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# New Fourth-Order Iterative Solver And Its Multi-Point Solver For Nonlinear Systems*

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**Abstract**

This manuscript presents a new two-step iterative algorithm having order of convergence four for approximating solutions of nonlinear system of equations. It requires one vector function evaluation and two Fréchet derivative evaluations per iteration. Also, the fourth order algorithm is extended into a general multi-point method with an additional vector function evaluation per step, having $2k+4$ order of convergence, $k \geq 1$. It is proved that the root of the nonlinear system is a point of attraction for the new iterative algorithms. Convergence analysis for the iterative process is derived from which order of convergence of the methods are obtained. Computational efficiency of the methods are provided based on the cost of computation. Numerical experimentation through some suitable examples are given and some known methods are compared with presented methods. Further, an application of these methods to solve boundary value problems for ordinary differential equations is also given. The presented algorithms perform better than many existing algorithms and equivalent to few available algorithms.

## 1 Introduction

The problem of solving equations and systems of nonlinear equations is among the most important in theory and practice, not only of applied mathematics, but also in many branches of science, engineering, physics, computer science, astronomy, finance, etc. In the light of this fact, there have been enormous contribution of iterative methods for solving scalar nonlinear equations [26]. Whereas all these methods cannot be extended to solve nonlinear system involving more than one variable. Even if some methods can be extended to solve nonlinear system, certain decisive factors like efficiency index, computational efficiency index, number of functional evaluations, number of Fréchet derivative and inverse of Fréchet derivative evaluations are to be given due importance. Moreover, when extending methods for single equation to solve system of nonlinear equations, due to increase in computational complexity they have no practical value. Chebyshev and Halley [2, 13] extended their methods to system of nonlinear equations and proved cubic convergence where first and second Fréchet derivatives are used. Due to evaluation of second Fréchet derivative, these methods are considered more costly from computational point of view. On the other hand, there have been considerable attempts to derive methods free from second derivative with higher order of convergence for single equation [4, 10, 20]. Extensions of these methods for system of nonlinear equations are found in [5, 11, 9].

Hence, finding a solution $\alpha$ of the nonlinear system $G(\mathtt{x}) = 0$ is a classical and difficult problem that unlocks the behavior pattern of many application problems in science and engineering. Consider $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$ which is a sufficiently Fréchet differentiable function in an open convex set $D$. Suppose the equation $G(\mathtt{x}) = 0$ has a solution $\alpha \in D$, that is $G(\alpha) = 0$, where $G(\mathtt{x}) = (g_1(\mathtt{x}), g_2(\mathtt{x}), ..., g_n(\mathtt{x}))^T$, $\mathtt{x} = (\mathtt{x}_1, \mathtt{x}_2, ..., \mathtt{x}_n)^T$, $g_i : \mathbb{R}^n \to \mathbb{R}, \forall i = 1, 2, \ldots, n$ are real valued functions.

Newton's method ($2^{nd}NM$) is the most used iterative technique for finding a solution $\alpha$ whose iterative expression is

$$\mathtt{x}^{(r+1)} = F_{2^{nd}NM}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - [G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\ r = 0, 1, 2, ..., \tag{1}$$

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where $F_{method}$ represents any iterative algorithm, $G'(\mathtt{x}^{(r)})$ denotes the Jacobian matrix of the function $G$ on $\mathtt{x}^{(r)}$ and $G'(\mathtt{x}^{(r)})^{-1}$ represents the inverse of $G'(\mathtt{x}^{(r)})$. This $2^{nd}NM$ method is proved to have convergence order two.

Another familiar method to solve nonlinear systems is a two-point Newton-like method ($3^{rd}TM$) of order three proposed in [26] is given by

$$\mathtt{x}^{(r+1)} = F_{3^{rd}TM}(\mathtt{x}^{(r)}) = F_{2^{nd}NM}(\mathtt{x}^{(r)}) - [G'(\mathtt{x}^{(r)})]^{-1}G(F_{2^{nd}NM}(\mathtt{x}^{(r)})).$$

In the recent past, many multi-point iterative algorithms whose convergence order $\geq 4$ have appeared for solving system of nonlinear equations. For example, some fourth-order schemes designed by Sharma et al. [24] and by Babajee et al. [3] are found in the literature which are Jarratt-type methods. Some fifth-order schemes are also found which are designed by Abad et al. [1], Grau-Sanchez et al. [12] and Madhu et al. [14]. Also, few sixth order methods for solving system of nonlinear equations were proposed by Cordero et al. [8], Madhu [16] and Madhu et al. [15].

We have extended the method given in [22] for nonlinear system, where an iterative solver with fourth-order convergence is presented in this paper. This method has one vector function evaluation and two Jacobian matrix evaluations per iteration. Also, we propose a general multi-point method which has 2k+4 order of convergence ($k \geq 1$), where it uses one more vector function evaluation in each step. Moreover, computational efficiency of the presented methods is compared with many equivalent methods.

The rest of this paper is arranged as follows. In Section 2, a new algorithm and its multi-point version for solving a system of nonlinear equations and some preliminaries are presented. Convergence analysis of the new methods are derived in section 3. Computational efficiency of the presented methods are computed based on the cost of computation and compared with other equivalent methods in terms of ratio is given in Section 4. In section 5, computational results for some examples are compared between different existing methods and presented methods. Finally, conclusions are given in section 6.

## 2 New Algorithms and Preliminaries

**New Fourth order solver ($4^{th}PM$):**
Consider the following iterative method of fourth order convergence to solve scalar nonlinear equation proposed in [22]:

$$w_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(w_n)}\left(1 + \frac{1}{4}\Big(\tau(x_n) - 1\Big) + \frac{3}{8}\Big(\tau(x_n) - 1\Big)^2\right),$$

where $\tau(x_n) = \frac{f'(w_n)}{f'(x_n)}$. This two-step method is extended to solve a nonlinear system which is given below:

$$\begin{aligned} y(\mathtt{x}^{(r)}) &= \mathtt{x}^{(r)} - \frac{2}{3}[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}), \\ \mathtt{x}^{(r+1)} &= F_{4^{th}PM}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - H_1(\mathtt{x}^{(r)})[G'(y(\mathtt{x}^{(r)}))]^{-1}G(\mathtt{x}^{(r)}),\ \textit{where} \\ H_1(\mathtt{x}^{(r)}) &= I + \frac{1}{4}(\tau(\mathtt{x}^{(r)}) - I) + \frac{3}{8}(\tau(\mathtt{x}^{(r)}) - I)^2, \quad \tau(\mathtt{x}^{(r)}) = [G'(\mathtt{x}^{(r)})]^{-1}G'(y(\mathtt{x}^{(r)})), \end{aligned} \tag{2}$$

where $I$ represents $n \times n$ identity matrix. This algorithm is found to have fourth order convergence.

**$(2k+4)^{th}$ order solver ($(2k+4)^{th}$PM):**
The $4^{th}PM$ method is improved by added new function evaluations to get the multi-point algorithm, which

is given below:

$$\begin{aligned}
&\mathtt{x}^{(r+1)} = F_{(2k+4)^{th}PM}(\mathtt{x}^{(r)}) = \mu_k(\mathtt{x}^{(r)}),\\
&\mu_j(\mathtt{x}^{(r)}) = \mu_{j-1}(\mathtt{x}^{(r)}) - H_2(\mathtt{x}^{(r)})[G'(\mathtt{x}^{(r)})]^{-1}G(\mu_{j-1}(\mathtt{x}^{(r)})),\\
&H_2(\mathtt{x}^{(r)}) = I + \frac{3}{2}(\eta(\mathtt{x}^{(r)}) - I) + \frac{30}{16}(\eta(\mathtt{x}^{(r)}) - I)^2,\\
&\eta(\mathtt{x}^{(r)}) = [G'(y(\mathtt{x}^{(r)}))]^{-1}G'(\mathtt{x}^{(r)}),\ \mu_0(\mathtt{x}^{(r)}) = F_{4^{th}PM}(\mathtt{x}^{(r)}), \quad j = 1, 2, ..., r,\ r \geq 1.
\end{aligned} \tag{3}$$

This multi-point algorithm has convergence order $2k+4, k \geq 1$. For $k = 0$ produces the $4^{th}PM$.

The well-known $n$-dimensional Taylor's expansion and the point of attraction technique are used to obtain theoretical convergence. Hence, we recall some important definitions and theorems from [7]:

Let $G : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Fréchet differentiable function upto the required order in $D$. Assume that $i$th derivative of $G$ at $u \in \mathbb{R}^n$, $i \geq 1$, is the $i$-linear function $G^{(i)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $G^{(i)}(u)(v_1, \ldots, v_i) \in R^n$. It is easy to observe that

(i) $G^{(i)}(u)(v_1, \ldots, v_{i-1}) \in \mathcal{L}(\mathbb{R}^n)$, where $\mathcal{L}$ is a linear funtion.

