresponding standardized distribution. Based on the observed progressive Type-II right censored sample, the BLUP of $Y_{m+1:m+1:n}^{(r_1,\dots,r_{m-1},0,r_m-1)}$ is given by

$$Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)^*} = \boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \frac{(\alpha_{m+1:m+1:n} - \boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}},$$

and its variance is given by

$$\sigma^2\left\{\sigma_{m+1,m+1:m+1:n} - \boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega} + \frac{(\alpha_{m+1:m+1:n} - \boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})^2}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}\right\},$$

where

$$\mathbf{Y}' = (Y_{1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}, \cdots, Y_{m:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}),$$

$$E(\mathbf{Y}) = \sigma\boldsymbol{\alpha},\ Var(\mathbf{Y}) = \sigma^2\boldsymbol{\Sigma},$$

$$E(Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}) = \sigma\alpha_{m+1:m+1:n},$$

$$Var(Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}) = \sigma^2\sigma_{m+1,m+1:m+1:n},$$

and

$$\boldsymbol{\omega}' = (\sigma_{m+1,1:m+1:n}, \cdots, \sigma_{m+1,m:m+1:n}),$$

with $\sigma_{i,m+1:m+1:n} = Cov(Y_{i:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}, Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)})$, $i = 1, 2, \cdots, m$.

Now, suppose $Y_{1:m:n}^{(r_1,\cdots,r_m)}, Y_{2:m:n}^{(r_1,\cdots,r_m)}, \cdots, Y_{m:m:n}^{(r_1,\cdots,r_m)}$ be a progressively Type-II right censored sample observed from location-scale WG distribution in (31), and let $X_{i:m:n} = (Y_{i:m:n} - \mu)/\sigma$, $i = 1, 2, \cdots, m$, be the corresponding progressively Type II right censored order sample from the standard distribution. Based on the observed progressive right censored sample $Y_{1:m+1:n}^{(r_1,\dots,r_{m-1},0,r_m-1)}, \dots, Y_{m:m+1:n}^{(r_1,\dots,r_{m-1},0,r_m-1)}$, the BLUP of $Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}$ is given by

$$Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)^*} = \mu^* + \alpha_{m+1:m+1:n}\sigma^* + \boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mu^*\mathbf{1} - \sigma^*\boldsymbol{\alpha}),$$

and its variance is given by

$$\sigma^2\left\{\sigma_{m+1,m+1:m+1:n} - \boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega} + \lambda_1^2\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} + \lambda_2^2\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha} + 2\lambda_1\lambda_2\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\right\},$$

where

$$\mathbf{Y}' = (Y_{1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}, \cdots, Y_{m:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}),$$

$$E(\mathbf{Y}) = \mu\mathbf{1} + \sigma\boldsymbol{\alpha},\ Var(\mathbf{Y}) = \sigma^2\boldsymbol{\Sigma},$$

$$E(Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}) = \mu + \sigma\alpha_{m+1:m+1:n},$$

$$Var(Y_{m+1:m+1:n}^{(r_1,\cdots,r_{m-1},0,r_m-1)}) = \sigma^2\sigma_{m+1,m+1:m+1:n},$$

$$\lambda_1 = \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha} - \boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\mathbf{1} - \alpha_{m+1:m+1:n}\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1} + \boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2},$$

$$\lambda_2 = \frac{-\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1} + \boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\mathbf{1} + \alpha_{m+1:m+1:n}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} - \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2},$$

where μ^* and σ^* are the BLUEs of μ and σ based on the progressively Type-II censored sample $\mathbf{Y}$ (see Balakrishnan and Aggarwala, [6]).

7 Illustrative Example

In this section, we consider the data set represent the fatigue life (rounded to the closest thousand cycles) for 67 specimens of Alloy T7987 that failed before having accumulated 300,000 cycles of testing (Meeker and Escobar, [34], p. 149). Barreto-Souza et al. [18] indicated that the WG distribution provides the best fit to this data when compared to the EEG and Weibull distributions.
For illustrative purposes, a random sample of size 12 is selected as: 94, 96, 99, 99, 104, 108, 112, 114, 117, 117, 118, 121. Assuming the data come from the WG distribution with $\theta = 2$, $\beta = 0.50$ and $p = 0.80$, we generate a progressively censored sample using the censoring scheme $r = (2, 0, 0, 0, 0, 4)$ from the above data with $n = 12$ and $m = 6$. The generated progressively censored sample is given in Table 3. Then, we plotted the progressively Type-II censored data $y_{i:6:12}$ against the values $\alpha_{i:6:12}$ for $i = 1, 2, \cdots, 6$ obtained by the computational algorithm in Section 4 as (0.2342, 0.3903, 0.5203, 0.6557, 0.7921, 0.9513). We also determined the correlation coefficient and found the p-value of this correlation coefficient using Monte Carlo simulations. It is observed that the correlation coefficient is 0.9725 with a corresponding p-value=0.6509 . This indicates that the WG model with $\theta = 2$, $\beta = 0.50$ and $p = 0.80$ is a good fit for these data.

Table 3: Generated progressively Type-II censored sample from data of Meeker and Escobar [34].

i	1	2	3	4	5	6
$y_{i:6:12}$	94	99	99	108	112	114
r_i	2	0	0	0	0	4

Table 4: BLUPs of $y_{m:m:n}$ and their standard errors for the location-scale WG distribution

m	Scheme	$y_{m:m:n}$	$SE(y_{m:m:n})$
7	2, 5*0 , 3	119.44	6.1339
8	2, 6*0 , 2	152.049	7.8662
9	2, 7*0 , 1	194.372	11.2509
10	2, 8*0 , 0	254.742	19.9608

Based on the above progressive Type II censored sample, we get the BLUEs of μ and σ to be

$$\begin{aligned} \mu^* &= (1.2377 \times 94) + (0.1118 \times 99) + (0.0461 \times 99) \\ &\quad + (-0.0425 \times 108) + (-0.0395 \times 112) + (-0.3136 \times 114) \\ &= 87.2115, \end{aligned}$$

and

$$\begin{aligned}\sigma^* &= (-1.6980 \times 94) + (0.3253 \times 99) + (-0.3765 \times 99) \\ &\quad +(0.6838 \times 108) + (-0.0297 \times 112) + (1.0951 \times 114) \\ &= 30.6850,\end{aligned}$$

and their standard errors are $SE(\mu^*) = 5.2411$ and $SE(\sigma^*) = 11.0976$. From Equation (32), the BLUPs of $y_{m:m:n}$, $m = 7, ..., 10$ and their standard errors were obtained (by taking $\boldsymbol{\omega}' = (\sigma_{m+1,1:m+1:n}, ..., \sigma_{m+1,m:m+1:n})$ and the progressive censoring scheme as $(r_1, ..., r_{m-1}, 0, r_m)$), and these results are given in Table 4.

8 Conclusion

In this paper, we have established the explicit expressions and some recurrence relations for both single and product moments of progressively Type-II right censored order statistics from the WG distribution. By using these moments, we have determined the BLUEs of parameters of scaled WG and location-scale WG distributions based on progressively Type-II censored samples for some specific values of the parameters. The BLUPs of future failure times have also been introduced. In addition, we have presented an example to illustrate the application of our results. A future work may be to derive the explicit expressions for the moments of the WG distribution based on generalized order statistics and dual generalized order statistics. Another possible future work may be to establish recurrence relations for the triple and quadruple moments of order statistics from the WG distribution. These results can then be utilized to develop approximate confidence intervals for the location and scale parameters using the Edgeworth approximation; see, for example, Sultan et al. [37].

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