(ii) $G^{(i)}(u)(v_{\omega(1)}, \ldots, v_{\omega(i)})) = G^{(i)}(u)(v_1, \ldots, v_i)$, for all permutation $\omega$ of $1, 2, ...i$.

From the above results (i)–(ii), we use the following notations:

(a) $G^{(i)}(u)(v_1, \ldots, v_i) = G^{(i)}(u)v_1, \ldots, v_i$.

(b) $G^{(i)}(u)v^{i-1}G^{(p)}v^p = G^{(i)}(u)G^{(p)}(u)v^{i+p-1}$.

For $\alpha + h \in \mathbb{R}^n$, lying in a neighborhood of the solution $\alpha$ of the system of nonlinear equations $G(x) = 0$ and assuming that the Fréchet derivative $G'(\alpha)$ is nonsingular, Taylor's expansion can be applied, to get

$$G(\alpha + h) = G'(\alpha)\left[h + \sum_{i=2}^{p-1} C_i h^i\right] + O(h^p), \tag{4}$$

where $C_i = (1/i!)[G'(\alpha)]^{-1}G^{(i)}(\alpha)$, $i \geq 2$. It is noted that $C_iG^i \in \mathbb{R}^n$ since $G^{(i)}(\alpha) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[G'(\alpha)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. Differentiating the Taylor series of $G(\alpha + h)$ with respect to $h$, we get

$$G'(\alpha + h) = G'(\alpha)\left[I + \sum_{i=2}^{p-1} iC_i h^{i-1}\right] + O(h^p), \tag{5}$$

where $I$ denotes the identity matrix. We remark that $iC_ih^{i-1} \in \mathcal{L}(\mathbb{R}^n)$. The error is denoted as $E^{(r)} = x^{(r)} - \alpha$ for the $r$th iteration. The equation $E^{(r+1)} = LE^{(r)^p} + O(E^{(r)^{p+1}})$ is called the *error equation*, where $L$ is a $p$-linear function $L \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $p$ denotes *order of convergence*. Also, $E^{(r)^p} = (E_1^{(r)}, E_2^{(r)}, \cdots, E_n^{(r)})$.

**Definition 1 (Point of Attraction [18])** *Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$. Then $\alpha$ is a point of attraction of the iteration*

$$\mathbf{x}^{(r+1)} = F(\mathbf{x}^{(r)}),\ r = 0, 1, ... \tag{6}$$

*if there is an open neighbourhood $S$ of $\alpha$ defined by*

$$S(\alpha) = \{\mathbf{x} \in \mathbb{R}^n \Big| \ \|\mathbf{x} - \alpha\| < \delta\},\ \delta > 0,$$

*such that $S \subset D$ and for any $\mathbf{x}^{(0)} \in S$, the iterates $\{\mathbf{x}^{(r)}\}$ defined by equation (6) all lie in $D$ and converge to $\alpha$.*

**Theorem 1 (Ostrowski Theorem on fixed points [18])** *Assume that $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point $\alpha \in int(D)$ and $F$ is Fréchet differentiable on $\alpha$. If*

$$\rho(F'(\alpha)) = \sigma < 1,$$

*then $\alpha$ is a point of attraction for $x^{(k+1)} = F(x^{(k)})$, where $\rho$ denotes the spectral radius and $\sigma$ is a constant such that $0 \leq \sigma < 1$.*

We now prove a general result that shows $\alpha$ is a point of attraction of a general iteration function $F(x) = P(x) - Q(x)R(x)$, where the values of $P(x)$, $Q(x)$ and $R(x)$ represent the corresponding terms in the proposed methods (2) and (3).

**Theorem 2 ([5])** *Let $G : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Fréchet differentiable upto the required order at each point of an open convex neighborhood $D$ of $\alpha \in D$, which is a solution of the system $G(x) = 0$. Suppose that $P, Q, R : D \subset \mathbb{R}^n \to \mathbb{R}^n$ are Fréchet differentiable functionals upto the required order (depending on $G$) at each point in $D$ with $P(\alpha) = \alpha$, $Q(\alpha) \neq 0$ and $R(\alpha) = 0$.*

*Then, there exists a ball*

$$S = \overline{S}(\alpha, \delta) = \Big\{\|\alpha - x\| \leq \delta\Big\} \subset S_0,\ \delta > 0,$$

*on which the mapping*

$$F : S \to \mathbb{R}^n, \quad F(x) = P(x) - Q(x)R(x), \text{ for all } x \in \ S$$

*is well-defined; moreover, $F$ is Fréchet differentiable at $\alpha$, thus*

$$F'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha).$$

**Proof.** Here, we reproduce the proof given in [5] for the purpose of clarity. Clearly, $F(\alpha) = \alpha$.

$$\begin{aligned} & \|F(x) - F(\alpha) - F'(\alpha)(x - \alpha)\| \\ = \ & \|P(x) - Q(x)R(x) - \alpha - (P'(\alpha) - Q(\alpha)R'(\alpha))(x - \alpha)\| \\ \leq \ & \|P(x) - \alpha - P'(\alpha)(x - \alpha)\| + \| - Q(x)R(x) + Q(\alpha)R'(\alpha)(x - \alpha)\|, \text{ using triangle inequality.} \end{aligned}$$

Since $P(x)$ is differentiable in $\alpha$ and $P(\alpha) = \alpha$, we can assume that $\delta$ was chosen sufficiently small such that

$$\|P(x) - \alpha - P'(\alpha)(x - \alpha)\| \leq \epsilon\|x - \alpha\|,$$

for all $x \in S$ with $\epsilon > 0$ depending on $\delta$ and $\epsilon = 0$ in case $P(x) = x$. Since $P$, $Q$ and $R$ are continuously differentiable functions, then $Q'$, $R'$ and $R''$ are bounded:

$$\|Q'(x)\| \leq K_1, \ \ \|R'(x)\| \leq K_2, \ \ \|R''(x)\| \leq K_3.$$

Now by mean value theorem for integrals, we have

$$Q(x) = Q(\alpha) + \int_0^1 Q'(\alpha + t(x - \alpha))\ dt\ (x - \alpha)$$

and

$$R(x) = \int_0^1 \ R'(\alpha + s(x - \alpha))\ ds\ (x - \alpha)$$

so that

$$\begin{aligned}
& \|Q(x)R(x) - Q(\alpha)R'(\alpha)(x-\alpha)\| \\
= & \left\| Q(\alpha)\left(\int_0^1 R'(\alpha+s(x-\alpha)) - R'(\alpha)\ ds\right)(x-\alpha)^2 \right. \\
& \left. + \int_0^1\int_0^1 Q'(\alpha+t(x-\alpha))\ R'(\alpha+s(x-\alpha))\ dt\ ds\ (x-\alpha)^2 \right\| \\
\leq & \left\| Q(\alpha)\left(\int_0^1\int_0^1 R''(\alpha+s\lambda(x-\alpha))\ ds\ d\lambda\right)\ s\ (x-\alpha)^2 \right. \\
& \left. + \int_0^1\int_0^1 Q'(\alpha+t(x-\alpha))\ R'(\alpha+s(x-\alpha))\ dt\ ds\ (x-\alpha)^2 \right\|, \text{ using triangle inequality} \\
\leq & \|Q(\alpha)\| \int_0^1\int_0^1 \|R''(\alpha+s\lambda(x-\alpha))\|\ ds\ d\lambda\ |s|\ \ \|x-\alpha\|^2 \\
& + \int_0^1\int_0^1 \|Q'(\alpha+t(x-\alpha))\|\ \|R'(\alpha+s(x-\alpha))\|\ dt\ ds\ \|x-\alpha\|^2, \text{ using Schwartz inequality,} \\
\leq & \left(\frac{K_3}{2}\|Q(\alpha)\| + K_1K_2\right)\|x-\alpha\|^2, \text{ since } Q',\ R' \text{ and } R'' \text{ are bounded,} \\
\leq & \ \delta\left(\frac{K_3}{2}\|Q(\alpha)\| + K_1K_2\right)\|x-\alpha\|, \text{ since } \|x-\alpha\| \leq \delta.
\end{aligned}$$

Combining, we have

$$\|F(x) - F(\alpha) - F'(\alpha)(x-\alpha)\| \leq \delta\left(\epsilon + \frac{K_3}{2}\|Q(\alpha)\| + K_1K_2\right)\|x-\alpha\|,$$

which shows that $F(x)$ is differentiable in $\alpha$ since $\delta$ and $\epsilon$ are arbitrary and $\|Q(\alpha)\|$, $K_1$, $K_2$ and $K_3$ are constants. Thus $F'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha)$. ■

# 3 Analysis of Convergence

**Theorem 3** *Let $G : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Fréchet differentiable upto the required order at each point of an open convex neighborhood $D$ of $\alpha \in \mathbb{R}^n$, where $\alpha$ is a solution of the system $G(\mathbf{x}) = 0$. Let us suppose that $\mathbf{x} \in S = \overline{S}(\alpha, \delta)$ and $G'(\mathbf{x})$ is continuous and nonsingular in $\alpha$, and $\mathbf{x}^{(0)}$ nearer to $\alpha$. Then $\alpha$ is a point of attraction of the sequence $\{\mathbf{x}^{(r)}\}_{r\geq 0}$ obtained using the iterative expression (2). Furthermore, this sequence $\{\mathbf{x}^{(r)}\}$ converges to $\alpha$ with order four, where the error equation obtained is*

$$E^{(r+1)} = F_{4^{th}PM}(\mathtt{x}^{(r)}) - \alpha = L_1 {E^{(r)}}^4 + O({E^{(r)}}^5),\ L_1 = \Big(\frac{1}{9}C_4 - \frac{14}{9}C_2C_3 + \frac{7}{3}C_2^3 + \frac{5}{9}C_3C_2\Big).$$

**Proof.** First we show that $\alpha$ is a point of attraction using Theorem 2. In this case,

$$P(\mathtt{x}) = \mathtt{x}, \quad Q(\mathtt{x}) = H_1(\mathtt{x})[G'(y(\mathtt{x}))]^{-1}, \quad R(\mathtt{x}) = G(\mathtt{x}).$$

Since $G(\alpha) = 0$, we have

$$y(\alpha) = \alpha - \frac{2}{3}[G'(\alpha)]^{-1}G(\alpha) = \alpha,$$

$$\tau(\alpha) = [G'(\alpha)]^{-1}G'(y(\alpha)) = [G'(\alpha)]^{-1}G'(\alpha) = I, \quad H_1(\alpha) = I,$$

$$P(\alpha) = \alpha, \quad P'(\alpha) = I,$$

$$Q(\alpha) = H_1(\alpha)[G'(\alpha)]^{-1} = I[G'(\alpha)]^{-1} = [G'(\alpha)]^{-1} \neq 0,$$
$$R(\alpha) = G(\alpha), \quad R'(\alpha) = G'(\alpha),$$
$$F'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha) = I - [G'(\alpha)]^{-1}G'(\alpha) = 0.$$

So $\rho(F'(\alpha)) = 0 < 1$. Hence, by Ostrowski's Theorem $\alpha$ is a point of attraction for the iteration function (2). We next establish the fourth order convergence of this method. From (4) and (5), we obtain

$$G(\mathtt{x}^{(r)}) = G'(\alpha)\left[E^{(r)} + C_2{E^{(r)}}^2 + C_3{E^{(r)}}^3 + C_4{E^{(r)}}^4\right] + O({E^{(r)}}^5), \tag{7}$$

and we express the differential of first order as

$$G'(\mathtt{x}^{(r)}) = G'(\alpha)\left[I + 2C_2E^{(r)} + 3C_3{E^{(r)}}^2 + 4C_4{E^{(r)}}^3 + 5C_5{E^{(r)}}^4\right] + O({E^{(r)}}^5),$$

where $C_i = (1/i!)[G'(\alpha)]^{-1}G^{(i)}(\alpha)$, $i = 2, 3, \ldots$, and $E^{(r)} = \mathtt{x}^{(r)} - \alpha$.

In order to write in simple form, we use the following notations; we use different constants like $B_i, M_i, R_i$ and $N_i$ to represent the different combinations of $C_i,\ i = 2, 3, \ldots$. Taking inverse for $G'(\mathtt{x}^{(r)})$, we get

$$[G'(\mathtt{x}^{(r)})]^{-1} = [G'(\alpha)]^{-1}\left[I + B_2E^{(r)} + B_3{E^{(r)}}^2 + B_4{E^{(r)}}^3\right] + O({E^{(r)}}^4), \tag{8}$$

where $B_2 = -2C_2,\ \ B_3 = 4C_2^2 - 3C_3,\ \ B_4 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4$. Multiplying equations (7) and (8), we get

$$[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}) = E^{(r)} + M_0{E^{(r)}}^2 + M_1{E^{(r)}}^3 + M_2{E^{(r)}}^4 + O({E^{(r)}}^5), \tag{9}$$

where $M_0 = -C_2,\ \ M_1 = 2C_2^2 - 2C_3,\ \ M_2 = -4C_2^3 + 4C_2C_3 + 3C_3C_2 - 3C_4$. Then by using (9) we get the expression

$$y(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - \frac{2}{3}[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}) = \alpha + \frac{1}{3}E^{(r)} - \frac{2}{3}M_0{E^{(r)}}^2 + M_1{E^{(r)}}^3 + M_2{E^{(r)}}^4.$$

Taylor's expression of the Jacobian matrix $G'(y^{(r)})$ is

$$\begin{aligned} G'(y(\mathtt{x}^{(r)})) &= G'(\alpha)\Big[I + 2C_2(y(\mathtt{x}^{(r)}) - \alpha) + 3C_3(y(\mathtt{x}^{(r)}) - \alpha)^2 + 4C_4(y(\mathtt{x}^{(r)}) - \alpha)^3 \\ &\quad + 5C_5(y(\mathtt{x}^{(r)}) - \alpha)^4\Big] + O({E^{(r)}}^5) \\ &= G'(\alpha)\left[I + N_1E^{(r)} + N_2{E^{(r)}}^2 + N_3{E^{(r)}}^3\right] + O({E^{(r)}}^4), \end{aligned}$$

where $N_1 = \frac{2}{3}C_2,\ N_2 = \frac{4}{3}C_2^2 + \frac{1}{3}C_3,\ N_3 = -\frac{8}{3}C_2^3 + \frac{8}{3}C_2C_3 + \frac{4}{3}C_3C_2 + \frac{4}{27}C_4$. Therefore,

$$\begin{aligned} \tau(x^{(r)}) &= [G'(x^{(r)})]^{-1}G'(y(x^{(r)})) \\ &= I + (N_1 + B_2)E^{(k)} + (N_2 + B_2N_1 + B_3)E^{(r)2} + (N_3 + B_2N_2 + B_3N_1 + B_4)E^{(r)3} \\ &\quad + O(E^{(r)4}), \end{aligned} \tag{10}$$

and then

$$\begin{aligned} H_1(\tau(\mathtt{x}^{(r)})) &= I + \frac{1}{4}\left(\tau(\mathtt{x}^{(r)}) - I\right) + \frac{3}{8}\left(\tau(\mathtt{x}^{(r)}) - I\right)^2 \\ &= I + R_1E^{(r)} + R_2{E^{(r)}}^2 + R_3{E^{(r)}}^3 + O({E^{(r)}}^4), \end{aligned} \tag{11}$$

where

$$R_1 = -\frac{1}{3}C_2,\ \ R_2 = \frac{5}{3}C_2^2 - \frac{2}{3}C_3,\ \ R_3 = -\frac{20}{3}C_2^3 + \frac{14}{3}C_2C_3 + \frac{4}{3}C_3C_2 - \frac{26}{27}C_4.$$

Then

$$[G'(y(\mathtt{x}^{(r)}))]^{-1} = [G'(\alpha)]^{-1}\Big[I-\frac{2}{3}C_2E^{(r)}+\Big(-\frac{8}{9}C_2^2-\frac{1}{3}C_3\Big)E^{(r)^2} +\Big(\frac{112}{27}C_2^3-\frac{20}{9}C_2C_3-\frac{4}{3}C_3C_2-\frac{4}{27}C_4\Big)E^{(r)^3}\Big]+O(e^{(r)^4}). \quad (12)$$

Using equations (7), (12) and (11), we have

$$H_1(\mathtt{x}^{(r)})[G'(y(\mathtt{x}^{(r)}))]^{-1}G(\mathtt{x}^{(r)}) = E^{(r)}+\Big(-\frac{63}{27}C_2^3+\frac{14}{9}C_2C_3-\frac{5}{9}C_3C_2-\frac{1}{9}C_4\Big)E^{(r)^4}+O(E^{(r)^5}). \quad (13)$$

Finally, by using equations (13) in (2), we get the required error estimate

$$E^{(r+1)} = F_{4^{th}PM}(\mathtt{x}^{(r)})-\alpha = \Big(\frac{1}{9}C_4-\frac{14}{9}C_2C_3+\frac{7}{3}C_2^3+\frac{5}{9}C_3C_2\Big)E^{(r)^4}+O(E^{(r)^5}),$$

which shows fourth order convergence, where $F_{4^{th}PM}$ represents the proposed iterative algorithm. ■

**Theorem 4** *Let $G : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Fréchet differentiable upto the required order at each point of an open convex neighborhood $D$ of $\alpha \in \mathbb{R}^n$, where $\alpha$ is a solution of the system $G(\mathbf{x}) = 0$. Let us suppose that $\mathbf{x} \in S = \overline{S}(\alpha, \delta)$ and $G'(\mathbf{x})$ is continuous and nonsingular in $\alpha$, and $\mathbf{x}^{(0)}$ is nearer to $\alpha$. Then $\alpha$ is a point of attraction of the sequence $\{\mathbf{x}^{(r)}\}_{r\geq 0}$ obtained using the iterative expression (3). Furthermore, this sequence $\{\mathbf{x}^{(r)}\}$ converges to $\alpha$ with order $2k+4$, where $k$ is a positive integer and $k \geq 1$.*

**Proof.** Here $P(\mathtt{x}) = \mu_{j-1}(\mathtt{x}),\ Q(\mathtt{x}) = H_2(\mathtt{x})[G'(\mathtt{x})]^{-1},\ R(\mathtt{x}) = G(\mu_{j-1}(\mathtt{x})),\ j = 1, ..., k$. We can show by induction that

$$\mu_{j-1}(\alpha) = \alpha, \quad \mu'_{j-1}(\alpha) = 0, \quad \forall\, j = 1, ..., k,$$

so that

$$P(\alpha) = \mu_{j-1}(\alpha) = \alpha,\ H_2(\alpha) = I,\ Q(\alpha) = I[G'(\alpha)]^{-1} = [G'(\alpha)]^{-1} \neq 0,$$

$$R(\alpha) = G(\mu_{j-1}(\alpha)) = G(\alpha) = 0,$$

$$P'(\alpha) = \mu'_{j-1}(\alpha) = 0,\ R'(\alpha) = G'(\mu_{j-1}(\alpha))\mu'_{j-1}(\alpha) = 0,$$

$$F'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha) = 0.$$

So $\rho(F'(\alpha)) = 0 < 1$. Hence, by Ostrowski's Theorem, $\alpha$ is a point of attraction for the iteration function (3). Taylor's expansion of $G(\mu_{j-1}(\mathtt{x}^{(k)}))$ about $\alpha$ yields

$$G(\mu_{j-1}(\mathtt{x}^{(r)})) = G'(\alpha)\left[(\mu_{j-1}(\mathtt{x}^{(r)})-\alpha)+C_2(\mu_{j-1}(\mathtt{x}^{(r)})-\alpha)^2+...\right]. \quad (14)$$

Also, we find that

$$H_2(\mathtt{x}^{(r)}) = I+\frac{3}{2}(\eta(\mathtt{x}^{(r)})-I)+\frac{30}{16}(\eta(\mathtt{x}^{(r)})-I)^2 = I+2C_2E^{(r)}+4C_3E^{(r)^2}+(7C_2C_3-\frac{68}{9}C_2^3-3C_3C_2+\frac{52}{9}C_4)E^{(r)^3}+... \quad (15)$$

Using equations (8) and (15), we have

$$H_2(\mathtt{x}^{(r)})[G'(\mathtt{x}^{(r)})]^{-1} = \left[I+L_2\ E^{(r)^2}+...\right][G'(\alpha)]^{-1}, \quad L_2 = C_3. \quad (16)$$

Using equations (16) and (14), we have

$$\begin{aligned} & H_2(\mathtt{x}^{(r)})[G'(\mathtt{x}^{(r)})]^{-1}G(\mu_{j-1}(\mathtt{x}^{(r)})) \\ = & \Big(I + L_2\ {E^{(r)}}^3 + ...\Big)\Big((\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha) + C_2(\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha)^2 + ...\Big) \\ = & \big[\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha + L_2\ {E^{(r)}}^2(\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha) + C_2(\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha)^2 + ...\big]. \end{aligned} \tag{17}$$

Using (17) in (3), we obtain

$$\begin{aligned} \mu_j(\mathtt{x}^{(r)}) - \alpha &= (\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha) - \Big((\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha) + L_2\ {E^{(r)}}^2(\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha) \\ &\quad + C_2(\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha)^2 + ...\Big) \\ &= -L_2\ {E^{(r)}}^2(\mu_{j-1}(\mathtt{x}^{(r)}) - \alpha) + ... \end{aligned} \tag{18}$$

As we know that $\mu_0(\mathtt{x}^{(r)}) - \alpha = L_1 {E^{(r)}}^4 + O({E^{(r)}}^5)$ and from (18), for $j = 1, 2, ...,$

$$\mu_1(\mathtt{x}^{(r)}) - \alpha = -L_2({E^{(r)}}^{(2)})\Big(\mu_0(\mathtt{x}^{(r)}) - \alpha\Big) + ... = -L_2 L_1 {E^{(r)}}^6 + ...,$$

$$\mu_2(\mathtt{x}^{(r)}) - \alpha = -L_2({E^{(r)}}^{(2)})\Big(\mu_1(\mathtt{x}^{(r)}) - \alpha\Big) + ... = L_2^2 L_1 {E^{(r)}}^8 + ....$$

Proceeding by induction, we get the required error estimate

$$\mu_k(\mathtt{x}^{(r)}) - \alpha = (-L_2)^k L_1\ ({E^{(r)}}^{(2k+4)}) + O({E^{(r)}}^{(2k+4)}),\ k \geq 1,$$

which shows $(2k+4)th$ order convergence. ■

# 4 Computational Efficiency

The efficiency index of any iterative method is measured using the Ostrowski's definition [19], $EI = p^{\frac{1}{d}}$, where $p$ denotes the order of convergence and $d$ denotes the number of functional evaluations per iteration. The proposed algorithms are compared with different algorithms given below in terms of computational cost. For evaluating the Jacobian $G'$ and $G$, $n^2$ evaluation of functions and $n$ scalar function evaluations are required. Also, for any iterative method solving a nonlinear system, we need one or more inversion of matrix. That means, few system of linear equations should be solved. Therefore, the number of operations needed for solving the system is taken into account while determining the computational cost of an iterative scheme. Hence, Cordero et al. [7] proposed the idea of computational efficiency index ($CE$), where the efficiency index given by Ostrowski is combined with the number of products-quotients required per iteration. Computational efficiency index is defined as $CE = p^{1/(d+op)}$, where $op$ is the number of products-quotients per iteration and the details of its calculation is given in [21].

**Some Existing Methods:**

For the purpose of comparing computational efficiency and numerical calculations, some well-known available iterative methods for solving systems of nonlinear equations are given below:

Method of order four by Noor *et al.* [17] ($4^{th}NR$) :

$$\mathtt{x}^{(r+1)} = F_{4^{th}NR}(\mathtt{x}^{(r)}) = F_{2^{nd}NM}(\mathtt{x}^{(r)}) - G'(F_{2^{nd}NM}(\mathtt{x}^{(r)}))^{-1}G(F_{2^{nd}NM}(\mathtt{x}^{(r)})). \tag{19}$$

Method of order four by Babajee et al. [3] ($4^{th}BCST$):

$$\begin{aligned} & y(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - \tfrac{2}{3}[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}), \\ & \mathtt{x}^{(r+1)} = F_{4^{th}BCST}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - W(\mathtt{x}^{(r)})[A_1(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}), \quad \text{where} \\ & A_1(\mathtt{x}^{(r)}) = \tfrac{1}{2}\big[G'(\mathtt{x}^{(r)}) + G'(y(\mathtt{x}^{(r)}))\big], \\ & W(\mathtt{x}^{(r)}) = I - \tfrac{1}{4}(\tau(\mathtt{x}^{(r)}) - I) + \tfrac{3}{4}(\tau(\mathtt{x}^{(r)}) - I)^2,\ \tau(\mathtt{x}^{(r)}) = G'(\mathtt{x}^{(r)})^{-1}G'(y(\mathtt{x}^{(r)})). \end{aligned} \tag{20}$$

Fourth order method by Sharma et al. [24] ($4^{th}SGS$):

$$\begin{aligned} &y(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - \tfrac{2}{3}[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\\ &\mathtt{x}^{(r+1)} = F_{4^{th}SGS}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - W(\mathtt{x}^{(r)})[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}), \quad \text{where}\\ &W(\mathtt{x}^{(r)}) = -\tfrac{1}{2}I + \tfrac{9}{8}[G'(y(\mathtt{x}^{(r)}))]^{-1}G'(\mathtt{x}^{(r)}) + \tfrac{3}{8}[G'(\mathtt{x}^{(r)})]^{-1}G'(y(\mathtt{x}^{(r)})). \end{aligned} \tag{21}$$

Fourth order method by Babajee et al. [5] ($4^{th}BMJ$):

$$\begin{aligned} &y(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - \tfrac{2}{3}[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\\ &\mathtt{x}^{(r+1)} = F_{4^{th}BMJ}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - H(\mathtt{x}^{(r)})A(\mathtt{x}^{(r)})G(\mathtt{x}^{(r)}), \quad \text{where}\\ &H(\mathtt{x}^{(r)}) = I - \tfrac{1}{4}(\tau(\mathtt{x}^{(r)}) - I) + \tfrac{1}{2}(\tau(\mathtt{x}^{(r)}) - I)^2, \quad \tau(\mathtt{x}^{(r)}) = [G'(\mathtt{x}^{(r)})]^{-1}G'(y(\mathtt{x}^{(r)})),\\ &A(\mathtt{x}^{(r)}) = \tfrac{1}{2}\left([G'(\mathtt{x}^{(r)})]^{-1} + [G'(y(\mathtt{x}^{(r)}))]^{-1}\right). \end{aligned} \tag{22}$$

Sixth order method by Cordero *et al.* [8] ($6^{th}CHMT$):

$$\begin{aligned} &F_{2^{nd}NM}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - [G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\\ &z(\mathtt{x}^{(r)}) = F_{2^{nd}NM}(\mathtt{x}^{(r)}) - \left[2I - G'(\mathtt{x}^{(r)})^{-1}G'(F_{2^{nd}NM}(\mathtt{x}^{(r)}))\right][G'(\mathtt{x}^{(r)})]^{-1}G(F_{2^{nd}NM}(\mathtt{x}^{(r)})),\\ &\mathtt{x}^{(r+1)} = F_{6^{th}CHMT}(\mathtt{x}^{(r)}) = z(\mathtt{x}^{(r)}) - [G'(F_{2^{nd}NM}(\mathtt{x}^{(r)}))]^{-1}G(z(\mathtt{x}^{(r)})). \end{aligned} \tag{23}$$

Eighth order method by Sharma and Arora [23] ($8^{th}SA$):

$$\begin{aligned} &F_{2^{nd}NM}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - [G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\\ &z(\mathtt{x}^{(r)}) = F_{2^{nd}NM}(\mathtt{x}^{(r)}) - \left[\tfrac{13}{4}I - F(\mathtt{x}^{(r)})(\tfrac{7}{2}I - \tfrac{5}{4}F(\mathtt{x}^{(r)}))\right][G'(\mathtt{x}^{(r)})]^{-1}G(F_{2^{nd}NM}(\mathtt{x}^{(r)})),\\ &\mathtt{x}^{(r+1)} = F_{8^{th}SA}(\mathtt{x}^{(r)}) = z(\mathtt{x}^{(r)}) - \left[\tfrac{7}{2}I - F(\mathtt{x}^{(r)})(4I - \tfrac{3}{2}K(\mathtt{x}^{(r)})\right]G'(\mathtt{x}^{(r)})^{-1}G(z^{(r)}),\\ &\text{where} \;\; K(\mathtt{x}^{(r)}) = [G'(\mathtt{x}^{(r)})]^{-1}G'[F_{2^{nd}NM}(\mathtt{x}^{(r)})]. \end{aligned} \tag{24}$$

The following algorithms presented recently are considered only for the purpose of comparing computational efficiency:
A sixth order method by Bhel et al. [6] ($6^{th}BCMT$):

$$\begin{aligned} &y(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - a\;[G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\\ &z(\mathtt{x}^{(r)}) = y(\mathtt{x}^{(r)}) - \left[b[G'(\mathtt{x}^{(r)})]^{-1} + [cG'(\mathtt{x}^{(r)}) + dG'(\mathtt{x}^{(r)})]^{-1}\right]G(\mathtt{x}^{(r)}),\\ &\mathtt{x}^{(r+1)} = F_{6^{th}BCMT}(\mathtt{x}^{(r)}) = z(\mathtt{x}^{(r)}) - \left[g[G'(\mathtt{x}^{(r)})]^{-1} + [eG'(\mathtt{x}^{(r)}) + hG'^{(r)})]^{-1}\right]G(z^{(r)}),\\ &\text{where} \;\; a = \tfrac{2}{3},\; b = -\tfrac{1}{6},\; c = -1,\; d = 3,\; g = \tfrac{1}{2},\; e = \tfrac{2g+1}{2(g-1)^2}. \end{aligned} \tag{25}$$

An eighth order four-step method by Sharma and Kumar [25] ($8^{th}SD$):

$$\begin{aligned} &F_{2^{nd}NM}(\mathtt{x}^{(r)}) = \mathtt{x}^{(r)} - [G'(\mathtt{x}^{(r)})]^{-1}G(\mathtt{x}^{(r)}),\\ &z(\mathtt{x}^{(r)}) = F_{2^{nd}NM}(\mathtt{x}^{(r)}) - \left(3I - 2G'(\mathtt{x}^{(r)})^{-1}[F_{2^{nd}NM}, \mathtt{x}; G]\right),\\ &w(\mathtt{x}^{(r)}) = z(\mathtt{x}^{(r)}) - \psi(\mathtt{x}, F_{2^{nd}NM})G(z^{(r)}),\\ &\mathtt{x}^{(r+1)} = F_{8^{th}SD}(\mathtt{x}^{(r)}) = w(\mathtt{x}^{(r)}) - \psi(\mathtt{x}, F_{2^{nd}NM})G(w^{(r)}),\\ &\textit{where} \;\; \psi(\mathtt{x}, F_{2^{nd}NM}) = \left(2I - G'(\mathtt{x}^{(r)})^{-1}[z, F_{2^{nd}NM}; G]\right)[G'(\mathtt{x}^{(r)})]^{-1}. \end{aligned} \tag{26}$$

Table 1 displays the computational cost ($C_{method}$) and computational efficiency ($CE_{method}$) of various methods. The formulas in computational cost in the second column of Table (1) are given in [7]. To compare the CE of considered iterative methods, we calculate the following ratio [25]:

$$R_{method1;method2} = \frac{\log(CE_{method1})}{\log(CE_{method2})} = \frac{C_{method2}\;\log(order\;of\;method1)}{C_{method1}\;\log(order\;of\;method2)}. \tag{27}$$

Table 1: Formula for Computational Cost and Computational Efficiency

| **Method** | **Computational Cost** | **Computational Efficiency** |
|---|---|---|
| $2^{nd}NM$ | $\frac{1}{3}n^3+2n^2+\frac{2}{3}n$ | $2^{\frac{1}{C_{2^{nd}NM}}}$ |
| $4^{th}NR$ | $\frac{2}{3}n^3+4n^2+\frac{4}{3}n$ | $4^{\frac{1}{C_{4^{th}NR}}}$ |
| $4^{th}BCST$ | $\frac{2}{3}n^3+9n^2+\frac{13}{3}n$ | $4^{\frac{1}{C_{4^{th}BCST}}}$ |
| $4^{th}SGS$ | $\frac{2}{3}n^3+8n^2+\frac{13}{3}n$ | $4^{\frac{1}{C_{4^{th}SGS}}}$ |
| $4^{th}BMJ$ | $\frac{2}{3}n^3+8n^2+\frac{13}{3}n$ | $4^{\frac{1}{C_{4^{th}BMJ}}}$ |
| $4^{th}PM$ | $\frac{2}{3}n^3+8n^2+\frac{10}{3}n$ | $4^{\frac{1}{C_{4^{th}PM}}}$ |
| $6^{th}CHMT$ | $\frac{2}{3}n^3+7n^2+\frac{10}{3}n$ | $6^{\frac{1}{C_{6^{th}CHMT}}}$ |
| $6^{th}BCMT$ | $n^3+9n^2+3n$ | $6^{\frac{1}{C_{6^{th}BCMT}}}$ |
| $6^{th}PM$ | $\frac{2}{3}n^3+13n^2+\frac{19}{3}n$ | $6^{\frac{1}{C_{6^{th}PM}}}$ |
| $8^{th}SA$ | $\frac{1}{3}n^3+10n^2+\frac{23}{3}n$ | $8^{\frac{1}{C_{8^{th}SA}}}$ |
| $8^{th}SD$ | $\frac{1}{3}n^3+15n^2+\frac{17}{3}n$ | $8^{\frac{1}{C_{8^{th}SD}}}$ |
| $8^{th}PM$ | $\frac{2}{3}n^3+15n^2+\frac{22}{3}n$ | $8^{\frac{1}{C_{8^{th}PM}}}$ |

It is clear that when $R_{method1;method2} > 1$, the iterative method1 is more efficient than method2. The ratio (27) for all the discussed methods is given below:

$4^{th}PM$ **versus** $4^{th}BCST$**:**

$$R_{4^{th}PM;4^{th}BCST} = \frac{(\frac{2}{3}n^3+9n^2+\frac{13}{3}n)\ \log(4)}{(\frac{2}{3}n^3+8n^2+\frac{10}{3}n)\ \log(4)} > 1 \text{ for } n \geq 2.$$

Hence, we conclude that $CE_{4^{th}PM} > CE_{4^{th}BCST}$ for $n \geq 2$.

$4^{th}PM$ **versus** $4^{th}SGS$**:**

$$R_{4^{th}PM;4^{th}SGS} = \frac{(\frac{2}{3}n^3+9n^2+\frac{13}{3}n)\ \log(4)}{(\frac{2}{3}n^3+8n^2+\frac{10}{3}n)\ \log(4)} > 1 \text{ for } n \geq 2.$$

Hence, we conclude that $CE_{4^{th}PM} > CE_{4^{th}SGS}$ for $n \geq 2$.

$4^{th}PM$ **versus** $4^{th}BMJ$**:**

$$R_{4^{th}PM;4^{th}BMJ} = \frac{(\frac{2}{3}n^3+8n^2+\frac{13}{3}n)\ \log(4)}{(\frac{2}{3}n^3+8n^2+\frac{10}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 2.$$

Hence, we have $CE_{4^{th}PM} > CE_{4^{th}BMJ}$ for $n \geq 2$.

$4^{th}PM$ **versus** $6^{th}BCMT$**:**

$$R_{4^{th}PM;6^{th}BCMT} = \frac{(n^3+9n^2+3n)\ \log(4)}{(\frac{2}{3}n^3+8n^2+\frac{10}{3}n)\ \log(6)} > 1 \quad \text{for } n \geq \ 11.$$

Thus, we conclude that $CE_{4^{th}PM} > CE_{6^{th}BCMT}$ for $n \geq 11$.

$4^{th}PM$ **versus** $6^{th}PM$**:**

$$R_{4^{th}PM;6^{th}PM} = \frac{(\frac{2}{3}n^3+13n^2+\frac{19}{3}n)\ \log(4)}{(\frac{2}{3}n^3+8n^2+\frac{10}{3}n)\ \log(6)} > 1 \quad \text{for } 2 \leq n \leq 14.$$

Hence, we have $CE_{4^{th}PM} > CE_{6^{th}PM}$ for $2 \leq n \leq 14$.

$4^{th}PM$ **versus** $8^{th}SD$**:**

$$R_{4^{th}PM;8^{th}SD} = \frac{(\frac{1}{3}n^3+15n^2+\frac{17}{3}n)\ \log(4)}{(\frac{2}{3}n^3+8n^2+\frac{10}{3}n)\ \log(8)} > 1 \quad \text{for } 2 \leq n \leq 4.$$

Thus, we conclude that $CE_{4^{th}PM} > CE_{8^{th}SD}$ for $2 \leq n \leq 4$.
$4^{th}PM$ **versus** $8^{th}PM$**:**

$$R_{4^{th}PM;8^{th}PM} = \frac{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(4)}{(\frac{2}{3}n^3 + 8n^2 + \frac{10}{3}n)\ \log(8)} > 1 \quad \text{for } 2 \leq n \leq 9.$$

Hence, we conclude that $CE_{4^{th}PM} > CE_{8^{th}PM}$ for $2 \leq n \leq 9$.
$6^{th}PM$ **versus** $2^{nd}NM$**:**

$$R_{6^{th}PM;2^{nd}NM} = \frac{(\frac{1}{3}n^3 + 2n^2 + \frac{2}{3}n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(2)} > 1 \quad \text{for } n \geq 41.$$

Thus, we conclude that $CE_{6^{th}PM} > CE_{2^{nd}NM}$ for $n \geq 41$.
$6^{th}PM$ **versus** $4^{th}NR$**:**

$$R_{6^{th}PM;4^{th}NR} = \frac{(\frac{2}{3}n^3 + 4n^2 + \frac{4}{3}n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 41.$$

Hence, we have $CE_{6^{th}PM} > CE_{4^{th}NR}$ for $n \geq 41$.
$6^{th}PM$ **versus** $4^{th}BCST$**:**

$$R_{6^{th}PM;4^{th}BCST} = \frac{(\frac{2}{3}n^3 + 9n^2 + \frac{13}{3}n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 8.$$

Hence, we conclude that $E_{6^{th}PM} > E_{4^{th}BCST}$ for $n \geq 8$.
$6^{th}PM$ **versus** $4^{th}SGS$**:**

$$R_{6^{th}PM;4^{th}SGS} = \frac{(\frac{2}{3}n^3 + 8n^2 + \frac{13}{3}n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 14.$$

Hence, we conclude that $CE_{6^{th}PM} > CE_{4^{th}SGS}$ for $n \geq 14$.
$6^{th}PM$ **versus** $4^{th}BMJ$**:**

$$R_{6^{th}PM;4^{th}BMJ} = \frac{(\frac{2}{3}n^3 + 8n^2 + \frac{13}{3}n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 14.$$

Hence, we conclude that $CE_{6^{th}PM} > CE_{4^{th}BMJ}$ for $n \geq 14$.
$6^{th}PM$ **versus** $4^{th}PM$**:**

$$R_{6^{th}PM;4^{th}PM} = \frac{(\frac{2}{3}n^3 + 8n^2 + \frac{10}{3}n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 15.$$

Thus, we have $E_{6^{th}PM} > E_{4^{th}PM}$ for $n \geq 15$.
$6^{th}PM$ **versus** $6^{th}BCMT$**:**

$$R_{6^{th}PM;6^{th}BCMT} = \frac{(n^3 + 9n^2 + 3n)\ \log(6)}{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 13.$$

Hence, we conclude that $CE_{6^{th}PM} > CE_{6^{th}BCMT}$ for $n \geq 13$.
$8^{th}PM$ **versus** $2^{nd}NM$**:**

$$R_{8^{th}PM;2^{nd}NM} = \frac{(\frac{1}{3}n^3 + 2n^2 + \frac{2}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(2)} > 1 \quad \text{for } n \geq 28.$$

Thus, we have $E_{8^{th}PM} > E_{2^{nd}NM}$ for $n \geq 28$.
$8^{th}PM$ **versus** $4^{th}NR$**:**

$$R_{8^{th}PM;4^{th}NR} = \frac{(\frac{2}{3}n^3 + 4n^2 + \frac{4}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 28.$$

Hence, we conclude that $E_{8^{th}PM} > E_{4^{th}NR}$ for $n \geq 28$.
$8^{th}PM$ **versus** $4^{th}BCST$**:**

$$R_{8^{th}PM;4^{th}BCST} = \frac{(\frac{2}{3}n^3 + 9n^2 + \frac{13}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 6.$$

Hence, we have $E_{8^{th}PM} > E_{4^{th}BCST}$ for $n \geq 6$.
$8^{th}PM$ **versus** $4^{th}SGS$**:**

$$R_{8^{th}PM;4^{th}SGS} = \frac{(\frac{2}{3}n^3 + 8n^2 + \frac{13}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 10.$$

Hence, we conclude that $E_{8^{th}PM} > E_{4^{th}SGS}$ for $n \geq 10$.
$8^{th}PM$ **versus** $4^{th}BMJ$**:**

$$R_{8^{th}PM;4^{th}BMJ} = \frac{(\frac{2}{3}n^3 + 8n^2 + \frac{13}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 10.$$

Thus, we have $E_{8^{th}PM} > E_{4^{th}BMJ}$ for $n \geq 10$.
$8^{th}PM$ **versus** $4^{th}PM$**:**

$$R_{8^{th}PM;4^{th}PM} = \frac{(\frac{2}{3}n^3 + 8n^2 + \frac{10}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(4)} > 1 \quad \text{for } n \geq 10.$$

Hence, we conclude that $E_{8^{th}PM} > E_{4^{th}PM}$ for $n \geq 10$.
$8^{th}PM$ **versus** $6^{th}CHMT$**:**

$$R_{8^{th}PM;6^{th}CHMT} = \frac{(\frac{2}{3}n^3 + 7n^2 + \frac{10}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(6)} > 1 \quad \text{for } n \geq 65.$$

Thus, we have $E_{8^{th}PM} > E_{6^{th}CHMT}$ for $n \geq 65$.
$8^{th}PM$ **versus** $6^{th}BCMT$**:**

$$R_{8^{th}PM;6^{th}BCMT} = \frac{(n^3 + 9n^2 + 3n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(6)} > 1 \quad \text{for } n \geq 11.$$

Hence, we have $E_{8^{th}PM} > E_{6^{th}BCMT}$ for $n \geq 11$.
$8^{th}PM$ **versus** $6^{th}PM$**:**

$$R_{8^{th}PM;6^{th}PM} = \frac{(\frac{2}{3}n^3 + 13n^2 + \frac{19}{3}n)\ \log(8)}{(\frac{2}{3}n^3 + 15n^2 + \frac{22}{3}n)\ \log(6)} > 1 \quad \text{for } n \geq 2.$$

Hence, we conclude that $E_{8^{th}PM} > E_{6^{th}PM}$ for $n \geq 2$.
Consolidating the above ratios, the following theorem is stated to show the superiority of the proposed methods.

**Theorem 5** *Computational efficiency of $4^{th}PM$, $6^{th}PM$ and $8^{th}PM$ methods satisfy:*

*(a) $CE_{4^{th}PM} > CE_{4^{th}BCST}, CE_{4^{th}SGS}, CE_{4^{th}BMJ}, CE_{6^{th}BCMT}, CE_{6^{th}PM}, CE_{8^{th}SD}$ and $CE_{8^{th}PM}$ for $n \geq 2,\ n \geq 2,\ n \geq 2,\ n \geq 11,\ 2 \leq n \leq 14,\ 2 \leq n \leq 4$ and $2 \leq n \leq 9$ respectively.*

*(b) $CE_{6^{th}PM} > CE_{2^{nd}NM}, CE_{4^{th}NR}, CE_{4^{th}BCST}, CE_{4^{th}SGS}, CE_{4^{th}BMJ}, CE_{4^{th}PM}$ and $CE_{6^{th}BCMT}$ for $n \geq 41,\ n \geq 41,\ n \geq 8,\ n \geq 14,\ n \geq 14,\ n \geq 15$ and $n \geq 13$, respectively.*

*(c) $CE_{8^{th}PM} > CE_{2^{nd}NM}, CE_{4^{th}NR}, CE_{4^{th}BCST}, CE_{4^{th}SGS}, CE_{4^{th}BMJ}, CE_{4^{th}PM}, CE_{6^{th}CHMT}, CE_{6^{th}BCMT}$ and $CE_{6^{th}PM}$ for $n \geq 28,\ n \geq 28,\ n \geq 6,\ n \geq 10,\ n \geq 10,\ n \geq 10,\ n \geq 65, n \geq 11,$ and $n \geq 2$, respectively.*

It is noted that the following ratios do not satisfy the required condition $R_{method1;method2} > 1$:

(i) $4^{th}PM$ respectively with $2^{nd}NM$, $4^{th}NR$, $6^{th}CHMT$, $8^{th}SA$;

(ii) $6^{th}PM$ respectively with $6^{th}CHMT$, $8^{th}SA$, $8^{th}SD$, $8^{th}PM$;

(iii) $8^{th}PM$ respectively with $8^{th}SA$, $8^{th}SD$.

# 5 Numerical Results

The performance of the proposed methods is compared with Newton's method and few existing methods such as $4^{th}NR$ (19), $4^{th}BCST$ (20), $4^{th}SGS$ (21), $4^{th}BMJ$ (22), $6^{th}CHMT$ (23) and $8^{th}SA$ (24). The numerical computations are performed using MATLAB software for the test problems given below. The numerical solutions are computed correct to 500 digits by using variable precision arithmetic. The following stopping criterion is used for the iteration scheme:

$$err_{min} = \|\mathtt{x}^{(r+1)} - \mathtt{x}^{(r)}\|_2 < 10^{-100}.$$

The approximated computational order of convergence $p_c$ is calculated as follows:

$$p_c \approx \frac{\log\left(\|\mathtt{x}^{(r+1)} - \mathtt{x}^{(r)}\|_2 / \|\mathtt{x}^{(r)} - \mathtt{x}^{(r-1)}\|_2\right)}{\log\left(\|\mathtt{x}^{(r)} - \mathtt{x}^{(r-1)}\|_2 / \|\mathtt{x}^{(r-1)} - \mathtt{x}^{(r-2)}\|_2\right)}. \tag{28}$$

We give below few examples along with starting vector and exact solution, on which the methods are experimented.

**Test Problem 1** (TP1): The following nonlinear system is taken for study:

$$G(\mathtt{x}_1, \mathtt{x}_2) = (\mathtt{x}_1 + exp(\mathtt{x}_2) - cos(\mathtt{x}_2), \quad 3\mathtt{x}_1 - \mathtt{x}_2 - sin(\mathtt{x}_2)).$$

The Jacobian matrix is given by $G'(\mathtt{x}) = \begin{pmatrix} 1 & \exp(\mathtt{x}_2) + \sin(\mathtt{x}_2) \\ 3 & -1 - \cos(\mathtt{x}_2) \end{pmatrix}$. Initial approximation is taken as $\mathtt{x}^{(0)} = (1.5, 2)^T$ and the analytic solution is given by $\alpha = (0, 0)^T$.

**Test Problem 2** (TP2): The following nonlinear system is considered:

$$\begin{cases} \mathtt{x}_2\mathtt{x}_3 + \mathtt{x}_4(\mathtt{x}_2 + \mathtt{x}_3) = 0, \\ \mathtt{x}_1\mathtt{x}_3 + \mathtt{x}_4(\mathtt{x}_1 + \mathtt{x}_3) = 0, \\ \mathtt{x}_1\mathtt{x}_2 + \mathtt{x}_4(\mathtt{x}_1 + \mathtt{x}_2) = 0, \\ \mathtt{x}_1\mathtt{x}_2 + \mathtt{x}_1\mathtt{x}_3 + \mathtt{x}_2\mathtt{x}_3 = 1. \end{cases}$$

The above system is solved by taking the starting approximation
$\mathtt{x}^{(0)} = (0.5, 0.5, 0.5, -0.2)^T$. The solution is given by $\alpha \approx (0.577350, 0.577350, 0.577350, -0.288675)^T$. The Jacobian matrix is given by

$$G'(\mathtt{x}) = \begin{pmatrix} 0 & \mathtt{x}_3 + \mathtt{x}_4 & \mathtt{x}_2 + \mathtt{x}_4 & \mathtt{x}_2 + \mathtt{x}_3 \\ \mathtt{x}_3 + \mathtt{x}_4 & 0 & \mathtt{x}_1 + \mathtt{x}_4 & \mathtt{x}_1 + \mathtt{x}_3 \\ \mathtt{x}_2 + \mathtt{x}_4 & \mathtt{x}_1 + \mathtt{x}_4 & 0 & \mathtt{x}_1 + \mathtt{x}_2 \\ \mathtt{x}_2 + \mathtt{x}_3 & \mathtt{x}_1 + \mathtt{x}_3 & \mathtt{x}_1 + \mathtt{x}_2 & 0 \end{pmatrix}.$$

**Test Problem 3** (TP3): The following huge nonlinear system is considered:

$$\begin{cases} \mathtt{x}_i\mathtt{x}_{i+1} - 1 = 0, \quad i = 1, 2, 3, ...15, \\ \mathtt{x}_{15}\mathtt{x}_1 - 1 = 0. \end{cases}$$

The solution is $\alpha = (1, 1, 1, ..., 1)^T$. Choosing the initial vector as $\mathtt{x}^{(0)} = (1.5, 1.5, 1.5, ..., 1.5)^T$, we obtain the following Jacobian matrix.

$$G'(\mathtt{x}) = \begin{pmatrix} \mathtt{x}_2 & \mathtt{x}_1 & 0 & 0 & ... & 0 & 0 \\ 0 & \mathtt{x}_3 & \mathtt{x}_2 & 0 & ... & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & 0 & ... & \mathtt{x}_{15} & \mathtt{x}_{14} \\ \mathtt{x}_{15} & 0 & 0 & 0 & ... & 0 & \mathtt{x}_1 \end{pmatrix}.$$

**Test Problem 4** (TP4): Consider the following nonlinear system which has three equations:

$$\begin{cases} \cos \mathtt{x}_2 - \sin \mathtt{x}_1 = 0, \\ \mathtt{x}_3^{\mathtt{x}_1} - \frac{1}{\mathtt{x}_2} = 0, \\ \exp \mathtt{x}_1 - \mathtt{x}_3^2 = 0. \end{cases}$$

The solution for the above system is $\alpha \approx (0.909569, 0.661227, 1.575834)^T$. The initial vector for the iteration is taken as $^{(0)} = (1, 0.5, 1.5)^T$. The Jacobian matrix produced thus is given by

$$G'(\mathtt{x}) = \begin{pmatrix} -\cos \mathtt{x}_1 & -\sin \mathtt{x}_2 & 0 \\ \mathtt{x}_3^{\mathtt{x}_1} \ln \mathtt{x}_3 & 1/\mathtt{x}_2^2 & \mathtt{x}_3^{\mathtt{x}_1} \mathtt{x}_1/\mathtt{x}_3 \\ \exp \mathtt{x}_1 & 0 & -2\mathtt{x}_3 \end{pmatrix}.$$

**Test Problem 5** (TP5): The following nonlinear system is considered:

$$\begin{cases} \exp \mathtt{x}_1 + \mathtt{x}_1 \mathtt{x}_2 - 1 = 0, \\ \sin (\mathtt{x}_1 \mathtt{x}_2) + \mathtt{x}_1 + \mathtt{x}_2 - 1 = 0. \end{cases}$$

The starting value $\mathtt{x}^{(0)} = (0.7, 0.9)^T$ has been used for the calculations. The solution of this system is $\alpha \approx (0, 1)^T$. The Jacobian matrix is given by

$$G'(\mathtt{x}) = \begin{pmatrix} \exp \mathtt{x}_1 + \mathtt{x}_2 & \mathtt{x}_1 \\ 1 + \mathtt{x}_2 \cos (\mathtt{x}_1 \mathtt{x}_2) & 1 + \mathtt{x}_1 \cos (\mathtt{x}_1 \mathtt{x}_2) \end{pmatrix}.$$

**Test Problem 6** (TP6): The following boundary value problem is considered

$$y'' + y^3 = 0, \quad y(0) = 0, \quad y(1) = 1,$$

where equal mesh is used for dividing the interval [0, 1] which is given below

$$u_0 = 0 < u_1 < u_2 < ... < u_{m-1} < u_m = 1, \; u_{j+1} = u_j + h, \; h = 1/m.$$

Denote $y_0 = y(u_0) = 0, y_1 = y(u_1), ..., y_{m-1} = y(u_{m-1}), y_m = y(u_m) = 1$.

Discretizing the second derivative by the following difference formula

$$y'' \approx \frac{y_{r-1} - 2y_r + y_{r+1}}{h^2}, \; r = 1, 2, 3, ..., m-1,$$

we obtain $m-1$ nonlinear equations in $m-1$ variables as given below

$$y_{r-1} - 2y_r + y_{r+1} + h^2 y_r^3 = 0, \; r = 1, 2, 3, ..., m-1.$$

The above equations are solved by taking $m = 16$ and $y^{(0)} = (1, 1, ..., 1)^T$ as the initial approximation, where we get the Jacobian matrix with 43 non-zero elements as below.

$$\begin{pmatrix} 3h^2y_1^2-2 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 3h^2y_2^2-2 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 3h^2y_3^2-2 & 1 & ... & 0 & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & ... & 1 & 3h^2y_{14}^2-2 & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 3h^2y_{15}^2-2 \end{pmatrix}.$$

$$\begin{aligned}\alpha = \{&0.065997633200364677..., \quad 0.131994143490292748..., \quad 0.197981670725993839..., \\ &0.263938884538034848..., \quad 0.329824274254574844..., \quad 0.395569509201723100..., \\ &0.461072959646730428..., \quad 0.526193524526372529..., \quad 0.590744978992414345..., \\ &0.654491128910354268..., \quad 0.717142134576548678..., \quad 0.778352432953974123..., \\ &0.837720734425024994..., \quad 0.894792581480763658..., \quad 0.949065916629282713...\}^T.\end{aligned}$$

Tables 2 to 4 display number of iterations ($N$), $err_{min}$, ACOC ($p_c$) and CPU time for the test problems

Table 2: Comparison of numerical results of different methods

| Methods | TP1 | | | | TP2 | | | |
|---|---|---|---|---|---|---|---|---|
| | $N$ | $err_{min}$ | $p_c$ | CPU | $N$ | $err_{min}$ | $p_c$ | CPU |
| $2^{nd}NM$ | 10 | 1.0385e−103 | 1.99 | 1.720 | 8 | 3.9287e−145 | 2.00 | 2.169 |
| $4^{th}NR$ | 6 | 5.3845e−207 | 3.99 | 1.395 | 5 | 2.9883e−291 | 4.03 | 2.010 |
| $4^{th}BCST$ | 6 | 5.0530e−139 | 3.99 | 1.779 | 5 | 3.4950e−238 | 4.03 | 2.860 |
| $4^{th}SGS$ | 6 | 2.2282e−170 | 3.99 | 1.574 | 5 | 8.8962e−257 | 4.03 | 2.553 |
| $4^{th}BMJ$ | 6 | 4.3350e−157 | 3.99 | 1.928 | 5 | 5.5234e−247 | 4.03 | 2.738 |
| $6^{th}CHMT$ | 5 | 9.0247e−163 | 5.84 | 1.531 | 4 | 4.6407e−199 | 6.12 | 2.318 |
| $8^{th}SA$ | 5 | 0 | 7.79 | 1.480 | 4 | 0 | 7.89 | 2.476 |
| $4^{th}PM, r=0$ | 6 | 2.2282e−170 | 3.99 | 1.987 | 5 | 8.8962e−257 | 4.03 | 2.876 |
| $6^{th}PM, r=1$ | 5 | 3.6913e−275 | 6.03 | 1.693 | 4 | 3.5368e−314 | 7.12 | 2.783 |
| $8^{th}PM, r=2$ | 4 | 2.0437e−132 | 8.59 | 1.700 | 4 | 0 | 10.71 | 3.468 |
| $10^{th}PM, r=3$ | 4 | 1.7203e−221 | 10.10 | 2.069 | 3 | 4.3253e−147 | 13.85 | 3.295 |

(TP1-TP6). From the tables, we conclude that the proposed methods are the most efficient methods with least number of iterations and residual error consuming less CPU time. In particular, for the test problems 2 and 3 we get improved numerical convergence than the theoretical convergence. For the test problem 5, the presented methods require less number of iteration than $2^{nd}NM$ and better than other compared methods.

# 6 Conclusion

In this work, we have proposed a fourth order algorithm and its multi-step version having higher order convergence using weight functions to solve systems of nonlinear equations. The merit of the presented algorithms is that they do not need second order Fréchet derivative which otherwise is proved to be computationally costly and more complicated. Computational efficiencies are found using the computational cost and compared with few existing methods by finding its ratio which shows that the present methods are superior to many other methods. Numerical experimentation for six test problems have been carried out in order to illustrate and practically check the validity of the theoretical results derived. The proposed methods are compared with Newton's method and some existing fourth, sixth and eighth order methods to validate their performance. Numerical results justify the robust and efficient convergence behavior of the

Table 3: Comparison of numerical results of different methods

| Methods | TP3 | | | | TP4 | | | |
|---|---|---|---|---|---|---|---|---|
| | $N$ | $err_{min}$ | $p_c$ | CPU | $N$ | $err_{min}$ | $p_c$ | CPU |
| $2^{nd}NM$ | 9 | 8.9692e−179 | 1.99 | 4.480 | 9 | 1.0104e−107 | 2.00 | 1.811 |
| $4^{th}NR$ | 5 | 8.9692e−179 | 4.00 | 4.100 | 5 | 1.0104e−107 | 4.00 | 1.417 |
| $4^{th}BCST$ | 5 | 1.6109e−142 | 3.99 | 7.375 | 6 | 1.5698e−299 | 3.99 | 2.562 |
| $4^{th}SGS$ | 5 | 6.0847e−155 | 3.99 | 7.036 | 6 | 0 | 3.99 | 2.404 |
| $4^{th}BMJ$ | 5 | 3.1534e−149 | 3.99 | 6.927 | 6 | 0 | 3.99 | 2.489 |
| $6^{th}CHMT$ | 4 | 1.1164e−117 | 5.99 | 6.195 | 5 | 0 | 6.01 | 2.149 |
| $8^{th}SA$ | 4 | 3.6805e−226 | 7.99 | 6.207 | 5 | 0 | 7.99 | 2.187 |
| $4^{th}PM$, $r = 0$ | 5 | 6.0847e−155 | 3.99 | 6.776 | 6 | 0 | 3.99 | 2.983 |
| $6^{th}PM$, $r = 1$ | 4 | 1.2424e−183 | 6.99 | 6.042 | 5 | 0 | 6.20 | 2.428 |
| $8^{th}PM$, $r = 2$ | 4 | 0 | 9.69 | 8.865 | 4 | 7.0539e−208 | 7.94 | 2.616 |
| $10^{th}PM$, $r = 3$ | 4 | 0 | 12.69 | 9.886 | 4 | 0 | 9.27 | 3.075 |

Table 4: Comparison of numerical results of different methods

| Methods | TP5 | | | | TP6 | | | |
|---|---|---|---|---|---|---|---|---|
| | $N$ | $err_{min}$ | $p_c$ | CPU | $N$ | $err_{min}$ | $p_c$ | CPU |
| $2^{nd}NM$ | 9 | 6.3439e−141 | 1.98 | 1.619 | 8 | 4.9636e−114 | 1.99 | 5.686 |
| $4^{th}NR$ | 5 | 6.3439e−141 | 3.90 | 1.580 | 5 | 1.4101e−228 | 3.99 | 5.805 |
| $4^{th}BCST$ | 5 | 4.0943e−111 | 3.96 | 1.784 | 5 | 6.2873e−146 | 3.99 | 10.889 |
| $4^{th}SGS$ | 5 | 1.6430e−124 | 3.93 | 1.641 | 5 | 1.5789e−159 | 3.99 | 9.384 |
| $4^{th}BMJ$ | 5 | 3.5939e−117 | 3.98 | 1.685 | 5 | 4.2697e−152 | 3.99 | 9.276 |
| $6^{th}CHMT$ | 5 | 0 | 5.97 | 1.828 | 4 | 1.0412e−141 | 6.00 | 8.472 |
| $8^{th}SA$ | 4 | 1.6882e−105 | 5.91 | 1.561 | 4 | 4.5119e−155 | 5.90 | 8.742 |
| $4^{th}PM$, $r = 0$ | 5 | 1.6430e−124 | 3.93 | 1.987 | 5 | 1.5789e−159 | 3.99 | 9.285 |
| $6^{th}PM$, $r = 1$ | 4 | 2.3672e−105 | 5.96 | 1.719 | 4 | 1.2213e−174 | 5.99 | 10.239 |
| $8^{th}PM$, $r = 2$ | 4 | 1.6491e−233 | 8.03 | 1.837 | 4 | 0 | 7.74 | 12.611 |
| $10^{th}PM$, $r = 3$ | 4 | 0 | 10.51 | 2.017 | 4 | 0 | 9.65 | 12.813 |

proposed methods. The applicability of the new methods is also tested on boundary value problems for ordinary differential equations. Hence, these new methods can be considered as good competitors to many existing equivalent methods.

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